Finite Markov chains with discrete
two-dimensional parameter

Mircea Bodnariu

Abstract. In the present paper we show that each stochastic process with finite state space and two-dimensional discrete parameter associated to a family of border probabilities and a t.t.f. has a Markov property called \( *\)Markov. We call such a stochastic process a finite Markov chain with discrete two-dimensional parameter.

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1 Introduction

In this work we intend to develop a theory of finite Markov chains with discrete two-dimensional parameter. Till now we did not find any paper which treats this subject. There are many works (see the list in References) that deal with stochastic processes with two-dimensional parameter and their Markov properties, but none of them follows a close analogy with the theory of finite Markov chains with discrete one-dimensional parameter. The main instrument in this theory is the concept of stochastic matrix and even the analogous concept for two-dimensional parameter is absent in the works that we had the possibility to study.

Our start point was the concept of three-point transition function (t.t.f. for short) (which can be found in papers [8], [13], [14], [15], [16]). Trying to adapt this concept to finite state space and discrete two-dimensional parameter we were led in a natural manner to the notions of four-dimensional stochastic matrix (4-s.m. for short) and of horizontal and vertical products of such matrices and we noted that the very analogous of the stochastic matrix is a 4-s.m. which can be composed with itself using both products ([3]). First, we were interested in knowing whether there are such 4-s.m.s. In paper [3], using the convolution product for functions defined on finite sets, we showed that such 4-s.m.s exist for every finite set. Although the class of 4-s.m.s found in [3] is rather a large one, it is still particular, so that we tried to find other such 4-s.m.s. In paper [5] we found all 4-s.m.s \( p \) on \( \{0, 1\} \) for which both the horizontal product \( p \circ p \) and the vertical product \( p \vee p \) can be defined. In paper [4] we studied the necessary and sufficient condition which must be fulfilled by a 4-s.m. so that its powers determined by the horizontal and the vertical products make up a t.t.f. and we found that this condition is that the 4-s.m. has the so called double product property. Then we showed ([6]) that all 4-s.m.s found in paper [5] have the double
product property. The result obtained in [6] determined us to ask ourselves whether all 4-s.m.s. $p$ for which both the horizontal product $p \circ p$ and the vertical product $p \lor p$ can be defined have the double product property. We gave the answer in paper [7] and this is affirmative.

After we studied the 4-s.m.s and their relation with t.t.f.s, we focussed our attention upon the Markov properties of stochastic processes with finite state space and two-dimensional discrete parameter. In the present paper we show that each stochastic process with finite state space and two-dimensional discrete parameter associated to a family of border probabilities and a t.t.f. has a Markov property called *-Markov. For this reason we propose to call such a stochastic process a finite Markov chain with discrete two-dimensional parameter.

2 Three-point transition functions with discrete parameter

Let $\Gamma$ be a finite set.

**Definition 2.1.** A function $p : \Gamma^4 \to [0, 1]$ which has the property
\[ \sum_{\eta \in \Gamma} p(\alpha, \beta, \gamma, \eta) = 1 \] for all $(\alpha, \beta, \gamma) \in \Gamma^3$ is called a four dimensional stochastic matrix on $\Gamma$ (4-s.m. for short).

**Definition 2.2.** Let $p$ and $q$ be two 4-s.m.s on $\Gamma$.

a) If for every $(\alpha, \beta, \gamma, \delta) \in \Gamma^4$ the sum $\sum_{\eta \in \Gamma} p(\alpha, \beta, \xi, \eta)q(\xi, \eta, \gamma, \delta)$ does not depend on $\xi \in \Gamma$, then we define the function $p \circ q : \Gamma^4 \to [0, 1]$ by means of the relation
\[ (p \circ q)(\alpha, \beta, \gamma, \delta) = \sum_{\eta \in \Gamma} p(\alpha, \beta, \xi, \eta)q(\xi, \eta, \gamma, \delta). \]
$p \circ q$ is called the horizontal product of $p$ and $q$.

b) If for every $(\alpha, \beta, \gamma, \delta) \in \Gamma^4$ the sum $\sum_{\eta \in \Gamma} p(\alpha, \xi, \gamma, \eta)q(\xi, \beta, \eta, \delta)$ does not depend on $\xi \in \Gamma$, then we define the function $p \lor q : \Gamma^4 \to [0, 1]$ by means of the relation
\[ (p \lor q)(\alpha, \beta, \gamma, \delta) = \sum_{\eta \in \Gamma} p(\alpha, \xi, \gamma, \eta)q(\xi, \beta, \eta, \delta). \]
$p \lor q$ is called the vertical product of $p$ and $q$.

**Definition 2.3.** A family of 4-s.m.s on $\Gamma$,
\[ \{P(i,j),(i+m,j+n)\}_{i,j \in \mathbb{N}, m,n \in \mathbb{N}^*} \] such that for all $i, j \in \mathbb{N}$, $m, n, k, l \in \mathbb{N}^*$,
\[ P(i,j),(i+m,j+n) \circ P(i+m,j),(i+m+k,j+n) \text{ and } P(i,j),(i+m,j+n) \lor P(i,j),(i+m,j+n+l) \text{ can be defined and} \]
\[ P(i,j),(i+m+k,j+n) = P(i,j),(i+m,j+n) \circ P(i+m,j),(i+m+k,j+n), \]
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\[ P(i,j),(i+m,j+n+l) = P(i,j),(i+m,j+n) \lor P(i,j),(i+m,j+n+l) \]

is called three-point transition function on \( \Gamma \) with discrete time (t.t.f. for short).

**Definition 2.4.** A family of 4-s.m.s on \( \Gamma \), \( (p_{m,n})_{m,n \in \mathbb{N}}^* \), such that for all \( m, n, k, l \in \mathbb{N}^* \), \( p_{m,n} \circ p_{k,n} \) and \( p_{m,n} \lor p_{m,l} \) can be defined and

\[ p_{m+k,n} = p_{m,n} \circ p_{k,n}, \quad p_{m,n+l} = p_{m,n} \lor p_{m,l}. \]

is called homogeneous three-point transition function on \( \Gamma \) with discrete time (h.t.t.f. for short).

**Theorem 2.1.** ([7]) Let \( \varphi = (p(i,j),(i+m,j+n))_{i,j \in \mathbb{N}, m,n \in \mathbb{N}}^* \) be a family of 4-s.m.s on \( \Gamma \) and \( p_{i,j} = p(i,j),(i+1,j+1) \) for \( i, j \in \mathbb{N} \). \( \varphi \) is a t.t.f. on \( \Gamma \) with discrete parameter if and only if for any \( i, j \in \mathbb{N} \), \( p_{i,j} \circ p_{i+1,j} \) and \( p_{i,j} \lor p_{i,j+1} \) can be defined and

\[ p_{i,j} = \lor_{k=0}^{n-1} (p_{i,j+k,l+1}). \]

**Corollary 2.1.** ([7]) Let \( \varphi = (p_{m,n})_{m,n \in \mathbb{N}}^* \) be a family of 4-s.m.s on \( \Gamma \) and \( p = p_{1,1} \). \( \varphi \) is a h.t.t.f. on \( \Gamma \) if and only if \( p \circ p \) and \( p \lor p \) can be defined and \( p_{m,n} = (p^m_o)_{i,j \in \mathbb{N}} \) for any \( m, n \in \mathbb{N}^* \). \( (p^m_o)_{i,j} = \sum_{i=1}^{m} p_i, p_i = p \) for \( i = 1, ..., k \) etc.

**Examples.**

1) ([3]) Let \( \Gamma = \mathbb{Z}, q \geq 2 \). If \( P \) is a probability on \( \Gamma \) (i.e. a function \( P : \Gamma \to [0,1] \) such that \( \sum_{\eta \in \Gamma} P(\eta) = 1 \), then we define \( p(P) : \Gamma^4 \to [0,1] \) by

\[ p(P)(\alpha, \beta, \gamma, \delta) = P(\alpha - \beta - \gamma + \delta), \quad (\alpha, \beta, \gamma, \delta) \in \Gamma^4. \]

\( p(P) \) is a 4-s.m. on \( \Gamma \). If \( P \) and \( Q \) are two probabilities on \( \Gamma \), then we define the convolution product of \( P \) and \( Q \) by means of the relation

\[ (P \ast Q)(\xi) = \sum_{\theta + \omega = \xi} P(\theta)Q(\omega), \quad \xi \in \Gamma. \]

\( p(P) \circ p(Q) \) and \( p(P) \lor p(Q) \) can be defined and \( p(P) \circ p(Q) = P(P \ast Q), p(P) \lor p(Q) = p(P \ast Q) \).

For this reason, if \( (P_{i,j})_{i,j \in \mathbb{N}} \) is a family of probabilities on \( \Gamma \), then \( (p_{i,j})_{i,j \in \mathbb{N}} \), where \( p_{i,j} = p(P_{i,j}) \), is a family of 4-s.m.s on \( \Gamma \) such that \( p_{i,j} \circ p_{i+1,j} \) and \( p_{i,j} \lor p_{i,j+1} \) can be defined. In view of Theorem 6 we obtain a t.t.f. on \( \Gamma \).

For the same reason, \( p(P) \) is a 4-s.m. on \( \Gamma \) such that \( p(P) \circ p(P) \) and \( p(P) \lor p(P) \) can be defined and, consequently, \( (p_{m,n})_{m,n \in \mathbb{N}}^* \), where \( p_{m,n} = p(P^m_o) \), \( (P^k_o) = P \ast \_ \ast P, \]

\( k \) times), is a h.t.t.f. on \( \Gamma \) (Corollary 1).
2) ([5], [6]) If \( p \) is a 4-s.m. on \( \Gamma = \{0, 1\} \), then we denote \( p(\alpha, \beta, \gamma, \delta) = p_t, \) where \( t = 2^3\alpha + 2^2\beta + 2\gamma + \delta. \) We define five sorts of 4-s.m. on \( \Gamma \) giving the values of \( p_0, p_2, p_4, p_6, p_8, p_{10}, p_{12}, p_{14} \) \((p_{2k+1} = 1 - p_{2k})\).

\[
P(a, u, v) : p_0 = a, p_2 = a - u, p_4 = a - v, p_6 = a - u - v, p_8 = a + uv, p_{10} = a + uv - u, p_{12} = a + uv - v, p_{14} = a + uv - u - v.
\]

\[
p(a, s) : p_0 = a, p_2 = s(1 - a), p_4 = s(1 - a), p_6 = s - s^2(1 - a), p_8 = 1 - \frac{a}{s}, p_{10} = a, p_{12} = a, p_{14} = s(1 - a), a \neq 1, s \neq 0, s \neq \frac{a}{1 - a}.
\]

\[
p(1) : p_0 = p_2 = p_4 = p_6 = p_{10} = p_{12} = p_{14} = 1, p_8 = a, a \neq 1.
\]

\[
p(0) : p_0 = p_2 = p_4 = p_6 = p_{10} = 0, p_8 = a, a \neq 0.
\]

\[
p(0) : p_0 = 1, p_2 = p_4 = 0, p_6 = 1, p_8 = 0, p_{10} = p_{12} = 1, p_{14} = 0.
\]

\( p \) is a 4-s.m. on \( \Gamma = \{0, 1\} \) for which \( p \circ p \) and \( p \lor p \) can be defined if and only if the form of \( p \) is one from the five forms described above (see Theorem 3 in [6]). So we can give five examples of t.t.f. on \( \Gamma = \{0, 1\} \).

In addition, because \( p(a, u, v) \circ p(b, u, w) = p(aw + b - u, u, vw) \), \( p(a, u, v) \lor p(b, v, w) = p(as + b - s, us, v) \), \( p(a, s) \circ p(A, s) = p(aA + s(1 - a)(1 - A), s) \), \( p(1) \circ p(1) = p(1) \lor p(1) = p(1, 0, 0) \), \( p(0) \circ p(0) = p(0) \lor p(0) = p(0, 0, 0) \), and \( p(0) \circ p(1) = p(0) \lor p(1) = p(1) \) we get five examples of t.t.f. on \( \Gamma = \{0, 1\} \) generated by the families of 4-s.m. \( p_{i,j} \) \((p_{i,j})_{i,j \in \mathbb{N}}^N\) where: 1) \( p_{i,j} = p(a_{i,j}, u_{i,j}, v_{i,j}) \), 2) \( p_{i,j} = p(a_{i,j}, s) \), 3) \( p_{i,j} = p(1) \), 4) \( p_{i,j} = p(0) \), 5) \( p_{i,j} = p(0) \) (we applied here again Theorem 6).

3) Let \( \Gamma = \Gamma_1 \times \Gamma_2 \), where \( \Gamma_1, \Gamma_2 \) are finite sets and let \( p_i \) be a transition probability from \( \Gamma_i \) to \( \Gamma_i, i = 1, 2. \) Define \( p : \Gamma^4 \rightarrow [0, 1] \) by means of the relation

\[
p((\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2), (\delta_1, \delta_2)) = p_1(\beta_1, \delta_1)p_2(\gamma_2, \delta_2).
\]

Then \( p \) is a 4-s.m. on \( \Gamma \) for which \( p \circ p \) and \( p \lor p \) can be defined.

It shows that, if \( p_{i_1,i_2} \) \((p_{i_1,i_2})_{i,j \in \mathbb{N}}^N\) and \( p_{i_2,i_3} \) \((p_{i_2,i_3})_{i,j \in \mathbb{N}}^N\) are two families of transition probabilities from \( \Gamma \) to \( \Gamma \), then \( (p_{i,j})_{i,j \in \mathbb{N}}^N \), where

\[
p_{i,j}((\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2), (\delta_1, \delta_2)) = p_{1,i,j}(\beta_1, \delta_1)p_{2,i,j}(\gamma_2, \delta_2),
\]

is a family of 4-s.m. on \( \Gamma \) such that \( p_{i,j} \circ p_{i,j+1} \) and \( p_{i,j} \lor p_{i,j+1} \) can be defined and, in view of Theorem 1.1 , it generates a t.t.f. on \( \Gamma \). In the same way it can be seen that, if \( p_1 \) and \( p_2 \) are transition probabilities from \( \Gamma \) to \( \Gamma \), then the family of 4-s.m. \( (p_{m,n})_{m,n \in \mathbb{N}}^N \), where

\[
p_{m,n}((\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2), (\delta_1, \delta_2)) = p_{1,m}(\beta_1, \delta_1)p_{2,n}(\gamma_2, \delta_2),
\]

is a h.t.t.f. on \( \Gamma \) (Corollary 1.1).
3 The Markov property

Let \((x_{m,n})_{(m,n)\in \mathbb{N}^2}\) be a stochastic process on the state space \(\Gamma\) and having the probability space \((\Omega, \mathcal{K}, P)\). If \((s, t) \in \mathbb{N}^2\), then \(T_{s,t}^* = \{(m, n) \in \mathbb{N}^2 \mid m \leq s\ or\ n \leq t\} \).

**Definition 3.1.** We say that the stochastic process \((x_{m,n})_{(m,n)\in \mathbb{N}^2}\) is \(*\)-Markov if for each \(m', m'', n', n''\) from \(\mathbb{N}\), \(m' < m'', n' < n''\) and each finite \(M, M \subset T_{m',n'}^*\) and \\(\{(m', n'), (m'', n'')\}\) \(\subset M\) one has

\[
P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m,n} = \eta_{m,n}, (m,n) \in M) = \]

\[
P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m',n'} = \eta_{m',n'}, x_{m',n''} = \eta_{m',n''}, x_{m'',n'} = \eta_{m'',n'})
\]

for each \(\eta_{m'',n''} \in \Gamma\) and each \((\eta_{m,n})_{(m,n)\in M} \in \Gamma^{|M|}\) (\(|M|\) is the number of elements of \(M\)).

If \(\Gamma\) is a finite set, \(\mu = \{\mu_{0,m_1,\ldots,m_k,n_1,\ldots,n_l} \mid 0 < m_1 < \ldots < m_k, 0 < n_1 < \ldots < n_l\}\) is a family of border probabilities on \(\Gamma\) (i.e. each \(\mu_{0,m_1,\ldots,m_k,n_1,\ldots,n_l}\) is a probability on \(\Gamma^{1+k+l}\) and the family is projective) and \(\overline{\mu} = (\mu_{i,j}, (i+m+j+n))_{i,j\in \mathbb{N}, m,n\in \mathbb{N}}^*\) a t.t.f. on \(\Gamma\), then there is a stochastic process \((x_{m,n})_{(m,n)\in \mathbb{N}^2}\) on the state space \(\Gamma\) and having the probability space \((\Omega, \mathcal{K}, P)\) such that

\[
P(x_{m,n} = \eta_{m,n}, m \in \{0, m_1, \ldots, m_k\}, n \in \{0, n_1, \ldots, n_l\}) =
\]

\[
\prod_{l=1}^{k-1} \prod_{j=0}^{1} \prod_{i=0}^{n_1-1} \prod_{m,s=0}^{m_1-1} \prod_{n,t=0}^{n_1-1} \mu_{0,m_1,\ldots,m_k,n_1,\ldots,n_l} (\eta_{0,0}, \eta_{m_1,0}, \ldots, \eta_{m_1,0}, \eta_{0,n_1}, \ldots, \eta_{0,n_1})
\]

for all \(0 = m_0 < m_1 < \ldots < m_k, 0 = n_0 < n_1 < \ldots < n_l\) and all \(\eta_{m,n} \in \Gamma\), \(m \in \{0,m_1,\ldots,m_k\}\), \(n \in \{0,n_1,\ldots,n_l\}\).

In the following we will show that each stochastic process \((x_{m,n})_{(m,n)\in \mathbb{N}^2}\) defined as above is \(*\)-Markov.

4 Three-point transition functions and the \(*\)-Markov property

**Theorem 4.1.** Let \(\Gamma\) be a finite set, \(\mu = \{\mu_{0,m_1,\ldots,m_k,n_1,\ldots,n_l} \mid 0 < m_1 < \ldots < m_k, 0 < n_1 < \ldots < n_l\}\) a family of border probabilities on \(\Gamma\), \(\overline{\mu} = (\mu_{i,j}, (i+m+j+n))_{i,j\in \mathbb{N}, m,n\in \mathbb{N}}^*\) a t.t.f. on \(\Gamma\) and \(x = (x_{m,n})_{(m,n)\in \mathbb{N}^2}\) the associated stochastic process. Then \(x\) is \(*\)-Markov.
Proof. Let \( m', m'', n', n'' \in \mathbb{N} \), \( m' < m'', n' < n'' \) and \( M \) a finite set, \( M \subset T_{m', n'}^\ast \) and \( \{(m', n'), (m', n''), (m'', n') \} \subset M \). We can find \( m_1, \ldots, m_k, n_1, \ldots, n_l \in \mathbb{N}, k \geq 2 \\ l \geq 2, m_1 < \ldots < m_k, n_1 < \ldots < n_l \) such that \( M \subset D(m_1, \ldots, m_k; n_1, \ldots, n_l) \), where \( D(m_1, \ldots, m_k; n_1, \ldots, n_l) = \{(m_i, n_j) \mid i = 1, \ldots, k, j = 1, \ldots, l\} \) and there are \( q, u, r, v \), \( 1 \leq q < u \leq k, 1 \leq r < v \leq l, m' = m_q, n' = n_r, m'' = m_u, n'' = n_v \).

Let \( A = (D(m_1, \ldots, m_k; n_1, \ldots, n_l) \setminus D(m_{q+1}, \ldots, m_k; n_{r+1}, \ldots, n_l)) \cup \{(m_u, n_v)\} \) and \( \overline{A} = D(m_1, \ldots, m_k; n_1, \ldots, n_l) \setminus D(m_{q+1}, \ldots, m_k; n_{r+1}, \ldots, n_l) \). We can write
\[
P(x_m = \eta_{m,n}, (m, n) \in A) = \sum_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A}} P(x_m = \eta_{m,n}, (m, n) \in A) = \sum_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A}} \sum_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A}} P(x_m = \eta_{m,n}, (m, n) \in A) = \sum_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A}} S = \sum_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A}} S
\]
where \( S = \sum_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A}} P(x_m = \eta_{m,n}, (m, n) \in A) \) and so we obtain \( P(x_m = \eta_{m,n}, (m, n) \in A) \) and the same relation can be written for \( \overline{A} \). From this relation we see that for every two finite sets \( M \) and \( M_1 \) so that \( M \subset T_{m', n'}^\ast \), \( M_1 \subset T_{m'', n'}^\ast \) and \( \{(m', n'), (m', n''), (m'', n') \} \subset M \cap M_1 \) the following equality holds \( P(x_m = \eta_{m,n}, (m, n) \in M) = P(x_m = \eta_{m,n}, (m, n) \in M) = P(x_m = \eta_{m,n}, (m, n) \in M) \) and we get the relation (1). Q.E.D.
Because Theorem 4.1 does not demand any conditions neither for the family of border probabilities \( \mu \), nor for the t.t.f \( \overline{p} \), all the t.t.f.s (and the h.t.t.f.s) described in section 1 are good as examples.

References


Mircea Bodnariu
University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independentei 313
RO-77206 Bucharest, Romania
e-mail: bodnariu@mathem.pub.ro