Finite Markov chains with discrete two-dimensional parameter

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Abstract. In the present paper we show that each stochastic process with finite state space and two-dimensional discrete parameter associated to a family of border probabilities and a t.t.f. has a Markov property called **Markov.* We call such a stochastic proces a *finite Markov chain with discrete two-dimensional parameter.*

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1 Introduction

In this work we intend to develop a theory of *finite Markov chains with discrete two-dimensional parameter.* Till now we did not find any paper which treats this subject. There are many works (see the list in References) that deal with stochastic processes with two-dimensional parameter and their Markov properties, but none of them follows a close analogy with the theory of finite Markov chains with discrete one-dimensional parameter. The main instrument in this theory is the concept of *stochastic matrix* and even the analogous concept for two-dimensional parameter is absent in the works that we had the possibility to study.

Our start point was the concept of *three-point transition function* (t.t.f. for short) (which can be found in papers [8], [13], [14], [15], [16]). Trying to adapt this concept to finite state space and discrete two-dimensional parameter we were led in a natural manner to the notions of *four-dimensional stochastic matrix* (4-s.m. for short) and of *horizontal and vertical products* of such matrices and we noted that the very analogous of the stochastic matrix is a 4-s.m. which can be composed with itself using both products ([3]). First, we were interested in knowing whether there are such 4-s.m.s. In paper [3], using the convolution product for functions defined on finite sets, we showed that such 4-s.m.s exist for every finite set. Although the class of 4-s.m.s found in [3] is rather a large one, it is still particular, so that we tried to find other such 4-s.m.s. In paper [5] we found all 4-s.m.s p on $\{0, 1\}$ for which both the horizontal product $p \circ p$ and the vertical product $p \vee p$ can be defined. In paper [4] we studied the necessary and sufficient condition which must be fulfilled by a 4-s.m. so that its powers determined by the horizontal and the vertical products make up a t.t.f. and we found that this condition is that the 4-s.m. has the so called *double product property.* Then we showed ([6]) that all 4-s.m.s found in paper [5] have the double

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product property. The result obtained in [6] determined us to ask ourselves whether all 4-s.m.s p for which both the horizontal product $p \circ p$ and the vertical product $p \vee p$ can be defined have the double product property. We gave the answer in paper [7] and this is affirmative.

After we studied the 4-s.m.s and their relation with t.t.f.s, we focussed our attention upon the *Markov properties* of stochastic processes with finite state space and two-dimensional discrete parameter. In the present paper we show that each stochastic process with finite state space and two-dimensional discrete parameter associated to a family of border probabilities and a t.t.f. has a Markov property called **-Markov.* For this reason we propose to call such a stochastic proces a *finite Markov chain with discrete two-dimensional parameter.*

2 Three-point transition functions with discrete parameter

Let Γ be a finite set.

Definition 2.1. A function $p : \Gamma^4 \to [0,1]$ which has the property $\sum\limits_{\eta\in\Gamma} p(\alpha,\beta,\gamma,\eta) = 1$ for all $(\alpha,\beta,\gamma) \in \Gamma^3$ is called *a four dimensional stochastic matrix on* Γ (4-s.m. for short).

Definition 2.2. Let p and q be two 4-s.m.s on Γ.

a) If for every $(\alpha,\beta,\gamma,\delta) \in \Gamma^4$ the sum $\sum\limits_{\eta\in\Gamma} p(\alpha,\beta,\xi,\eta)q(\xi,\eta,\gamma,\delta)$ does not depend on $\xi \in \Gamma$, then we define the function $p \circ q : \Gamma^4 \to [0,1]$ by means of the relation

$$(p \circ q)(\alpha,\beta,\gamma,\delta) = \sum_{\eta\in\Gamma} p(\alpha,\beta,\xi,\eta)q(\xi,\eta,\gamma,\delta).$$

$p \circ q$ is called *the horizontal product* of p and q.

b) If for every $(\alpha,\beta,\gamma,\delta) \in \Gamma^4$ the sum $\sum\limits_{\eta\in\Gamma} p(\alpha,\xi,\gamma,\eta)q(\xi,\beta,\eta,\delta)$ does not depend on $\xi \in \Gamma$, then we define the function $p \vee q : \Gamma^4 \to [0,1]$ by means of the relation

$$(p \vee q)(\alpha,\beta,\gamma,\delta) = \sum_{\eta\in\Gamma} p(\alpha,\xi,\gamma,\eta)q(\xi,\beta,\eta,\delta).$$

$p \vee q$ is called *the vertical product* of p and q.

Definition 2.3. A family of 4-s.m.s on Γ, $\left(p_{(i,j),(i+m,j+n)}\right)_{i,j\in\mathbb{N},m,n\in\mathbb{N}^*}$ such that for all $i,j \in \mathbb{N}$, $m,n,k,l \in \mathbb{N}^*$, $p_{(i,j),(i+m,j+n)} \circ p_{(i+m,j),(i+m+k,j+n)}$ and $p_{(i,j),(i+m,j+n)} \vee p_{(i,j+n),(i+m,j+n+l)}$ can be defined and

$$p_{(i,j),(i+m+k,j+n)} = p_{(i,j),(i+m,j+n)} \circ p_{(i+m,j),(i+m+k,j+n)},$$

$$p_{(i,j),(i+m,j+n+l)} = p_{(i,j),(i+m,j+n)} \vee p_{(i,j+n),(i+m,j+n+l)}$$

is called *three-point transition function on* Γ *with discrete time* (t.t.f. for short).

Definition 2.4. A family of 4-s.m.s on Γ , $(p_{m,n})_{m,n\in\mathbb{N}^*}$ such that for all $m,n,k,l \in \mathbb{N}^*$, $p_{m,n} \circ p_{k,n}$ and $p_{m,n} \vee p_{m,l}$ can be defined and

$$p_{m+k,n} = p_{m,n} \circ p_{k,n}, \quad p_{m,n+l} = p_{m,n} \vee p_{m,l}.$$

is called *homogeneous three-point transition function on* Γ *with discrete time* (h.t.t.f. for short).

Theorem 2.1. ([7]) *Let* $\overline{p} = (p_{(i,j),(i+m,j+n)})_{i,j\in\mathbb{N},m,n\in\mathbb{N}^*}$ *be a family of 4-s.m.s on* Γ *and* $p_{i,j} = p_{(i,j),(i+1,j+1)}$ *for* $i,j \in \mathbb{N}$. $\overline{p}$ *is a t.t.f. on* Γ *with discrete parameter if and only if for any* $i,j \in \mathbb{N}$, $p_{i,j} \circ p_{i+1,j}$ *and* $p_{i,j} \vee p_{i,j+1}$ *can be defined and* $p_{(i,j),(i+m,j+n)} = \vee_{l=0}^{n-1}(\circ_{k=0}^{m-1} p_{i+k,j+l})$.

Corollary 2.1. ([7]) *Let* $\overline{p} = (p_{m,n})_{m,n\in\mathbb{N}^*}$ *be a family of 4-s.m.s on* Γ *and* $p = p_{1,1}$. $\overline{p}$ *is a h.t.t.f. on* Γ *if and only if* $p \circ p$ *and* $p \vee p$ *can be defined and* $p_{m,n} = (p_\circ^m)_\vee^n$ *for any* $m,n \in \mathbb{N}^*$. *(*$p_\circ^m = \circ_{i=1}^m p_i, p_i = p$ *for* $i = 1,...,k$ *etc.).*

Examples.

1) ([3]) Let $\Gamma = \mathbb{Z}_q$, $q \geq 2$. If P is a probability on Γ (i.e. a function $P : \Gamma \to [0,1]$ such that $\sum\limits_{\eta\in\Gamma} P(\eta) = 1$), then we define $p(P) : \Gamma^4 \to [0,1]$ by

$$p(P)(\alpha,\beta,\gamma,\delta) = P(\alpha-\beta-\gamma+\delta),\ (\alpha,\beta,\gamma,\delta) \in \Gamma^4.$$

$p(P)$ is a 4-s.m. on Γ. If P and Q are two probabilities on Γ, then we define *the convolution product of* P *and* Q by means of the relation

$$(P*Q)(\xi) = \sum_{\theta+\omega=\xi} P(\theta)Q(\omega),\ \xi \in \Gamma.$$

$p(P)\circ p(Q)$ and $p(P)\vee p(Q)$ can be defined and $p(P)\circ p(Q) = p(P*Q)$, $p(P)\vee p(Q) = p(P*Q)$.

For this reason, if $(P_{i,j})_{i,j\in\mathbb{N}}$ is a family of probabilities on Γ, then $(p_{i,j})_{i,j\in\mathbb{N}}$, where $p_{i,j} = p(P_{i,j})$, is a family of 4-s.m.s on Γ such that $p_{i,j} \circ p_{i+1,j}$ and $p_{i,j} \vee p_{i,j+1}$ can be defined. In view of Theorem 6 we obtain a t.t.f. on Γ.

For the same reason, $p(P)$ is a 4-s.m. on Γ such that $p(P)\circ p(P)$ and $p(P)\vee p(P)$ can be defined and, consequently, $(p_{m,n})_{m,n\in\mathbb{N}^*}$, where $p_{m,n} = p(P_*^{mn})$ ($P_*^k = P*\ldots*P$, k times), is a h.t.t.f. on Γ (Corollary 1).

2) ([5], [6])) If p is a 4-s.m. on $\Gamma=\{0,1\}$, then we denote $p(\alpha,\beta,\gamma,\delta)=p_t$, where $t=2^3\alpha+2^2\beta+2\gamma+\delta$. We define five sorts of 4-s.m. on Γ giving the values of p_0, p_2, p_4, p_6, p_8, p_{10}, p_{12}, p_{14} ($p_{2k+1}=1-p_{2k}$).

$p(a,u,v): p_0=a$, $p_2=a-u$, $p_4=a-v$, $p_6=a-u-v$, $p_8=a+uv$, $p_{10}=a+uv-u$, $p_{12}=a+uv-v$, $p_{14}=a+uv-u-v$.

$p(a,s): p_0=a$, $p_2=s(1-a)$, $p_4=s(1-a)$, $p_6=s-s^2(1-a)$, $p_8=1-\dfrac{a}{s}$, $p_{10}=a$, $p_{12}=a$, $p_{14}=s(1-a)$, $a\neq 1$, $s\neq 0$, $s\neq\dfrac{a}{1-a}$.

$p1(a): p_0=p_2=p_4=p_8=p_{10}=p_{12}=p_{14}=1$, $p_6=a$, $a\neq 1$.

$p0(a): p_0=p_2=p_4=p_6=p_{10}=p_{12}=p_{14}=0$, $p_8=a$, $a\neq 0$.

$p01: p_0=1$, $p_2=p_4=0$, $p_6=1$, $p_8=0$, $p_{10}=p_{12}=1$, $p_{14}=0$.

p is a 4-s.m. on $\Gamma=\{0,1\}$ for which $p\circ p$ and $p\vee p$ can be defined if and only if the form of p is one from the five forms described above (see Theorem 3 in [6]). So we can give five examples of h.t.t.f. on $\Gamma=\{0,1\}$.

In addition, because $p(a,u,v)\circ p(b,u,w)=p(aw+b-w,u,vw)$, $p(a,u,v)\vee p(b,s,v)=p(as+b-s,us,v)$, $p(a,s)\circ p(A,s)=p(a,s)\vee p(A,s)=p(aA+s(1-a)(1-A),s)$, $p1(a_1)\circ p1(a_2)=p1(a_1)\vee p1(a_2)=p(1,0,0)$, $p0(a_1)\circ p0(a_2)=p0(a_1)\vee p0(a_2)=p(0,0,0)$, and $p01\circ p01=p01\vee p01=p01$ we get five examples of t.t.f. on $\Gamma=\{0,1\}$ generated by the families of 4-s.m. $(p_{i,j})_{i,j\in\mathbb{N}}$, where: 1) $p_{i,j}=p(a_{i,j},u_j,v_i)$, 2) $p_{i,j}=p(a_{i,j},s)$, 3) $p_{i,j}=p1(a_{i,j})$, 4) $p_{i,j}=p0(a_{i,j})$, 5) $p_{i,j}=p01$ (we applied here again Theorem 6).

3) Let $\Gamma=\Gamma_1\times\Gamma_2$, where Γ_1,Γ_2 are finite sets and let p_i be a transition probability from Γ_i to Γ_i, $i=1,2$. Define $p:\Gamma^4\to[0,1]$ by means of the relation

$$p((\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2),(\delta_1,\delta_2))=p_1(\beta_1,\delta_1)p_2(\gamma_2,\delta_2).$$

Then p is a 4-s.m. on Γ for which $p\circ p$ and $p\vee p$ can be defined.

It shows that, if $(p_{1;i,j})_{i,j\in\mathbb{N}}$ and $(p_{2;i,j})_{i,j\in\mathbb{N}}$ are two families of transition probabilities from Γ to Γ, then $(p_{i,j})_{i,j\in\mathbb{N}}$, where

$$p_{i,j}((\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2),(\delta_1,\delta_2))=p_{1;i,j}(\beta_1,\delta_1)p_{2;i,j}(\gamma_2,\delta_2),$$

is a family of 4-s.m. on Γ such that $p_{i,j}\circ p_{i+1,j}$ and $p_{i,j}\vee p_{i,j+1}$ can be defined and, in view of Theorem 1.1 , it generates a t.t.f. on Γ. In the same way it can be seen that, if p_1 and p_2 are transition probabilities from Γ to Γ, then the family of 4-s.m. $(p_{m,n})_{m,n\in\mathbb{N}^*}$, where

$$p_{m,n}((\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2),(\delta_1,\delta_2))=p_1^m(\beta_1,\delta_1)p_2^n(\gamma_2,\delta_2),$$

is a h.t.t.f. on Γ (Corollary 1.1).

3 The Markov property

Let $(x_{m,n})_{(m,n)\in\mathbb{N}^2}$ be a stochastic process on the state space Γ and having the probability space $(\Omega, \mathcal{K}, P)$. If $(s,t) \in \mathbb{N}^2$, then $T^*_{s,t} = \{(m,n) \in \mathbb{N}^2 \mid m \le s \text{ or } n \le t\}$.

Definition 3.1. We say that the stochastic process $(x_{m,n})_{(m,n)\in\mathbb{N}^2}$ is **-Markov* if for each m', m'', n', n'' from $\mathbb{N}$, $m' < m''$, $n' < n''$ and each finite M, $M \subset T^*_{m',n'}$ and $\{(m',n'), (m',n''), (m'',n')\} \subset M$ one has

$$\begin{aligned}&P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m,n} = \eta_{m,n},\, (m,n) \in M) = \\ &P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m',n'} = \eta_{m',n'},\, x_{m',n''} = \eta_{m',n''},\, x_{m'',n'} = \eta_{m'',n'})\end{aligned} \tag{1}$$

for each $\eta_{m'',n''} \in \Gamma$ and each $(\eta_{m,n})_{(m,n)\in M} \in \Gamma^{|M|}$ ($|M|$ is the number of elements of M).

If Γ is a finite set, $\mu = \{\mu_{0;m_1,\ldots,m_k;n_1,\ldots,n_l} \mid 0 < m_1 < \ldots < m_k,\ 0 < n_1 < \ldots < n_l\}$ is a family of border probabilities on Γ (i.e. each $\mu_{0;m_1,\ldots,m_k;n_1,\ldots,n_l}$ is a probability on Γ^{1+k+l} and the family is projective) and $\overline{p} = (p_{(i,j),(i+m,j+n)})_{i,j\in\mathbb{N},m,n\in\mathbb{N}^*}$ a t.t.f. on Γ, then there is a stochastic process $(x_{m,n})_{(m,n)\in\mathbb{N}^2}$ on the state space Γ and having the probability space $(\Omega, \mathcal{K}, P)$ such that

$$\begin{aligned}&P(x_{m,n} = \eta_{m,n}, m \in \{0, m_1, \ldots, m_k\}, n \in \{0, n_1, \ldots, n_l\}) = \\ &\mu_{0;m_1,\ldots,m_k;n_1,\ldots,n_l}(\eta_{0,0}, \eta_{m_1,0}, \ldots, \eta_{m_k,0}, \eta_{0,n_1}, \ldots, \eta_{0,n_l}) \times \\ &\prod_{j=0}^{l-1}\prod_{i=0}^{k-1} p_{(m_i,n_j),(m_{i+1},n_{j+1})}(\eta_{m_i,n_j}, \eta_{m_i,n_{j+1}}, \eta_{m_{i+1},n_j}, \eta_{m_{i+1},n_{j+1}})\end{aligned} \tag{2}$$

for all $0 = m_0 < m_1 < \ldots < m_k$, $0 = n_0 < n_1 < \ldots < n_l$ and all $\eta_{m,n} \in \Gamma$, $m \in \{0, m_1, \ldots, m_k\}$, $n \in \{0, n_1, \ldots, n_l\}$.

In the following we will show that each stochastic process $(x_{m,n})_{(m,n)\in\mathbb{N}^2}$ defined as above is *-Markov.

4 Three-point transition functions and the *-Markov property

Theorem 4.1. *Let* Γ be a finite set, $\mu = \{\mu_{0;m_1,\ldots,m_k;n_1,\ldots,n_l} \mid 0 < m_1 < \ldots < m_k,\ 0 < n_1 < \ldots < n_l\}$ a family of border probabilities on Γ, $\overline{p} = (p_{(i,j),(i+m,j+n)})_{i,j\in\mathbb{N},m,n\in\mathbb{N}^*}$ *a t.t.f. on* Γ *and* $x = (x_{m,n})_{(m,n)\in\mathbb{N}^2}$ *the associated stochastic process. Then* x *is *-Markov.*

Proof. Let m', m'', n', $n'' \in \mathbb{N}$, $m' < m''$, $n' < n''$ and M a finite set, $M \subset T^*_{m',n'}$ and $\{(m',n'), (m',n''), (m'',n')\} \subset M$. We can find $m_1, \ldots, m_k, n_1, \ldots, n_l \in \mathbb{N}$, $k \geq 2$, $l \geq 2$, $m_1 < \ldots < m_k$, $n_1 < \ldots < n_l$ such that $M \subset D(m_1, \ldots, m_k; n_1, \ldots, n_l)$, where $D(m_1, \ldots, m_k;\, n_1, \ldots, n_l) = \{(m_i, n_j) \mid i = 1, \ldots, k,\ j = 1, \ldots, l\}$ and there are q, u, r, v, $1 \leq q < u \leq k$, $1 \leq r < v \leq l$, $m' = m_q$, $n' = n_r$, $m'' = m_u$, $n'' = n_v$.

Let $A = (D(m_1, \ldots, m_k; n_1, \ldots, n_l) \setminus D(m_{q+1}, \ldots, m_k; n_{r+1}, \ldots, n_l)) \cup \{(m_u, n_v)\}$ and $\overline{A} = D(m_1, \ldots, m_k; n_1, \ldots, n_l) \setminus D(m_{q+1}, \ldots, m_k; n_{r+1}, \ldots, n_l)$. We can write $P(x_{m,n} = \eta_{m,n},\, (m,n) \in A) =$
$$\sum_{\eta_{s,t} \in \Gamma,\, (s,t) \in D(m_1,\ldots,m_k;n_1,\ldots,n_l) \setminus A} P(x_{m,n} = \eta_{m,n},\, (m,n) \in A,$$
$x_{s,t} = \eta_{s,t},\ (s,t) \in D(m_1, \ldots, m_k; n_1, \ldots, n_l) \setminus A)$ and the same relation can be written for $\overline{A}$. Taking into consideration the relation (2), we get
$P(x_{m,n} = \eta_{m,n},\, (m,n) \in A) = P(x_{m,n} = \eta_{m,n},\, (m,n) \in \overline{A})\cdot$
$p_{(m_q,n_r),(m_u,n_v)}(\eta_{(m_q,n_r)}, \eta_{(m_q,n_v)}, \eta_{(m_u,n_r)}, \eta_{(m_u,n_v)})$.

Since $M \subset \overline{A}$, we have $P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m,n} = \eta_{m,n},\ (m,n) \in M) = P(x_{m_u,n_v} = \eta_{m_u,n_v} \mid x_{m,n} = \eta_{m,n},\ (m,n) \in M) =$
$$\frac{P(x_{m_u,n_v} = \eta_{m_u,n_v},\ x_{m,n} = \eta_{m,n}, (m,n) \in M)}{P(x_{m,n} = \eta_{m,n}, (m,n) \in M)} =$$
$$\frac{\sum\limits_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A} \setminus M} P(x_{m_u,n_v} = \eta_{m_u,n_v},\ x_{m,n} = \eta_{m,n}, (m,n) \in \overline{A})}{\sum\limits_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A} \setminus M} P(x_{m,n} = \eta_{m,n}, (m,n) \in \overline{A})} =$$
$$\frac{\sum\limits_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A} \setminus M} P(x_{m,n} = \eta_{m,n}, (m,n) \in A)}{\sum\limits_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A} \setminus M} P(x_{m,n} = \eta_{m,n}, (m,n) \in \overline{A})} =$$
$$\frac{S}{\sum\limits_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A} \setminus M} P(x_{m,n} = \eta_{m,n}, (m,n) \in \overline{A})},$$
where $S = \sum\limits_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A} \setminus M} P(x_{m,n} = \eta_{m,n}, (m,n) \in \overline{A})\cdot$
$p_{(m_q,n_r),(m_u,n_v)}(\eta_{(m_q,n_r)}, \eta_{(m_q,n_v)}, \eta_{(m_u,n_r)}, \eta_{(m_u,n_v)})$

Since $\{(m_q, n_r), (m_q, n_v), (m_u, n_v)\} \subset M$, we can write
$S = p_{(m_q,n_r),(m_u,n_v)}(\eta_{(m_q,n_r)}, \eta_{(m_q,n_v)}, \eta_{(m_u,n_r)}, \eta_{(m_u,n_v)})\cdot$
$\sum\limits_{\eta_{m,n} \in \Gamma, (m,n) \in \overline{A} \setminus M} P(x_{m,n} = \eta_{m,n}, (m,n) \in \overline{A})$ and so we obtain $P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m,n} = \eta_{m,n},\ (m,n) \in M) = p_{(m_q,n_r),(m_u,n_v)}(\eta_{(m_q,n_r)}, \eta_{(m_q,n_v)}, \eta_{(m_u,n_r)}, \eta_{(m_u,n_v)})$.

From this relation we see that for every two finite sets M and M_1 so that $M \subset T^*_{m',n'}$, $M_1 \subset T^*_{m',n'}$ and $\{(m',n'), (m',n''), (m'',n')\} \subset M \cap M_1$ the following equality holds $P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m,n} = \eta_{m,n},\ (m,n) \in M) = P(x_{m'',n''} = \eta_{m'',n''} \mid x_{m,n} = \eta_{m,n},\ (m,n) \in M_1)$. Particularly, we can take $M_1 = \{(m',n'), (m',n''), (m'',n')\}$ and we get the relation (1). Q.E.D.

Because Theorem 4.1 does not demand any conditions neither for the family of border probabilities μ, nor for the t.t.f $\overline{p}$, all the t.t.f.s (and the h.t.t.f.s) described in section 1 are good as examples.

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