Discrete geometric dynamics

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Abstract. Section 1 summarizes our recent discovery of continuous geometric dynamics. Section 2 introduces the idea of discrete geometric dynamics, as a framework for the computational science and engineering of the future. Paraphrasing Rüdiger [7], this will allow to predict and simulate the geometric dynamics of states in the biological systems we consist of, in the ecological and economical systems we live in, and in the technical systems we make use of. Section 3 gives the mathematical frames describing Pendulum, Lorentz, and ABC geometric dynamics. Section 4 exhibits some computer-generated plots for geometric dynamics mentioned in Section 3, using a specialized MAPLE software designed by our research team.

Mathematics Subject Classification 2000: 70H03, 78A35, 65P10
Key words: Geometric dynamics, variational integrator, graphical simulations

1 Continuous Geometric Dynamics

Let M be an n-dimensional differentiable manifold. A C^∞ vector field X on M defines the flow

$$\frac{dx}{dt} = X(x). \tag{1}$$

A semi-Riemann metric g on the manifold M is a C^∞ symmetric tensor field of type (0,2) which assigns to each point $x \in M$ a nondegenerate inner product $g(x)$ on the tangent space T_xM of signature (r, s). The pair (M, g) is called a *semi-Riemann manifold.*

The vector field X and the semi-Riemann metric g determine the *energy*

$$f : M \to R, \quad f = \frac{1}{2}g(X, X).$$

The vector field (flow) X on (M, g) is called:
1) *timelike*, if $f < 0$;
2) *nonspacelike or causal*, if $f \leq 0$;
3) *null or lightlike*, if $f = 0$;
4) *spacelike*, if $f > 0$.

Let X be a nonwhere vanishing vector field, of everywhere constant energy. Upon rescaling, it may be supposed that $f \in \{-1, 0, 1\}$. Generally, $\mathcal{E}$ is the set of zeros of the vector field X. If $\mathcal{E} \neq \emptyset$, then this rescaling is possible only on $M \setminus \mathcal{E}$.

Proceedings of The First French-Romanian Colloquium of Numerical Physics, October 30-31, 2000, Bucharest, Romania.

Let ∇ be the Levi-Civita connection of (M, g). Using the operator $\frac{\nabla}{dt}$ (covariant differentiation along a solution) we obtain the prolongation

$$\frac{\nabla}{dt}\frac{dx}{dt} = \nabla_{\frac{dx}{dt}} X \tag{2}$$

of the differential system (1) or of any perturbation of the system (1) obtained adding to the second member X a parallel vector field Y with respect to the covariant derivative ∇. The prolongation by derivation represents the general dynamics of the flow. The vector field Y can be used to illustrate a progression from stable to unstable flows, or converse.

The vector field X, the metric g, and the connection ∇ determine the external (1,1)-tensor field

$$F = \nabla X - g^{-1} \otimes g(\nabla X),$$
$$F_j{}^i = \nabla_j X^i - g^{ih} g_{kj} \nabla_h X^k, \quad i, j, h, k = 1, \dots, n,$$

which characterizes the *helicity* of vector field (flow) X.

First we write the differential system (2) in the equivalent form

$$\frac{\nabla}{dt}\frac{dx}{dt} = g^{-1} \otimes g(\nabla X)\left(\frac{dx}{dt}\right) + F\left(\frac{dx}{dt}\right). \tag{2'}$$

Successively we modify the differential system (2′) as follows:

$$\frac{\nabla}{dt}\frac{dx}{dt} = g^{-1} \otimes g(\nabla X)(X) + F\left(\frac{dx}{dt}\right), \tag{3}$$

$$\frac{\nabla}{dt}\frac{dx}{dt} = g^{-1} \otimes g(\nabla X)\left(\frac{dx}{dt}\right) + F(X), \tag{4}$$

$$\frac{\nabla}{dt}\frac{dx}{dt} = g^{-1} \otimes g(\nabla X)(X) + F(X). \tag{5}$$

Obviously, the second order systems (3), (4), (5) are prolongations of the first order system (1). Each of them is connected either to the dynamics of the field X or to the dynamics of a particle which is sensitive to the vector field X. Since

$$g^{-1} \otimes g(\nabla X)(X) = grad\, f,$$

we shall show that the prolongation (3) describes a conservative dynamics of the vector field X or of a particle which is sensitive to the vector field X. The physical phenomenon produced by (4) or (5) was not yet studied.

Theorem. 1) *If $F = 0$, then the kinematic system (1) prolonges to a potential dynamical system with n degrees of freedom, namely*

$$\frac{\nabla}{dt}\frac{dx}{dt} = grad\, f. \tag{3'}$$

2) *If $F \neq 0$, then the kinematic system (1) prolonges to a non-potential dynamical system with n degrees of freedom, namely*

$$\frac{\nabla}{dt}\frac{dx}{dt} = grad\, f + F\left(\frac{dx}{dt}\right). \tag{3''}$$

Corollary. *If the metric g is chosen such that $f \in \{-1, 0, 1\}$ on $M \setminus \mathcal{E}$, then the flow generated by X is global and the dynamical systems (3′), (3″) are reduced to*

$$\frac{\nabla^2 x}{dt^2} = 0, \ \frac{\nabla^2 x}{dt^2} = F\left(\frac{dx}{dt}\right).$$

Let us show that the dynamical systems (3′) and (3″) are conservative. To simplify the exposition we identity the tangent bundle TM with the cotangent bundle T^*M using the semi-Riemann metric g.

Theorem. 1) *The trajectories of the dynamical system (3′) are the extremals of the Lagrangian*

$$L = \frac{1}{2} g\left(\frac{dx}{dt}, \frac{dx}{dt}\right) + f(x).$$

2) *The trajectories of the dynamical system (3″) are the extremals of the Lagrangian*

$$L = \frac{1}{2} g\left(\frac{dx}{dt} - X, \ \frac{dx}{dt} - X\right) = \frac{1}{2} g\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - g\left(X, \frac{dx}{dt}\right) + f(x).$$

3) *The dynamical systems (3′) and (3″) are conservative, the Hamiltonian being the same for both cases, namely*

$$H = \frac{1}{2} g\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - f(x).$$

The restriction of the Hamiltonian H to the flow of the vector field X is zero. Obviously the values of the Hamiltonian H can be positive, negative or zero, even if the metric g is a Riemannian metric; therefore, just in this case, there exist boundary-value problems associated to the differential system (3), having three solutions (for example, the first corresponding to constant total energy $H < 0$, the second for $H = 0$, and the third for $H > 0$).

For the next theorem we recall that a *pregeodesic* is a smooth curve which may be reparametrized to be a geodesic.

Theorem (Lorentz-Udrişte World-Force Law). 1) *Every non-cons-tant trajectory of the dynamical system (3′), which corresponds to a constant value H_0 of the Hamiltonian, is a pregeodesic of the semi-Riemann-Jacobi manifold*

$$(M \setminus \mathcal{E}, \ \bar{g} = (H_0 + f)g).$$

2) *Let g_{ij} be the local components of the metric g and let Γ^i_{jk}, $i, j, k = 1, \dots, n$ be the local components of the connection ∇. Every non-constant trajectory of the*

dynamical system (3′′), which corresponds to a constant value H_0 of the Hamiltonian, is a horizontal pregeodesic of the semi-Riemann-Jacobi-Lagrange manifold

$$(M \setminus \mathcal{E},\ \bar{g} = (H_0 + f)g,\ N_j{}^i = \Gamma^i_{jk} y^k - F_j{}^i, \quad i, j, k = 1, \ldots, n).$$

Corollary. *If the metric g is chosen such that $f \in \{-1, 0, 1\}$ on $M \setminus \mathcal{E}$ and if we denote*

$$\alpha^2 = g\left(\frac{dx}{dt}, \frac{dx}{dt}\right), \quad \beta = k^{1/2} g\left(X, \frac{dx}{dt}\right),$$

then every trajectory of the dynamical system (3′′), with $\alpha^2 k^{1/2} + \beta = 0$, is a pregeodesic of the semi-Finsler-Jacobi manifold

$$(M \setminus \mathcal{E},\ L = F_w^2 = w\alpha\beta, \quad w = \text{constant}).$$

For the theory in this section see also [8]-[14], [17], [18].

2 Discrete Geometric Dynamics

Let (M, g) be a Riemannian manifold and ∇ be Levi-Civita connection. Let $L : TM \to R$, $L = L(x, \dot{x})$ be a Lagrangian density energy. The discretization of L can be made by using the midpoint rule which consists in the substitution of the point x with $\dfrac{x_{k+1} + x_k}{2}$ and of the velocity $\dot{x}$ by $\dfrac{x_{k+1} - x_k}{h}$, where h is the time step. One obtains the discrete Lagrangian

$$L_d : M \times M \to R, \quad L_d(u, v) = h^2 L\left(\frac{u + v}{2}, \frac{v - u}{h}\right).$$

This produces the discrete Euler-Lagrange equations (variational integrator)

$$\frac{\partial L_d}{\partial x_k^i}(x_{k-1}, x_k) + \frac{\partial L_d}{\partial x_k^i}(x_k, x_{k+1}) = 0,$$

$$i = 1, \ldots, n; \quad k = 1, \ldots, N - 1.$$

Denoting

$$A_i(k) = \frac{\partial L_d}{\partial x_k^i}(x_{k-1}, x_k), \quad f_i(u) = \frac{\partial L_d}{\partial x_k^i}(x_k, u) + A_i(k),$$

$$i = 1, \ldots, n; \quad u = (u^1, \ldots, u^n), \quad F = (f_1, \ldots, f_n),$$

the preceding equations transfer into a nonlinear equation system (6) $F(u) = 0$, at each step k. The solution of the system (6) can be approximated by using $u(e) = x_k$ in the Newton method

$$J_F(u(e)) \begin{pmatrix} u^1(e+1) \\ \cdots \\ u^n(e+1) \end{pmatrix} = J_F(u(e)) \begin{pmatrix} u^1(e) \\ \cdots \\ u^n(e) \end{pmatrix} - \begin{pmatrix} f_1(u(e)) \\ \cdots \\ f_n(u(e)) \end{pmatrix},$$

$e = 1, 2, \ldots, \bar{e}$. The matrix J_F is the Jacobi matrix of the function F, and the integer number $\bar{e}$ is determined by the condition

$$\frac{\left|\sum_{i=1}^{n} f_i^2(u(e)) - \sum_{i=1}^{n} f_i^2(u(e+1))\right|}{1 + \sum_{i=1}^{n} f_i^2(u(e+1))} < \varepsilon$$

Next, we put $x_{k+1} = u(\bar{e})$.

The discrete Lagrangian L_d produces the discrete Hamiltonian

$$H(k) = h(x_k^i - x_{k-1}^i)\frac{\partial L_d}{\partial \dot{x}^i}(x_{k-1}, x_k) - L_d(x_{k-1}, x_k).$$

Let $X = (X^i)$ be a C^∞ vector field on M. For the first order differential system (1) the midpoint rule is an usual method of discretization. The discretized system can be written

$$(7) \qquad x_{k+1}^i - x_k^i - hX^i\left(\frac{x_{k+1} + x_k}{2}\right) = 0, \quad i = 1, \ldots, n; \quad k = 1, \ldots, N-1.$$

The left hand members of (7) and the Riemannian metric $g = (g_{ij})$ produce a discrete Lagrangian density energy as $L_d : M \times M \to R$

$$\begin{aligned} L_d(x_{k-1}, x_k) &= \frac{1}{2} g_{rs}\left(\frac{x_k + x_{k-1}}{2}\right)\left(x_k^r - x_{k-1}^r - hX^r\left(\frac{x_k + x_{k-1}}{2}\right)\right) \cdot \\ &\cdot \left(x_k^s - x_{k-1}^s - hX^s\left(\frac{x_k + x_{k-1}}{2}\right)\right) \end{aligned}$$

Suppose Γ_{jk}^i are the components of the Levi-Civita connection. Computing the covariant derivative of $L_d(x_{k-1}, x_k)$ we find

$$\frac{\partial L_d}{\partial x_k^i}(x_{k-1}, x_k) = g_{rs}\left(\frac{x_k + x_{k-1}}{2}\right)\left(\delta_i^r + \frac{1}{2}\Gamma_{it}^r\left(\frac{x_k + x_{k-1}}{2}\right)(x_k^t - x_{k-1}^t) - \right.$$
$$\left. - \frac{h}{2}\nabla_i X^r\left(\frac{x_k + x_{k-1}}{2}\right)\right)\left(x_k^s - x_{k-1}^s - hX^s\left(\frac{x_k + x_{k-1}}{2}\right)\right).$$

Similarly, using

$$L_d(x_k, x_{k+1}) = \frac{1}{2} g_{rs}\left(\frac{x_{k+1} + x_k}{2}\right)\left(x_{k+1}^r - x_k^r - hX^r\left(\frac{x_{k+1} + x_k}{2}\right)\right) \cdot$$
$$\cdot \left(x_{k+1}^s - x_k^s - hX^s\left(\frac{x_{k+1} + x_k}{2}\right)\right)$$

we have

$$\frac{\partial L_d}{\partial x_k^i}(x_k, x_{k+1}) = g_{rs}\left(\frac{x_{k+1}+x_k}{2}\right)\left(-\delta_i^r + \frac{1}{2}\Gamma_{it}^r\left(\frac{x_{k+1}+x_k}{2}\right)(x_{k+1}^t - x_k^t) - \right.$$

$$\left. -\frac{h}{2}\nabla_i X^r\left(\frac{x_k+x_{k+1}}{2}\right)\right)\left(x_{k+1}^s - x_k^s - hX^s\left(\frac{x_{k+1}+x_k}{2}\right)\right).$$

The associated discrete Euler-Lagrange equations are

$$g_{rs}\left(\frac{x_{k+1}+x_k}{2}\right)\left(-\delta_i^r + \frac{1}{2}\Gamma_{it}^r\left(\frac{x_{k+1}+x_k}{2}\right)(x_{k+1}^t - x_k^t) - \frac{h}{2}\nabla_i X^r\left(\frac{x_{k+1}+x_k}{2}\right)\right)\cdot$$

$$\cdot\left(x_{k+1}^s - x_k^s - hX^s\left(\frac{x_{k+1}+x_k}{2}\right)\right) + A_i(k) = 0,$$

where

$$A_i(k) = \frac{\partial L_d}{\partial x_k^i}(x_{k-1}, x_k).$$

These discrete equations generate an *Algorithm for Geometric Dynamics.*

The discrete Hamiltonian produced by the discrete Lagrangian $L_d(x_{k-1}, x_k)$ is

$$H(x_{k-1}, x_k) = \frac{1}{2}g_{rs}\left(\frac{x_k+x_{k-1}}{2}\right)(x_k^r - x_{k-1}^r)(x_k^s - x_{k-1}^s) -$$

$$-\frac{h^2}{2}g_{rs}\left(\frac{x_k+x_{k-1}}{2}\right)X^r\left(\frac{x_k+x_{k-1}}{2}\right)X^s\left(\frac{x_k+x_{k-1}}{2}\right).$$

Assumption. For the next simulations (see Section 5) we use the Euclidean metric $g_{rs} = \delta_{rs}$, we neglect the terms containing h^2 and we approximate the Jacobian matrix by $J_F = I$ (multiplying the discrete Euler-Lagrange equations by -1).

For the theory in this section see also [6], [15], [16].

3 Case Studies

Pendulum Geometric Dynamics. We use the Riemannian manifold (R^2, δ_{ij}). The small oscillations of a plane pendulum are described as solutions of the differential system (plane pendulum flow)

$$\frac{dx_1}{dt} = -x_2, \quad \frac{dx_2}{dt} = x_1. \tag{6}$$

In this case $x_1(t) = 0$, $x_2(t) = 0$, $t \in R$ is the equilibrium point and $x_1(t) = c_1 \cos t + c_2 \sin t$, $x_2(t) = c_1 \sin t - c_2 \cos t$, $t \in R$ is the general solution (family of circles with same centre).

Let

$$X = (X_1, X_2),\ X_1(x_1, x_2) = -x_2, \quad X_2(x_1, x_2) = x_1,$$

$$f(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2), \quad \text{rot}\, X = (0, 0, 2),\ div\, X = 0.$$

The pendulum flow conserves the areas. The prolongation by derivation of the kinematic system (6) is

$$\frac{d^2 x_i}{dt^2} = \sum_j \frac{\partial X_i}{\partial x_j} \frac{dx_j}{dt}, \quad i, j = 1, 2$$

or

$$\frac{d^2 x_1}{dt^2} = -\frac{dx_2}{dt}, \quad \frac{d^2 x_2}{dt^2} = \frac{dx_1}{dt}. \tag{7}$$

This prolongation admits the general solution

$$x_1(t) = a_1 \cos t + a_2 \sin t + h$$
$$x_2(t) = a_1 \sin t - a_2 \cos t + k,\ t \in R.$$

(family of circles).

The pendulum geometric dynamics is described by

$$\frac{d^2 x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_j \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) \frac{dx_j}{dt}, \quad i, j = 1, 2$$

or

$$\frac{d^2 x_1}{dt^2} = x_1 - 2\frac{dx_2}{dt}, \quad \frac{d^2 x_2}{dt^2} = x_2 + 2\frac{dx_1}{dt}, \tag{8}$$

with the general solution

$$x_1(t) = b_1 \cos t + b_2 \sin t + b_3 t \cos t + b_4 t \sin t$$
$$x_2(t) = b_1 \sin t - b_2 \cos t + b_3 t \sin t - b_4 t \cos t,\ t \in R$$

(family of spirals).

Using

$$L = \frac{1}{2}\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right] + x_2\frac{dx_1}{dt} - x_1\frac{dx_2}{dt} + f$$
$$H = \frac{1}{2}\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right] - f$$
$$g_{ij} = (H + f)\delta_{ij}, \quad N_j{}^i = \Gamma^i_{jk} y^k - F_j{}^i,$$
$$F_j{}^i = \delta^{ih} F_{jh}, \quad F_{ij} = \frac{\partial X_j}{\partial x_i} - \frac{\partial X_i}{\partial x_j},\ i, j, h = 1, 2,$$

the solutions of the differential system (8) are horizontal pregeodesics of the Riemann-Jacobi-Lagrange manifold

$$(R^2 \setminus \{0\},\ g_{ij}, {N_j}^i).$$

Lorenz Geometric Dynamics. We use the Riemannian manifold (R^3, δ_{ij}). The Lorenz flow is a first dissipative model with chaotic behaviour discovered in numerical experiment. Its state equations are

$$\frac{dx_1}{dt} = -\sigma x_1 + \sigma x_2$$
$$\frac{dx_2}{dt} = -x_1 x_3 + r x_1 - x_2 \tag{9}$$
$$\frac{dx_3}{dt} = x_1 x_2 - b x_3,$$

where σ, r, b are real parameters. Usually σ, b are kept fixed whereas r is varied. At

$$r > r_0 = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

chaotic behaviour is observed. With $\sigma = 10,\ b = \frac{8}{3}$, the preceding inequality yields $r_0 = 24,7368$. If $\sigma \neq 0$ and $b(r-1) > 0$, then the equilibrium points of the Lorenz flow are

$$x = 0,\ y = 0,\ z = 0;$$
$$x = \pm\sqrt{b(r-1)},\ y = \pm\sqrt{b(r-1)},\ z = r - 1.$$

Let

$$X = (X_1, X_2, X_3),\ X_1(x_1, x_2, x_3) = -\sigma x_1 + \sigma x_2,$$
$$X_2(x_1, x_2, x_3) = -x_1 x_3 + r x_1 - x_2,\ X_3(x_1, x_2, x_3) = x_1 x_2 - b x_3,$$
$$f = \frac{1}{2}[(-\sigma x_1 + \sigma x_2)^2 + (-x_1 x_3 + r x_1 - x_2)^2 + (x_1 x_2 - b x_3)^2],$$
$$rotX = (2x_1,\ -x_2,\ r - x_3 - \sigma).$$

The Lorenz geometric dynamics is described by

$$\frac{d^2 x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_j \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) \frac{dx_j}{dt},\ i, j = 1, 2, 3$$

or

$$\frac{d^2 x_1}{dt^2} = \frac{\partial f}{\partial x_1} + (\sigma + x_3 - r)\frac{dx_2}{dt} - x_2 \frac{dx_3}{dt}$$
$$\frac{d^2 x_2}{dt^2} = \frac{\partial f}{\partial x_2} + (r - x_3 - \sigma)\frac{dx_1}{dt} - 2x_1 \frac{dx_3}{dt} \tag{10}$$
$$\frac{d^2 x_3}{dt^2} = \frac{\partial f}{\partial x_3} + x_2 \frac{dx_1}{dt} + 2x_1 \frac{dx_2}{dt}.$$

Using

$$L = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx_i}{dt}\right)^2 - \sum_{i=1}^{3} X_i \frac{dx_i}{dt} + f$$

$$H = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx_i}{dt}\right)^2 - f$$

$$g_{ij} = (H+f)\delta_{ij}, \quad N_j{}^i = \Gamma^i_{jk}y^k - F_j{}^i$$

$$F_j{}^i = \delta^{ih}F_{jh} \quad F_{ij} = \frac{\partial X_j}{\partial x_i} - \frac{\partial X_i}{\partial x_j},\ i,j,h = 1,2,3,$$

the solutions of the differential system (10) are horizontal pregeodesics of the Riemann-Jacobi-Lagrange manifold

$$(R^3 \setminus \mathcal{E},\ g_{ij},\ N_j{}^i),$$

where $\mathcal{E}$ is the set of equilibrium points.

ABC Geometric Dynamics. We use the Riemannian manifold (R^3, δ_{ij}). One examples of a fluid velocity that contains exponential stretching and hence instability is the ABC flow,

$$(11) \qquad \begin{cases} \dfrac{dx_1}{dt} = A\sin x_3 + C\cos x_2 \\ \dfrac{dx_2}{dt} = B\sin x_1 + A\cos x_3 \\ \dfrac{dx_3}{dt} = C\sin x_2 + B\cos x_1. \end{cases}$$

This flow is named after the three mathematicians Arnold, Beltrami and Childress, who have contributed much to our understanding and appreciation of classes of "chaotic" flows of which the present one is an example. For nonzero values of the constants A, B, C the preceding system is not globally integrable. The topology of the flow lines is very complicated and can only be investigated numerically to reveal regions of chaotic behaviour. The ABC flow conserves the volumes since the ABC field is solenoidal.

The ABC geometric dynamics is described by

$$\frac{d^2x^i}{dt^2} = \frac{\partial f}{\partial x^i} + \sum_j \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right)\frac{dx_j}{dt},\ i,j = 1,2,3.$$

Since

$$f = \frac{1}{2}(A+B+C+2AC\sin x_3\cos x_2 + 2BA\sin x_1\cos x_3 + 2CB\sin x_2\cos x_1)$$

$$rotX = X,$$

the ABC geometric dynamics is given by the differential system (12):

$$\begin{aligned}\frac{d^2x_1}{dt^2} &= AB\cos x_1\cos x_3 - BC\sin x_1\sin x_2 - (B\cos x_1 + C\sin x_2)\frac{dx_2}{dt}+\\ &+ (B\sin x_1 + A\cos x_3)\frac{dx_3}{dt}\\ \frac{d^2x_2}{dt^2} &= -AC\sin x_2\sin x_3 + BC\cos x_1\cos x_2 + (B\cos x_1 + C\sin x_2)\frac{dx_1}{dt}-\\ &- (A\sin x_3 + C\cos x_2)\frac{dx_3}{dt}\\ \frac{d^2x_3}{dt^2} &= AC\cos x_3\cos x_2 - BA\sin x_1\sin x_3 - (B\sin x_1 + A\cos x_3)\frac{dx_1}{dt}+\\ &+ (C\cos x_2 + A\sin x_3)\frac{dx_2}{dt}.\end{aligned}$$

Using

$$\begin{aligned}&L = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx_i}{dt}\right)^2 - \sum_{i=1}^{3}X_i\frac{dx_i}{dt} + f\\ &H = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx_i}{dt}\right)^2 - f\\ &g_{ij} = (H+f)\delta_{ij}, \quad N_j{}^i = \Gamma^i_{jk}y^k - F_j{}^i\\ &F_j{}^i = \delta^{ih}F_{jh},\ F_{ij} = \frac{\partial X_j}{\partial x_i} - \frac{\partial X_i}{\partial x_j},\ i,j,h = 1,2,3\end{aligned}$$

the solutions of the differential system (12) are horizontal pregeodesics of the Riemann-Jacobi-Lagrange manifold

$$(R^3 \setminus \mathcal{E}, g_{ij}, N_j{}^i),$$

where $\mathcal{E}$ is the set of equilibrium points which is included in the surface of equation

$$\sin x_1\sin x_2\sin x_3 + \cos x_1\cos x_2\cos x_3 = 0.$$

4 Numerical and Graphical Simulations

The first set of 5 figures displays a spiral in the Pendulum Geometric Dynamics, for the following Data:
$x(1) = -1$, $y(1) = -1$; $x(2) = 1$, $y(2) = 1$,
$n = 300$; $h = 0.1$.

The drawings contain a discrete trajectory, its graph, graph of the discrete Hamiltonian, and the discrete Poincaré projections on the planes $(x, \dot{x})$, $(y, \dot{y})$.

The second set of 10 figures reflect the shape of a trajectory in the Lorenz Geometric Dynamics, using the Data:

$x(1) = 0$, $y(1) = 0.1$; $z(1) = -1$; $x(2) = 0$; $y(2) = 0$, $z(2) = 1$,

$x(1) = 0.1$; $y(1) = 0.2$; $z(1) = -0.1$; $x(2) = 0.1$; $y(2) = 0$; $z(2) = 0.1$,

$n = 250$; $h = 0.001$.

The pictures contain a discrete trajectory, graph of the discrete Hamiltonian, and the discrete Poincaré projections on the planes $(x, \dot{x})$, $(y, \dot{y})$, $(z, \dot{z})$.

The third set of 10 plots refer to the shape of a trajectory in the ABC Geometric Dynamics, using the Data:

$x(1) = 0$, $y(1) = 01$; $z(1) = 0$; $x(2) = -0.1$; $y(2) = 0$; $z(2) = 0$,

$x(1) = -0.5$, $y(1) = 0$; $z(1) = 0$; $x(2) = 0.5$; $y(2) = 0$; $z(2) = 0$,

$n = 200$; $h = 0.01$.

The plots contain a discrete trajectory, graph of the discrete Hamiltonian, and the discrete Poincaré projections on the planes $(x, \dot{x})$, $(y, \dot{y})$, $(z, \dot{z})$.

Similar plots are included now in an *Atlas of Geometric Magnetic Dynamics*.

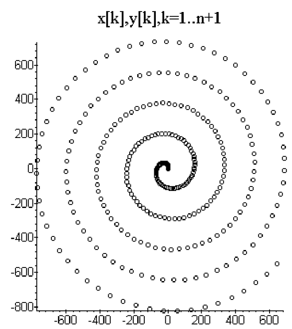


Fig.1. Discrete Trajectory in Pendulum Geometric Dynamics

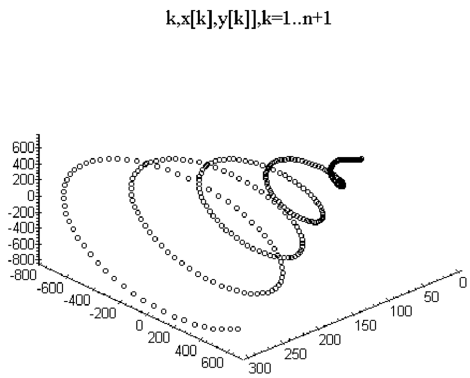


Fig.2. Graph of Discrete Trajectory in Pendulum Geometric Dynamics

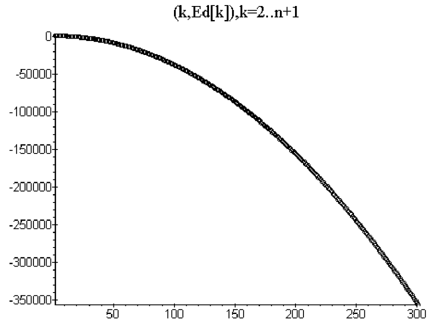
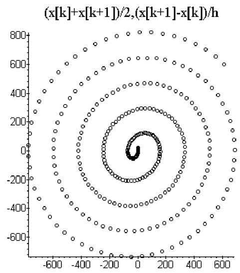


Fig.3. Graph of Discrete Hamiltonian in Trajectory in Pendulum Geometric Dynamics



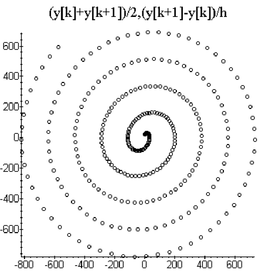


Fig.4. $(x, \dot{x})$-Discrete Poincare Projection Fig.5. $(y, \dot{y})$-Discrete Poincare Projection

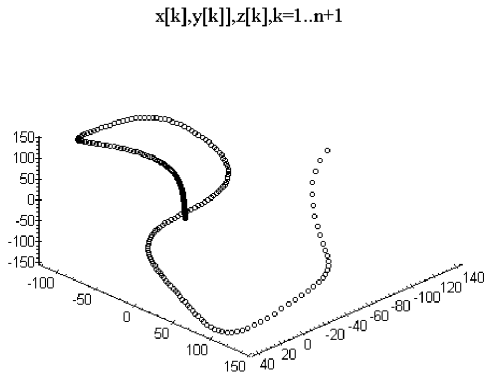


Fig.6. Discrete Trajectory in Lorentz Geometric Dynamics

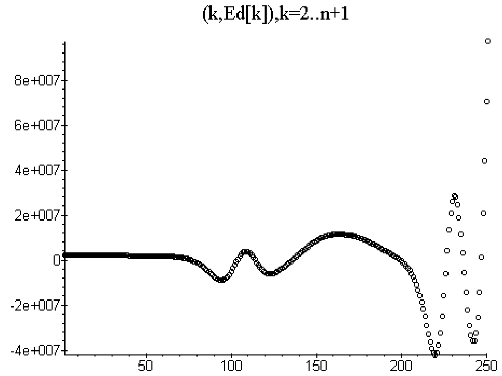
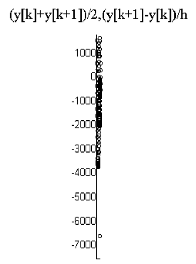


Fig.7. Graph of Discrete Hamiltonian in Geometric Dynamics



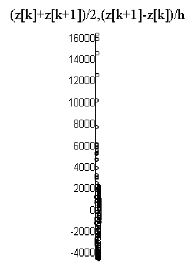


Fig.8. $(y, \dot{y})$-Discrete Poincare Projection Fig.9. $(z, \dot{z})$-Discrete Poincare Projection

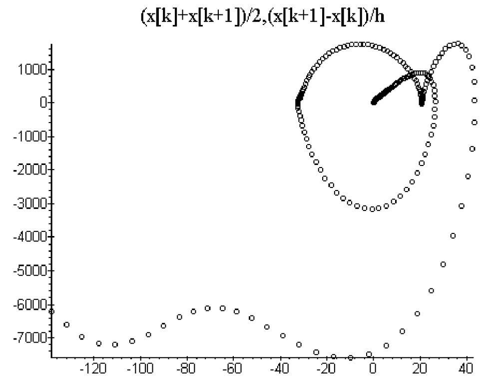


Fig.10. $(x, \dot{x})$-Discrete Poincare Projection

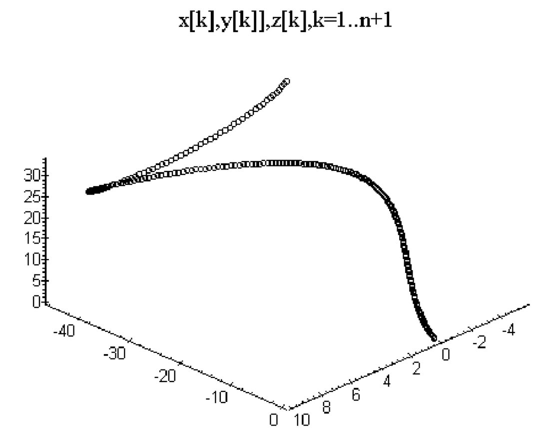


Fig.11. Discrete Trajectory in Lorentz Geometric Dynamics

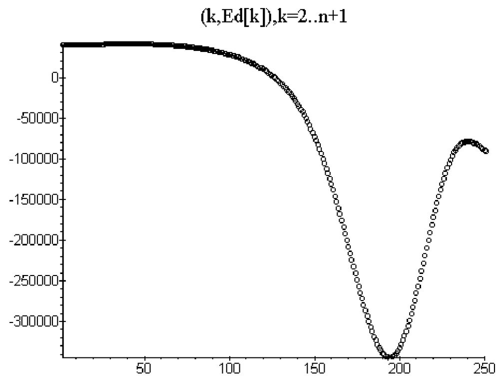


Fig.12. Discrete Trajectory in Lorentz Geometric Dynamics

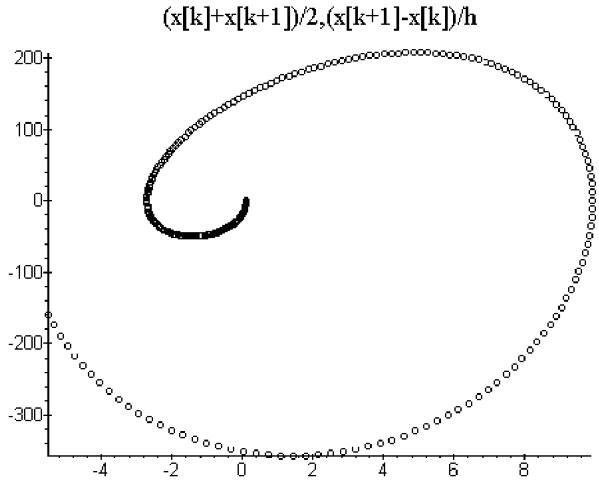
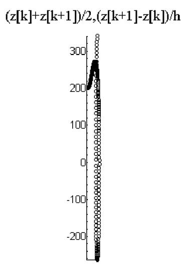


Fig.13. $(x, \dot{x})$ - Discrete Poincare Projection



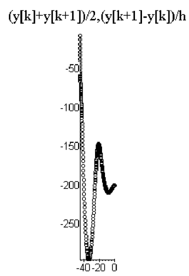


Fig.14. $(z, \dot{z})$ - Discrete Poincare Projection Fig.15. $(y, \dot{y})$ - Discrete Poincare Projection

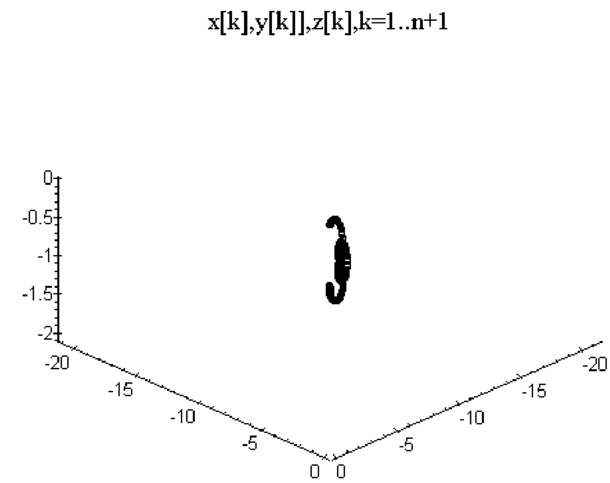


Fig.16. Discrete Trajectory in ABC Geometric Dynamics

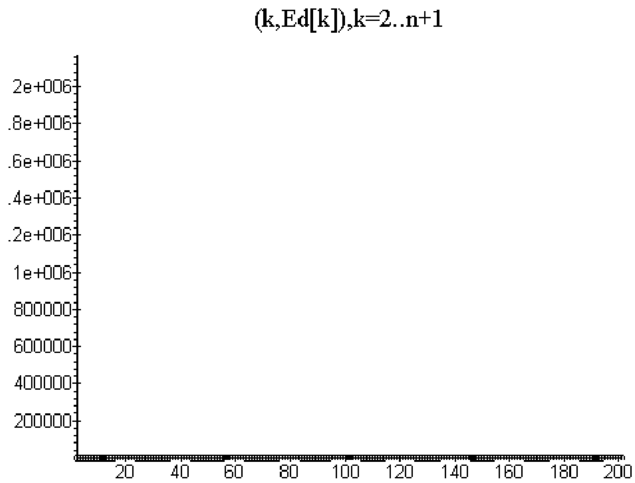


Fig.17. Graph of Discrete Hamiltonian in Geometric Dynamics

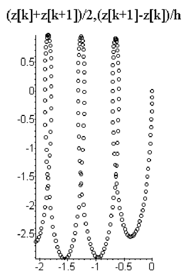


Fig.18. $(z, \dot{z})$ - Discrete Poincare Projection

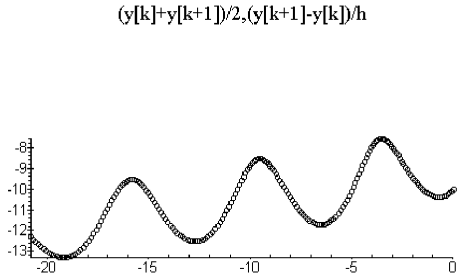


Fig.19. $(y, \dot{y})$ - Discrete Poincare Projection

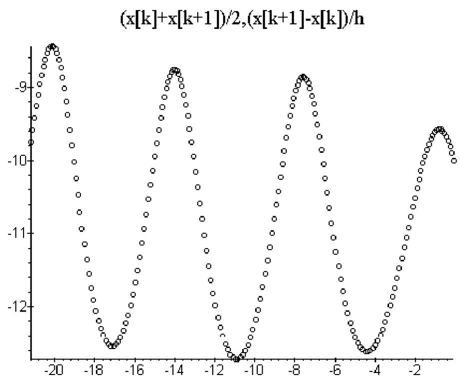


Fig.20. $(z, \dot{z})$ - Discrete Poincare Projection

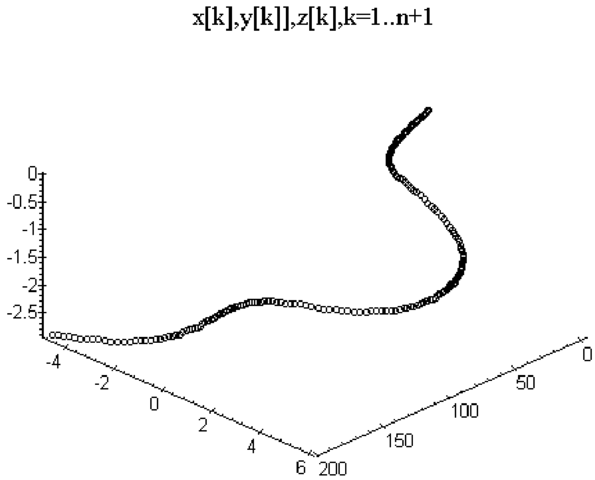


Fig.21. Discrete Trajectory in ABC Geometric Dynamics

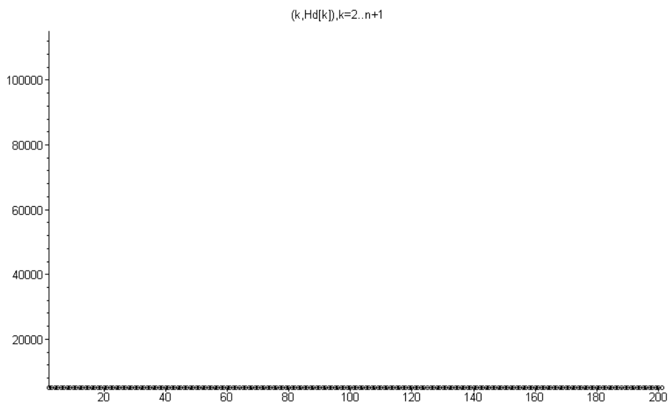
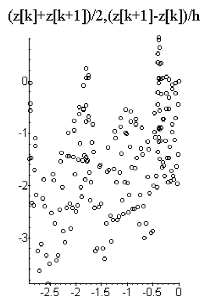


Fig.22. Discrete Trajectory in ABC Geometric Dynamics



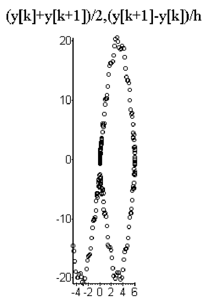


Fig.23. $(z, \dot{z})$ - Discrete Poincare Projection Fig. 24 $(y, \dot{y})$ - Discrete Poincare Projection

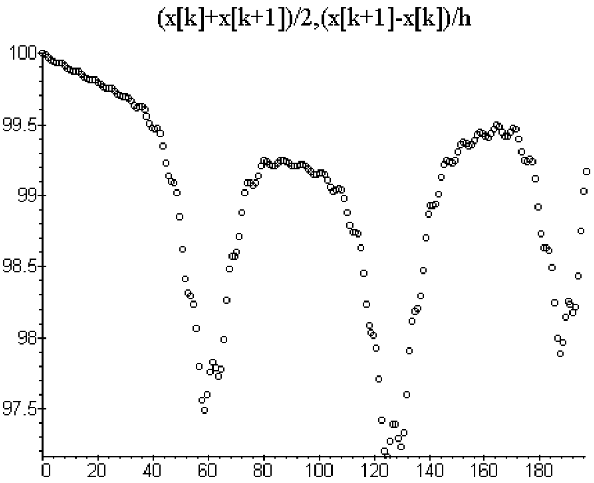


Fig.25. $(x, \dot{x})$ - Discrete Poincare Projection

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