Lorentz-type equations in first-order jet spaces endowed with nonlinear connection

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Abstract. Sections 1 and 2 describe the notions of nonlinear and N-linear connection in first order jet spaces $J^1(T, M)$, point out relevant subclasses which include the Cartan and Berwald connections and remind the 1-jet Lagrangian case for electromagnetism. In section 3 are derived the associated electromagnetic tensors and the corresponding generalized Lorentz equations. It is shown that in the uniparametric case these equations confine to the already known ones of the tangent bundle framework. Section 4 exemplifies by numerical simulation of the obtained Lorentz equations, the deformation of geodesics to Lorentz curves as effect of the electromagnetic field influence.

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1. Nonlinear connections on $J^1(T, M)$

Let $\xi = (E = J^1(T, M), \pi, T \times M)$ be the first order jet bundle of mappings $\varphi : T \to M$, where T and M are $\mathcal{C}^\infty$ real differentiable manifolds with $\dim T = m$, $\dim M = n$ respectively. We shall denote the local coordinates in E by

$$(t^\alpha, x^i, y^A)_{(\alpha,i,A)\in I_*} \equiv (y^\mu)_{\mu\in I},$$

where we use the notations

$$I_* = I_{h_1} \times I_{h_2} \times I_v,\ I_{h_1} = \overline{1, m}, \quad I_{h_2} = \overline{m+1, m+n},$$
$$I_v = \overline{m+n+1, m+n+mn},\ I_h = I_{h_1} \cup I_{h_2}, I = I_h \cup I_v.$$

The indices will implicitly take values as follows:

$$\alpha, \beta, \ldots \in I_{h_1}; \ \ i, j, \ldots \in I_{h_2}; \ \ A, B, \ldots \in I_v; \ \ \lambda, \mu, \ldots \in I.$$

When appropriate, for $A = m + n + n(i - m - 1) + \alpha$, we identify $A \equiv \binom{i}{\alpha}$ and $y^A \equiv x^{\binom{i}{\alpha}} = \frac{\partial x^i}{\partial t^\alpha}$.

A *non-linear connection* $N = \{N^A_\mu\}_{\mu\in I_h, A\in I_v}$ on E provides a splitting [3]

$$TE = HE \oplus VE, \tag{1}$$

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and provides the local (adapted) basis of $\mathcal{X}(E)$

$$\mathcal{B} = \{\delta_\alpha, \delta_i, \delta_A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta_\mu\}_{\mu\in I}, \tag{2}$$

where we write briefly

$$\delta_\alpha = \partial_\alpha - N_\alpha^A \delta_A, \quad \delta_i = \partial_i - N_i^A \delta_A, \quad \delta_A = \dot{\partial}_A = \frac{\partial}{\partial y^A}, \quad \partial_\alpha = \frac{\partial}{\partial t^\alpha}, \partial_i = \frac{\partial}{\partial x^i}.$$

We denote as well the module of local sections of the sub-bundles HE and VE by $\mathcal{S}(HE) = Span(\{\delta_\mu\})_{\mu\in I_h}$ and $\mathcal{S}(VE) = Span(\{\delta_\mu\})_{\mu\in I_v}$ respectively. The dual basis of $\mathcal{B}$ writes then

$$\mathcal{B}^* = \{\delta^\alpha, \delta^i, \delta^A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta^\mu\}_{\mu\in I},$$

where $\delta^\alpha = dt^\alpha, \delta^i = dx^i, \delta^A \equiv \delta y^A = dy^A + N_\alpha^A dt^\alpha + N_i^A dx^i$.

2. N-linear connections on $J^1(T, M)$. Special cases

Consider on E a fixed non-linear connection and let $\nabla = \{L^\lambda_{\mu\nu}\}_{\lambda,\mu,\nu\in I}$ be a linear connection on E. Then its coefficients relative to the adapted basis (2) given by

$$\delta^\lambda(\nabla_{\delta_\nu}\delta_\mu) = L^\lambda_{\mu\nu}, \ \ \forall \lambda, \mu, \nu \in I. \tag{3}$$

form $3^3 = 27$ distinct subsets, according to the three sets of indices. Then the torsion and curvature of ∇ are given respectively by

$$\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$
$$\mathcal{R}(X, Y)Z = \nabla_{[X}\nabla_{Y]}Z - \nabla_{[X,Y]}Z, \ \ \forall\, X, Y, Z \in \mathcal{S}(TE)$$

and have the adapted coefficients defined by the relations

$$\delta^\lambda(\mathcal{T}(\delta_\nu, \delta_\mu)) = T^\lambda_{\mu\nu}, \ \ \delta^\lambda(\mathcal{R}(\delta_\nu, \delta_\mu)\delta_\rho) = R_\rho{}^\lambda{}_{\mu\nu}, \quad \forall\, \lambda, \mu, \nu, \rho \in I.$$

In the following we describe the N-linear connections on E which preserve the distributions related to the adapted basis, and point out some of their relevant classes.

1. As first particular case, we denote by $\Gamma(N)$ the set of connections ∇ (called "N-connections"), whose coefficients form $3 \cdot 2^2 + 3 = 15$ generally nonvanishing subsets related to the three index classes, due to the relations

$$L^\lambda_{\mu\nu} = 0, \ \ \forall\, (\lambda, \mu) \in (I_h \times I_v) \cup (I_v \times I_h). \tag{4}$$

Any such connection provides a covariant derivative which preserves $\mathcal{S}(HE)$ and $\mathcal{S}(VE)$. Note that if E carries a metric structure, the Levi-Civita (metric and torsionless) connection is not generally a member of $\Gamma(N)$ [3]. It can be easily shown that the associated torsion and curvature of a connection $\nabla \in \Gamma(N)$ satisfy

$$\begin{aligned} &T^\omega_{BC} = 0, && \forall\, \omega \in I_h, B, C \in I_v, \\ &R_\rho{}^\lambda{}_{\mu\nu} = 0, && \forall\, \mu, \nu \in I, (\lambda, \rho) \in (I_h \times I_v) \cup (I_v \times I_h), \end{aligned} \tag{5}$$

and hence the torsion subsets reduce from 3^3 to 25 and the curvature ones from 3^4 to $5 \cdot 3^2$.

2. As second further subcase, we consider the *special N-connections*, i.e., the connections ∇ whose covariant derivations preserve the distributions $Span(\delta_\alpha)_{\alpha \in I_{h_1}}$ and $Span(\delta_i)_{i \in I_{h_2}}$. They have just 9 sets of generally nonvanishing coefficients, since besides (4) they satisfy as well

$$L^\lambda_{\mu\nu} = 0, \ \ \forall\, (\lambda, \mu) \in (I_{h_1} \times I_{h_2}) \cup (I_{h_2} \times I_{h_1}). \tag{6}$$

A part of the curvature distinguished tensors vanish,

$$R_\rho{}^\lambda{}_{\mu\nu} = 0, \quad \forall\, \mu, \nu \in I, (\lambda, \rho) \in (I_{h_1} \times I_{h_2}) \cup (I_{h_2} \times I_{h_1}),$$

and the number of torsion and curvature sets reduce to 12 and 18 respectively. Their family, which we shall further denote by $\Gamma_*(N)$, was intensively studied in [7], [5].

3. Among the connections $\Gamma_*(N)$ we evidentiate the so-called "Γ-linear *h*-normal connections" denoted $\Gamma_n(N)$, which depend on the four essential components

$$\nabla \equiv (L^\alpha_{\beta\gamma}, L^i_{j\gamma}, L^i_{jk}, L^i_{jA}), \tag{7}$$

and with the other 5 components provided further via

$$L^A_{B\gamma} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = \delta^\beta_\alpha L^i_{j\gamma} - \delta^i_j \left|{}^{\beta}_{\alpha\gamma}\right|, \ \ L^A_{Bk} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = \delta^\beta_\alpha \left|{}^{i}_{jk}\right|,$$
$$L^A_{BC} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}C} = \delta^\beta_\alpha L^i_{jC}, \ \ L^\alpha_{\beta j} = 0, \quad L^\alpha_{\beta C} = 0.$$

A connection $\nabla \in \Gamma_n(N)$ has the number of torsion and curvature sets reduced to 9 and respectively to 7. Further, we endow E with a semi-Riemannian metric

$$G = \underbrace{h_{\alpha\beta}(t,x) dt^\alpha \otimes dt^\beta}_{h} + \underbrace{g_{ij}(t,x,y) dx^i \otimes dx^j}_{g} + \underbrace{\tilde{g}(t,x,y)\delta y^A \otimes \delta y^B}_{\tilde{g}}, \tag{8}$$

with $\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = g_{ij}(t,x,y) h^{\alpha\beta}(t)$. Within $\Gamma_n(N)$, we evidentiate two linear connections [7]:

a) *The Cartan connection* which is metrical and exhibits generally (for $m > 1$) just 8 torsion sets and 7 curvature sets, and 5 torsion sets provided that g is y-independent. Its essential coefficients (7) specify to

$$\begin{aligned} L^\alpha_{\beta\gamma} &= \left|{}^{\alpha}_{\beta\gamma}\right| = \tfrac{1}{2} h^{\alpha\varepsilon}(\delta_{\{\beta} h_{\varepsilon\}\gamma} - \delta_\varepsilon h_{\beta\gamma}),\\ L^i_{j\gamma} &= \tfrac{1}{2} g^{ik}\delta_\gamma g_{kj}, \ \ L^i_{jk} = \tfrac{1}{2} g^{il}(\delta_{\{k} g_{j\}l} - \delta_l g_{jk}),\\ L^i_{jA} &\equiv L^i_{j\binom{k}{\gamma}} = \tfrac{1}{2} g^{il}(\delta_{\binom{k}{\gamma}} g_{jl} + \delta_{\binom{j}{\gamma}} g_{kl} - \delta_{\binom{l}{\gamma}} g_{jk}), \end{aligned} \tag{9}$$

where we use the notations $\tau_{[i...j]} = \tau_{i...j} - \tau_{j...i}$, $\tau_{\{i...j\}} = \tau_{i...j} + \tau_{j...i}$.

b) *The Berwald connection*, whose essential coefficients for g dependent on x only, are

$$L^{\alpha}_{\beta\gamma} = \left|{}^{\alpha}_{\beta\gamma}\right|, \quad L^{i}_{j\gamma} = 0, \quad L^{i}_{jk} = \left|{}^{i}_{jk}\right|, \quad L^{i}_{jA} = 0, \tag{10}$$

where we denoted by $\left|{}^{\alpha}_{\beta\gamma}\right|$ and $\left|{}^{i}_{jk}\right|$ the Christoffel symbols of the metrics h and g respectively.

On the other hand, regarding the construction of the nonlinear connection N on E from preexistent geometric structures, we note that there are at least two special cases when existing structures on E lead in a natural way to such a connection, as the following two cases do.

a) If E is endowed with a metric structure whose (v, v)-block is nondegenerate then there exists ([3, Prop. 6.1.1, p. 85]) a uniquely determined nonlinear connection N such that in the adapted coordinates provided by N the metric rewrites as a 2-cell metric

$$G = \sum_{\lambda,\mu \in I_h} g_{\lambda\mu} \delta y^{\lambda} \otimes \delta y^{\mu} + \tilde{g}_{AB} \delta y^{A} \otimes \delta y^{B}. \tag{11}$$

A particular case is the one in which E possesses a Lagrangian $L : E \to \mathbb{R}$ which satisfies the regularity condition

$$rank \left(\frac{\partial L}{\partial y^A \partial y^B} \stackrel{not}{=} \tilde{g}_{AB} \right)_{A,B \in I_v} = mn; \tag{12}$$

such Lagrangians were considered in [6], [7], [5]. b) If E is endowed with a Lagrangian for which the metric $\tilde{g} = \tilde{g}_{AB} \delta y^A \otimes \delta y^B$ in (11) splits as

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = g_{ij}(t, x) h^{\alpha\beta}(t), \tag{13}$$

where $h = h_{\alpha\beta}(t) dt^{\alpha} \otimes dt^{\beta}$, $\quad g = g_{ij}(t, x) dx^i \otimes dx^j$ are sub-Riemannian metrics on $T \times M$, then one can derive canonically a nonlinear connection from the considered Lagrangian L [7]. In particular, for given g, h and for $\tilde{g}$ as in (13) and for $U = U_i(x) dx^i$ a given 1-form on M, one may choose for L [6] the extended Lagrangian of electrodynamics

$$L(t, x, y) = \tilde{g}_{AB} \delta y^A \otimes \delta y^B + U_A(t, x) y^A + \Phi(t, x). \tag{14}$$

In this case, the split metric $\tilde{g}$ in (13) is produced as well by the Lagrangian via (12), and therefore L is called "Kronecker h-regular Lagrangian". Based on the two metrics g and h and on the potentials U_A only, one can build the nonlinear connection $N = (N^{\binom{1}{\alpha}}_{\beta}, N^{\binom{1}{\alpha}}_{j})$ of coefficients

$$N^{\binom{i}{\alpha}}_{\beta} = -\left|{}^{\gamma}_{\alpha\beta}\right| x^i_{\gamma}, \quad N^{\binom{i}{\alpha}}_{j} = \left|{}^{i}_{jk}\right| x^k_{\alpha} + \frac{1}{4} g^{ik} (2\partial_{\alpha} g_{jk} + h_{\alpha\beta} U_{\binom{k}{\beta} j}), \tag{15}$$

where we denoted the h_2-curl of U by $U_{\binom{k}{\beta}j} = \delta_j U_{\binom{k}{\beta}} - \delta_k U_{\binom{j}{\beta}}$.

3. The electromagnetic 2-form. The Lorentz equations

Consider that E is endowed with a nonlinear connection N, with the 3-cell semi-Riemannian metric (8) and a fixed N-linear connection $\nabla \in \Gamma_n(N)$. Then the *deflection tensor fields* produced by the Liouville field $\mathcal{C} = y^A \delta_A$ via

$$d^A_\mu = \delta^A \nabla_{\delta_\mu} \mathcal{C}, \ \ \mu \in I, \ A \in I_v.$$

The derived electromagnetic 2-form is then

$$\tilde{F} = \tilde{F}_{A\mu} \delta y^A \wedge dy^\mu, \tag{16}$$

and has the components

$$\left\{ \begin{array}{ll} \tilde{F}_{A\beta} & \equiv \tilde{F}_{\binom{i}{\alpha}\beta} = \left(h^{\alpha\gamma} g_{ik} x^k_{[\gamma} \right)_{|\beta]} \\ \tilde{F}_{Aj} & \equiv \tilde{F}_{\binom{i}{\alpha}j} = \frac{1}{2}(d_{\binom{i}{\alpha}j} - d_{\binom{j}{\alpha}i} = \\ & = \frac{1}{2}(y_{\binom{i}{\alpha}||j} - y_{\binom{j}{\alpha}||i} = \frac{1}{2}\left(x^k_\gamma h^{\alpha\gamma} g_{k[i} \right)_{||j]}, \\ \tilde{F}_{AB} & = \frac{1}{2} \tilde{g}_{[AC} d^C_{B]} = \frac{1}{2} \tilde{g}_{[AC} y^C_{|||B]}. \end{array} \right. \tag{17}$$

where the raising/lowering of the indices was performed according to the index I_{h_1} or I_v type by h and g respectively, and we indicated by $|\alpha$, $||i$ and $|||B$ the covariant derivations given by ∇_{δ_μ}, for $\mu \in I_{h_1}, I_{h_2}$ and I_v.

We consider the tensor field associated to (16) of essential components

$$F = F_A{}^\mu \delta_\mu \otimes \delta^A, \quad F^\alpha_A = h^{\alpha\beta} \tilde{F}_{A\beta}, \ F^i_A = g^{ij} \tilde{F}_{Aj}, \ F^B_A \equiv F^C_A = \tilde{g}^{CD} \tilde{F}_{AD}.$$

Then *the Lorentz equations* attached to G, N and ∇ have the generic shape

$$G_{\nu\rho} \frac{\nabla \mathcal{V}^\rho}{ds} = \tilde{F}_{A\nu} \mathcal{V}^A \quad \Leftrightarrow \quad \frac{\nabla \mathcal{V}^\mu}{ds} = \mathcal{F}_A{}^\mu \mathcal{V}^A, \tag{18}$$

where $\mathcal{V} = \mathcal{V}^\mu \delta_\mu$ is *the covariant velocity* along the considered extended path of the moving test-particle $c : J \subset \mathbb{R} \to E$, $c(s) = (t(s), x(s), y(s)), \forall s \in J$. The components of $\mathcal{V}$ are explicitely given by

$$\{\mathcal{V}^\mu\}_{\mu \in I} \equiv \left(\frac{dt^\alpha}{ds}, \frac{dx^i}{ds}, \frac{\delta y^A}{ds} = \frac{dy^A}{ds} + N^A_\beta \frac{dt^\beta}{ds} + N^A_j \frac{dx^j}{ds} \right)_{(\alpha,i,A) \in I_*}.$$

We have also denoted

$$\frac{\nabla \mathcal{V}^\mu}{ds} \overset{not}{=} \frac{\delta \mathcal{V}^\mu}{ds} + L^\mu_{\nu\rho} \mathcal{V}^\nu \mathcal{V}^\rho,$$

and we shall use the dot notation for expressing the s-derivation. In detail, the Lorentz equations have the form

$$\begin{aligned} \ddot{t}^\alpha \quad &+L^\alpha_{\beta C}\dot{t}^\beta\mathcal{V}^C + L^\alpha_{jC}\dot{x}^j\mathcal{V}^C + L^\alpha_{\beta\gamma}\dot{t}^\beta\dot{t}^\gamma + \\ &+L^\alpha_{j\gamma}\dot{x}^j\dot{t}^\gamma + L^\alpha_{\beta k}\dot{t}^\beta\dot{x}^k + L^\alpha_{jk}\dot{x}^j\dot{x}^k = F^\alpha_B\mathcal{V}^B \end{aligned} \tag{19}$$

$$\begin{aligned} \ddot{x}^i \quad &+L^i_{\beta C}\dot{t}^\beta\mathcal{V}^C + L^i_{jC}\dot{x}^j\mathcal{V}^C + L^i_{\beta\gamma}\dot{t}^\beta\dot{t}^\gamma + \\ &+L^i_{j\gamma}\dot{x}^j\dot{t}^\gamma + L^i_{\beta k}\dot{t}^\beta\dot{x}^k + L^i_{jk}\dot{x}^j\dot{x}^k = F^i_B\mathcal{V}^B \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{v}^A + N^A_\alpha\ddot{t}^\alpha \quad &+N^A_i\ddot{x}^i + L^A_{C\beta}\mathcal{V}^C\dot{t}^\beta + \\ &+L^A_{Cj}\mathcal{V}^C\dot{x}^j + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = F^A_B\mathcal{V}^B. \end{aligned} \tag{21}$$

As well, one may consider *the Lorentz h-paths*, characterized by the relations

$$\mathcal{V}^A = 0,\ A \in I_v \quad \Leftrightarrow \quad \frac{\delta y^A}{ds} = 0\ A \in I_v.$$

We note that, since the right side of (19)-(21) are identically vanishing, these curves *coincide with the usual h-paths of* (E, N, ∇).

As for the *Lorentz v-paths*, these have fixed base-point, i.e.,

$$\mathcal{V}^\mu = 0,\ \mu \in I_h \quad \Leftrightarrow \quad (t, x) = (t_0, x_0) \in T \times M,$$

and hence the associated equations rewrite

$$\begin{cases} F^\alpha_B\mathcal{V}^B = 0, \quad F^i_B\mathcal{V}^B = 0 \\ F^A_B\mathcal{V}^B = \dot{v}^A + L^A_{BC}\mathcal{V}^B\mathcal{V}^C. \end{cases}$$

As a particular case, when g depends on x only, the nonvanishing Cartan connection coefficients are

$$\begin{aligned} L^\alpha_{\beta\gamma} &= \left|{}^{\alpha}_{\beta\gamma}\right|, \quad L^i_{jk} = \left|{}^{i}_{jk}\right| \\ L^A_{B\gamma} &\equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = -\delta^i_j\left|{}^{\beta}_{\alpha\gamma}\right| \\ L^A_{Bk} &\equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = -\delta^\beta_\alpha\left|{}^{i}_{jk}\right|. \end{aligned}$$

We choose the nonlinear connection (15) induced by the Lagrangian (14). In this case, the electromagnetic tensors are

$$\begin{aligned} &F^\alpha_A \equiv F^\alpha_{\binom{i}{\beta}\gamma} = 0 \\ &F^i_A = g^{ij}\tilde{F}_{\binom{k}{al}j} = -\tfrac{1}{4}g^{ij}U_{\binom{k}{\alpha}j}mmF^B_A = -\tfrac{1}{2}\delta^B_A. \end{aligned}$$

Hence the Lorentz equations (19)-(21) rewrite

$$\ddot{t}^\alpha + \left|{}^{\alpha}_{\beta\gamma}\right| \dot{t}^\beta \dot{t}^\gamma = 0$$

$$\ddot{x}^i + \left|{}^{i}_{jk}\right| \dot{x}^j \dot{x}^k = -\tfrac{1}{4} g^{ij} U_{\binom{k}{\alpha} j} y^{\binom{k}{\alpha}}$$

$$\dot{v}^A = -\tfrac{1}{2} y^A.$$

Note that in this case (g dependent on x only), the Berwald connection has the same coefficients as the Cartan connection, and hence the associated Lorentz curves, h- and v-paths are described by the same equations. The Lorentz h-paths obey the extra equations $\dot{y}^A + N^A_\beta \dot{t}^\beta + N^A_j \dot{x}^j = 0$, which write explicitely

$$\dot{y}^{\binom{i}{\alpha}} - \left|{}^{\gamma}_{\alpha\beta}\right| y^{\binom{i}{\gamma}} \dot{t}^\beta + \left(\left|{}^{i}_{jk}\right| y^{\binom{k}{\alpha}} + \frac{1}{4} g^{ik} h_{\alpha\beta} U_{\binom{k}{\beta} j} \right) \dot{x}^j = 0.$$

The Lorentz v-paths for the Cartan connection satisfy just the extra condition $-\mathcal{V}^A = \dot{v}^A$, having as solutions the curves $(t_0, x_0, y^A = k^A_1 e^{-s} + k^A_2),\ s \in \mathbb{R}$, with $(k^A_{1,2} \in \mathbb{R}^{mn}$, semilines within the fibers of E, the linear geodesics of the flat fiber.

We consider a typical particular case, when $m = 1$ and $s = t^1 = t$, where we use *the Finsler-Lagrange tangent space notations* from [3]. Shifting the indices left by one unit ($I_{h_2} = \overline{1, n},\ I_v = \overline{n+1, 2n}$), we have $y^A \equiv y^{\binom{i}{1}} = \frac{dx^i}{dt} \overset{not}{=} y^i$, and set locally $h_{11} = 1$. For the Lagrangian (14) we consider its particular form

$$L = mc\,\gamma_{ij} y^i y^j + \frac{2e}{m} A_i(x) y^i, \tag{22}$$

where γ_{ij} is a pseudo-Riemannian metric and $A = A_i dx^i$ is a 1-form on M. The fundamental tensor derived from L via (11) is then

$$\tilde{g}_{\binom{i}{1}\binom{j}{1}}(t, x, y) = g_{ij}(x) = mc\,\gamma_{ij}(x).$$

The non-linear connection induced locally by the metrics $h = 1dt \otimes dt$ and g, has the components

$$N^A_1 = 0, N^A_i = \left|{}^{i}_{jk}\right| y^k + g^{ik} U_{\binom{k}{1} j}.$$

In this case, the Cartan and Berwald canonic connections have just null and Christoffel (re-indexed) components, as can be seen from (9) and (10) respectively. Choosing for ∇ the Cartan connection, the Lorentz generalized equations (20) confine to the known ones of Lagrange spaces ([3]) and coincide with the equations of the Lagrangian spray G^i described above. They have the equivalent form [2, p. 171]

$$\ddot{x}^i + 2G^i(x, y) = 0, \quad y^i = \frac{dx^i}{ds}, \tag{23}$$

with $G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k + \frac{e}{2m^2c} \gamma^{ij} A_{[j;k]} y^k$, where "$;k$" expresses the canonic covariant derivative on (M, γ_{ij}). We note that in the absence of the electromagnetic force F_{μ_A}, the equations (18) become the equations of stationary curves of the connection ∇. In the particular case $m = 1, s = t^1, h_{11} = 1$, we also note that in the absence of N for Cartan connection, or for the Levi-Civita connection on E, the equations (18) produce *the equations of geodesics* of the manifold M.

4. Numerical simulation for Lorentz curves in Lagrangian case

For $m = 1, T = \mathbb{R}, h_{11} = 1, n = 2$ and $M = H = \{(u, v) = (x^1, x^2) \mid x^2 > 0\} \subset \mathbb{R}^2$ *the Poincare semiplane* endowed with the Lagrangian (14) where the h_2-metric and the potentials are respectively given by

$$\gamma_{ij}(x) = \frac{1}{(x^2)^2}\delta_{ij}, \quad A = \varepsilon(-x^2 dx^1 + x^1 dx^2), \quad (i, j = \overline{1,2},\ \varepsilon \in \mathbb{R}), \tag{24}$$

we obtain $F^1_2 = -F^2_1 = \frac{\varepsilon e}{m^2 c}(x^2)^2$, and the Lorentz equations rewrite, after denoting $\lambda = \frac{\varepsilon e}{m^2 c}$ and appropriate rescaling of coordinates,

$$\begin{cases} \frac{d^2x^1}{ds^2} + \left|{}^{\,1}_{jk}\right| \frac{dx^j}{ds}\frac{dx^k}{ds} = \lambda (x^2)^2 \frac{dx^2}{ds} \\ \frac{d^2x^2}{ds^2} + \left|{}^{\,2}_{jk}\right| \frac{dx^j}{ds}\frac{dx^k}{ds} = -\lambda (x^2)^2 \frac{dx^1}{ds}. \end{cases} \tag{25}$$

Since the Christoffel symbols of (M, γ) are

$$\left|{}^{\,1}_{12}\right| = \left|{}^{\,1}_{21}\right| = \left|{}^{\,1}_{22}\right| = -\frac{1}{v}, \ \left|{}^{\,2}_{11}\right| = \frac{1}{v},$$

the Lorentz equations rewrite briefly

$$\begin{cases} \ddot{u} - \frac{2}{v}\dot{u}\dot{v} = \lambda \dot{v} v^2 \\ \ddot{v} + \frac{1}{v}(\dot{u}^2 - \dot{v}^2) = -\lambda \dot{u} v^2. \end{cases}$$

Using Maple V programming techniques one can visualize for different values of $\lambda \in \mathbb{R}$ the influence of the electric potentials $A_i(x)$ in (24) on the sheaves of geodesics of M (obtained for $\lambda = 0$, Fig.1), and perceive for nonvanishing values of λ their deformation into Lorentz curves ($\lambda \in \{\pm 512, \pm 1024\}$, Figs.2-5).[1]

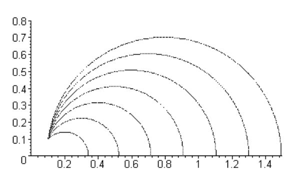

Fig.1. Lorentz Paths [case $\lambda = 0$: Poincare geodesics]

[1] In multiple-sheaf images, thick lines denote Poincare geodesics; the thin lines depict the Lorentz paths - Poincare geodesics deviated under the influence of electromagnetic force.

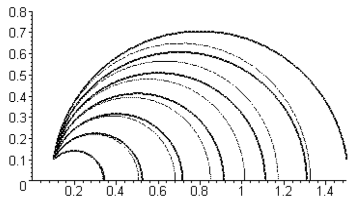

Fig.2. Lorentz Paths [case $\lambda \in \{0, 512\}$]

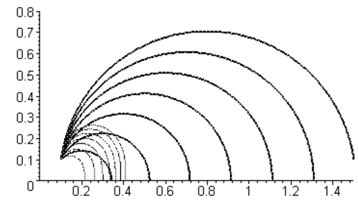

Fig.3. Lorentz Paths [case $\lambda \in \{0, 1024\}$]

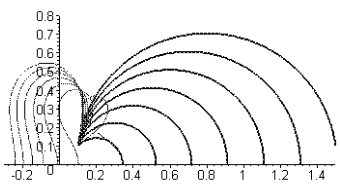

Fig.4. Lorentz Paths [case $\lambda \in \{0, -512\}$]

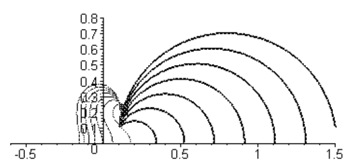

Fig.5. Lorentz Paths [case $\lambda \in \{0, -1024\}$]

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