ON A BLAIR-IANUS' QUESTION

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Abstract

In this paper we present some results of almost Hermitian manifolds of dimension four with J-invariant Ricci tensor and we restate some recent published results.

AMS Subject Classification: 53C15, 53C25.
Key words: Almost Hermitian manifold, almost Kähler manifold, Ricci tensor.

Much geometric information is carried by integrals of particular functions over a compact manifold, especially if such an integral is a critical value of a functional defined by such integrals on a space of certain geometric objects over the manifold.

The study of the integral of the scalar curvature: $A(g) = \int_M SdV$, as a functional on the set of all Riemannian metrics of the same total volume on a compact orientable manifold M is now classical (Hilbert 1915).

Let $M^{2n} \equiv (M^{2n}, g, J)$ be a $2n-$dimensional $(n \geq 2)$ almost Hermitian manifold equipped with the almost Hermitian structure (g, J) and Φ the Kähler form of M^{2n} defined by $\Phi(X, Y) = g(X, JY)$, for $X, Y \in \mathcal{X}(M^{2n})$ ($\mathcal{X}(M^{2n})$ denotes the Lie algebra of all smooth vector fields on M^{2n}). We denote by ∇, R , ρ, Q and S the Riemannian connection, the curvature tensor $(R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z)$, the Ricci tensor, the Ricci operator $(\rho(X,Y) = g(QX,Y))$ and the scalar curvature of M^{2n}, respectively.

We denote by ρ^* the $*$Ricci tensor of M^{2n} defined by $\rho^*(x,y) = \frac{1}{2}trace\ of$ $(z \to R(x,Jz)Jy)$ for $x,y,z \in T_p(M^{2n})$, $p \in M^{2n}$. We also denote by S^* the $*$scalar curvature of M^{2n}, which is the trace of the linear endomorphism Q^* defined by $g(Q^*x,y) = \rho^*(x,y)$, for $x,y \in T_p(M^{2n})$, $p \in M^{2n}$. If M^{2n} is a Kähler manifold, then $\rho = \rho^*$ (and therefore $S = S^*$).

On an almost Hermitian manifold M^{2n} we define the Weyl tensor W

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY +$$

Editor Gr.Tsagas *Proceedings of The Conference of Geometry and Its Applications in Technology and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1999,* 239-242

$$+\rho(Y,Z)X-\rho(X,Z)Y]+$$
$$+\frac{S}{2(2n-1)(n-1)}[g(Y,Z)X-g(X,Z)Y],$$

for all $X,Y,Z,W \in \mathcal{X}(M^{2n})$.

An almost Hermitian manifold M^{2n} is conformally flat if and only if $W(X,Y)Z \equiv 0$, or

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2(n-1)}[g(Y,Z)QX-g(X,Z)QY+\rho(Y,Z)X-\rho(X,Z)Y]+ \\ &+\frac{S}{2(2n-1)(n-1)}[g(Y,Z)X-g(X,Z)Y]. \end{aligned}$$

S.I. Goldberg in [8] proved that every almost Kähler manifold satisfying $g(R(X,Y)Z,W) = g(R(X,Y)JZ,JW)$ is a Kähler manifold. In this paper he stated the following conjecture:

"A compact Einstein almost Kähler manifold is a Kähler manifold " .

It is still an open problem in general. Important progress was made by K. Sekigawa ([15]), who proved that the conjecture is true if the scalar curvature is non-negative.

Let $M \equiv (M,g,\Phi)$ be a compact, symplectic manifold with scalar curvature S, $*$scalar curvature S^* and $\Phi(X,Y) = g(X,JY)$. Studing a variational problem, D.E.Blair and S. Ianus in [3] proved that an associated (to the symplectic form) metric on M is a critical point of $A(g) = \int_M SdV$ and $K(g) = \int_M (S-S^*)dV$ if and only if the Ricci operator is $J-invariant$. They also raised the question of whether a compact, almost Kähler manifold whose Ricci operator commutes with the almost complex structure must be Kähler. This question is stronger than Goldberg's conjecture. J. T. Davidov and O. Mushkarov in [4] gave a 6-dimensional example proving that the answer of this question is negative in general. Recently, P. Nurowski and M. Przanowski in [11] constucted an example of non-Kähler almost Kähler Ricci-flat space of dimension four. This example is also weakly $*$Einstein space which is not $*$Einstein.

It is worthwhile to discover some additional curvature condition on these manifolds such that Blair, Ianus' question has a positive answer.

In [16] we studied almost Hermitian manifolds of dimension 4 equipped with J-invariant Ricci tensor which also are conformally flat or heve harmonic curvature.

1Quite recently, the author knew through private communication with Prof. T. Draghici that in [1] (V. Apostolov and P. Gauduchon) a method is provided to obtain non-Kähler and non-Einstein Hermitian metrics with J-invariant Ricci tensor on compact 4-dimensional manifolds. These examples, which also appear by V. Apostolov, G. Ganchev and S. Ivanov in [2], condradict the statement of theorem 3.1 of [16].

This condradiction is due to the use of a result of Hamoui's paper [9]. In this paper it is proved that on every Hermitian manifold M it holds:

$$\begin{aligned} & g(R(X,Y)Z,W)-g(R(JX,JY)JZ,JW) \\ = & \frac{1}{2}[g((\nabla_{(\nabla_X J)Y-(\nabla_Y J)X}J)Z,W)+g((\nabla_{(\nabla_Z J)W-(\nabla_W J)Z}J)X,Y)] \end{aligned}$$

for all vector fields X, Y, Z, W on M. In the proof of this relation there exists a gap in the begining of the third paragraph (p. 206). Therefore, some of the results of [16] (Theorem 3.1, Corollary 3.3, Theorem 4.1 and Corollary 4.2) can be restated as follows.

Theorem 1 *Every conformally flat, almost Hermitian 4-manifold with J-invariant Ricci tensor is either a space of constant curvature or a Hermitian manifold.*

Corollary 2 *Every conformally flat, RK-manifold of dimension 4 is either a space of constant curvature or a Hermitian manifold.*

Theorem 3 *Every almost Hermitian 4-manifold with harmonic curvature and J-invariant Ricci tensor is either an Einstein manifold or a Hermitian manifold.*

Corollary 4 *Every RK-manifold of dimension 4 with harmonic curvature is either an Einstein manifold or a Hermitian manifold.*

Acknowledgement

The author would like to thank Professor Tedi Draghici for the useful suggestion.

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