Abstract

In this paper we present some results of almost Hermitian manifolds of dimension four with J-invariant Ricci tensor and we restate some recent published results.

AMS Subject Classification: 53C15, 53C25.

Key words: Almost Hermitian manifold, almost Kähler manifold, Ricci tensor.

Much geometric information is carried by integrals of particular functions over a compact manifold, especially if such an integral is a critical value of a functional defined by such integrals on a space of certain geometric objects over the manifold.

The study of the integral of the scalar curvature:

\[ A(g) = \int_M SdV, \]

as a functional on the set of all Riemannian metrics of the same total volume on a compact orientable manifold \( M \) is now classical (Hilbert 1915).

Let \( M^{2n} \equiv (M^{2n}, g, J) \) be a 2\( n \)-dimensional \((n \geq 2)\) almost Hermitian manifold equipped with the almost Hermitian structure \((g, J)\) and \( \Phi \) the Kähler form of \( M^{2n} \) defined by \( \Phi(X, Y) = g(X, JY) \), for \( X, Y \in \mathcal{X}(M^{2n}) \) \( (\mathcal{X}(M^{2n}) \) denotes the Lie algebra of all smooth vector fields on \( M^{2n} \). We denote by \( \nabla, R, \rho, Q \) and \( S \) the Riemannian connection, the curvature tensor \( (R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z) \), the Ricci tensor, the Ricci operator \( (\rho(X, Y) = g(QX, Y)) \) and the scalar curvature of \( M^{2n} \), respectively.

We denote by \( \rho^* \) the *Ricci tensor of \( M^{2n} \) defined by \( \rho^*(x, y) = \frac{1}{2}\text{trace of } (z \rightarrow R(z, Jx)y) \) for \( x, y, z \in T_p(M^{2n}) \), \( p \in M^{2n} \). We also denote by \( S^* \) the *scalar curvature of \( M^{2n} \), which is the trace of the linear endomorphism \( Q^* \) defined by \( g(Q^*x, y) = \rho^*(x, y) \), for \( x, y \in T_p(M^{2n}) \), \( p \in M^{2n} \). If \( M^{2n} \) is a Kähler manifold, then \( \rho = \rho^* \) (and therefore \( S = S^* \)).

On an almost Hermitian manifold \( M^{2n} \) we define the Weyl tensor \( W \)

\[ W(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)}[g(Y, Z)QX - g(X, Z)QY + \ldots] \]
\[ +\rho(Y, Z)X - \rho(X, Z)Y \] 
\[ + \frac{S}{2(2n - 1)(n - 1)}[g(Y, Z)X - g(X, Z)Y], \]
for all \( X, Y, Z, W \in \mathcal{X}(M^{2n}) \).

An almost Hermitian manifold \( M^{2n} \) is conformally flat if and only if \( W(X, Y)Z \equiv 0 \), or
\[ R(X, Y)Z = \frac{1}{2(n - 1)}[g(Y, Z)QX - g(X, Z)QY + \rho(Y, Z)X - \rho(X, Z)Y] + \]
\[ + \frac{S}{2(2n - 1)(n - 1)}[g(Y, Z)X - g(X, Z)Y]. \]

S.I. Goldberg in [8] proved that every almost Kähler manifold satisfying \( g(R(X, Y)Z, W) = g(R(X, Y)JZ, JW) \) is a Kähler manifold. In this paper he stated the following conjecture:

"A compact Einstein almost Kähler manifold is a Kähler manifold ".

It is still an open problem in general. Important progress was made by K. Sekigawa ([15]), who proved that the conjecture is true if the scalar curvature is non-negative.

Let \( M \equiv (M, g, \Phi) \) be a compact, symplectic manifold with scalar curvature \( S \), *scalar curvature \( S^* \) and \( \Phi(X, Y) = g(X, JY) \). Studing a variational problem, D.E.Blair and S. Ianus in [3] proved that an associated (to the symplectic form) metric on \( M \) is a critical point of \( A(g) = \int_M SdV \) and \( K(g) = \int_M (S - S^*)dV \) if and only if the Ricci operator is \( J \)-invariant. They also raised the question of whether a compact, almost Kähler manifold whose Ricci operator commutes with the almost complex structure must be Kähler. This question is stronger than Goldberg’s conjecture. J. T. Davidov and O. Mushkarov in [4] gave a 6-dimensional example proving that the answer of this question is negative in general. Recently, P. Nurowski and M. Przanowski in [11] constructed an example of non-Kähler almost Kähler Ricci-flat space of dimension four. This example is also weakly *Einstein space which is not *Einstein.

It is worthwhile to discover some additional curvature condition on these manifolds such that Blair, Ianus’ question has a positive answer.

In [16] we studied almost Hermitian manifolds of dimension 4 equipped with J-invariant Ricci tensor which also are conformally flat or have harmonic curvature.

Quite recently, the author knew through private communication with Prof. T. Draghici that in [1] (V. Apostolov and P. Gauduchon) a method is provided to obtain non-Kähler and non-Einstein Hermitian metrics with J-invariant Ricci tensor on compact 4-dimensional manifolds. These examples, which also appear by V. Apostolov, G. Ganchev and S. Ivanov in [2], contradict the statement of theorem 3.1 of [16].

This contradiction is due to the use of a result of Hamoui’s paper [9]. In this paper it is proved that on every Hermitian manifold \( M \) it holds:
\[ g(R(X, Y)Z, W) - g(R(JX, JY)JZ, JW) \] 
\[ = \frac{1}{2}g((\nabla_X J Y - (\nabla_Y J)X)J Z, W) + g((\nabla_Z J W - (\nabla_W J)Z)J X, Y) \]
for all vector fields $X, Y, Z, W$ on $M$. In the proof of this relation there exists a gap in the beginning of the third paragraph (p. 206). Therefore, some of the results of [16] (Theorem 3.1, Corollary 3.3, Theorem 4.1 and Corollary 4.2) can be restated as follows.

**Theorem 1** Every conformally flat, almost Hermitian 4-manifold with $J$-invariant Ricci tensor is either a space of constant curvature or a Hermitian manifold.

**Corollary 2** Every conformally flat, RK-manifold of dimension 4 is either a space of constant curvature or a Hermitian manifold.

**Theorem 3** Every almost Hermitian 4-manifold with harmonic curvature and $J$-invariant Ricci tensor is either an Einstein manifold or a Hermitian manifold.

**Corollary 4** Every RK-manifold of dimension 4 with harmonic curvature is either an Einstein manifold or a Hermitian manifold.

**Acknowledgement**

The author would like to thank Professor Tedi Draghici for the useful suggestion.

**References**


Author's address:

*Philippos J. Xenos*

*Mathematics Division-School of Technology*

*Aristotle University of Thessaloniki*

*Thessaloniki, 54006, GREECE*