A FAMILY OF METRICS WITH STRICTLY POSITIVE SECTIONAL CURVATURE ON A Riemannian Manifold

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Abstract

The aim of present paper is to find a family of metrics with strictly positive sectional curvature on the Riemannian manifold \( N_1^\kappa \times N_2^\lambda \), where \( N_1, N_2 \) are special subsets of \( IR^2 \).

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1 Introduction

The paper includes four sections which are organised as follows. The first section is the introduction. The second section includes a study of some special Riemannian metrics. In the third section we solve some special partial differential equations. The last section contains a study of the sectional curvature.

2 Study of special metrics

Let \( IR_1^2 \) be a Euclidian plane with an orthogonal coordinate system \((u_1, u_2)\) which as we already know is covered with an atlas with only one chart. Let us also consider a subset \( N_1 \subset IR_1^2 \) which is defined as follows:

\[
N_1 = \{(u_1, u_2) \in IR_1^2, 0 < u_1 < \infty, -\infty < u_2 < \infty \}. \tag{1}
\]

We consider a Riemannian metric on \( N_1 \) which is defined as below

\[
\omega_1 = \{\omega_{11} = 1, \omega_{12} = \omega_{21} = 0, \omega_{22} = u_1 \}. \tag{2}
\]
A family of metrics with strictly positive sectional curvature

The sectional curvature is given by the formula:

\[ K_P(\lambda) = \frac{-R_{ijkl}X^iY^jX^kY^l}{(\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk})X^iY^jX^kY^l}, i, j, k, l = 1, 2, \]

where \( P \in N_1 \) and \( \lambda \) is a plane of \( T_P(N_1) \).

The components of curvature tensor are given by:

\[ R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 \omega_{ik}}{\partial u_j \partial u_l} \right) + \frac{\partial^2 \omega_{jl}}{\partial u_i \partial u_k} - \frac{\partial^2 \omega_{il}}{\partial u_j \partial u_k} - \omega_{rs} \left\{ \Gamma^r_{jk} \Gamma^s_{il} - \Gamma^r_{jl} \Gamma^s_{ik} \right\}. \]

Using the above formula we have:

\[ R_{2121} = R_{1212} = - \frac{1}{4u_1} \\text{ and } R_{2112} = R_{1221} = \frac{1}{4u_1} \]

and all the rest components of curvature tensor are zero.

In the sequence we get:

\[ K_P(\lambda) = \frac{1}{4u_1^2} > 0, \forall P \in N_1, \forall \lambda \in T_P(N_1). \]

Hence \((N_1, \omega_1)\) has strictly positive sectional curvature. We have similar results for the subset:

\[ N_1 = \{(u_3, u_4) \in IR^2_2, 0 < u_3 < \infty, -\infty < u_4 < \infty\} \]

of another Euclidian plane \( IR^2_2 \) with orthogonal coordinate system \((u_3, u_4)\) and metric:

\[ \omega_2 = \{\omega_{33} = 1, \omega_{34} = \omega_{43} = 0, \omega_{44} = u_3\}. \]

Consequently we obtain the following:

**Proposition 1** The Riemannian manifolds \((N_1, \omega_1)\) and \((N_2, \omega_2)\) have strictly positive sectional curvature.

We now consider the Riemannian manifolds \( IR^{2\kappa}_1 \) with orthogonal coordinate system \( \{u_1, u_2, \ldots, u_{2\kappa}\} \) and \( IR^{2\lambda}_2 \) with orthogonal coordinate system \( \{u_{2\kappa+1}, u_{2\kappa+2}, \ldots, u_{2\kappa+2\lambda}\} \), \( \kappa, \lambda \in N^* \). Each of them is covered with an atlas with only one chart. Let us consider their subsets \( N^{\kappa}_1, N^{\lambda}_2 \), which are defined as follows:

\[ N^{\kappa}_1 = \{(u_1, u_2, \ldots, u_{2\kappa}) \in IR^{2\kappa}_1, 0 < u_1, u_3, \ldots, u_{2\kappa-1} < \infty, -\infty < u_2, u_4, \ldots, u_{2\kappa} < \infty\}, \]

\[ N^{\lambda}_2 = \{(u_{2\kappa+1}, u_{2\kappa+2}, \ldots, u_{2\kappa+2\lambda}) \in IR^{2\lambda}_2, 0 < u_{2\kappa+1}, u_{2\kappa+3}, \ldots, u_{2\kappa+2\lambda-1} < \infty, -\infty < u_{2\kappa+2}, u_{2\kappa+4}, \ldots, u_{2\kappa+2\lambda} < \infty\}. \]

We consider a Riemannian metric on \( N^{\kappa}_1 \) which is defined as below:
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\[ h_1 = \{ h_{ii} = 1, i = 1, 3, \ldots, 2\kappa - 1, h_{ii} = u_{i-1}, i = 2, 4, \ldots, 2\kappa, \]
\[ h_{ij} = 0, i \neq j, i, j = 1, 2, \ldots, 2\kappa \}. \quad (11) \]

Similarly on the Riemannian manifold \( N^2 \) we consider the metric:

\[ h_2 = \{ h_{ii} = 1, i = 2\kappa + 1, 2\kappa + 3, \ldots, 2\kappa + 2\lambda - 1, \]
\[ h_{ii} = u_{i-1}, i = 2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda, \]
\[ h_{ij} = 0, i \neq j, i, j = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda \}. \quad (12) \]

We easily find that the Riemannian manifolds \((N^1, h_1)\) and \((N^2, h_2)\) have positive sectional curvature.

From the above we get:

**Proposition 2** The Riemannian manifolds \((N^1, h_1)\) and \((N^2, h_2)\) have positive sectional curvature.

On the Cartesian product \( N^1 \times N^2 \) we consider a special monoparametrical family of Riemannian metrics:

\[ d(t) = \begin{cases} 
  d_{ii} = 1 + tf_i, i = 1, 3, \ldots, 2\kappa - 1, \\
  d_{ii} = u_{i-1}(1 + tf_i), i = 2, 4, \ldots, 2\kappa, \\
  d_{jj} = 1 + t\varphi_j, j = 2\kappa + 1, 2\kappa + 3, \ldots, 2\kappa + 2\lambda - 1, \\
  d_{jj} = u_{j-1}(1 + t\varphi_j), j = 2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda, \\
  d_{ij} = 0, i \neq j, i, j = 1, 2, \ldots, 2\kappa + 2\lambda, 
\end{cases} \quad (13) \]

where

\[ f_i = f_i(u_{2\kappa+1}, u_{2\kappa+2}, \ldots, u_{2\kappa+2\lambda}), i = 1, 2, \ldots, 2\kappa + 2\lambda, \]
\[ \varphi_j = \varphi_j(u_1, u_2, \ldots, u_{2\kappa}), j = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda \] \quad (14)

are arbitrary functions and \(-\varepsilon < t < \varepsilon\), \(\varepsilon\) is a small positive number.

Our aim is to define \( f_i, i = 1, 2, \ldots, 2\kappa \) and \( \varphi_j, j = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda \) in order that the Riemannian manifold \((N^1 \times N^2, d(t))\) had strictly positive sectional curvature.

It is obvious that:

\[ d(0) = h_1 \times h_2. \quad (15) \]

Let now \( P \) be an arbitrary point of \( N^1 \times N^2 \). We know that the sectional curvature of a plane which is spanned by the vectors \( X \) and \( Y \) of the tangent space \( T_P(N^1 \times N^2) \) is given by the formula:

\[ \sigma(X, Y)(t) = -\frac{(R(X, Y)X, Y)}{\|X\|^2 \|Y\|^2 - \langle X, Y \rangle}. \quad (16) \]

According to the Mac-Lauren theorem for the function of one variable \( \sigma(X, Y)(t) \) we can claim that:


\[ \sigma(X,Y)(t) = \sigma(X,Y)(0) + \frac{t}{11} + \frac{\sigma''(X,Y)(0)t^2}{2!} + \ldots \]  

(17)

From the formula (17) it is clear that the sign of \( \sigma(X,Y)(t) \) depends on the sign of \( \sigma(X,Y)(0) \), if \( \sigma(X,Y)(0) \neq 0 \) and \( t \) is a small positive number. In case where \( \sigma(X,Y)(0) = 0 \), the sign of \( \sigma(X,Y)(t) \) depends on the sign of \( \sigma'(X,Y)(0) \), since \( t > 0 \). We also know [1, p. 287] that:

\[ \sigma(X,Y)(0), \text{ if } X \in T_P(N^+_1) \text{ and } Y \in T_P(N^2_2). \]  

(18)

We have the following relation for \( \sigma(X,Y)(t) \):

\[ \sigma(X,Y)(t) = \frac{A(t)}{B(t)}, \]  

(19)

where

\[ A(t) = R_{ijkl}X^iX^jX^kX^l, \quad i, \rho = 1, 2, \ldots, 2\kappa, \]

\[ j, \sigma = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda. \]  

(20)

and

\[ B(t) = 4\left\{ \sum_{i=1}^{2\kappa} d_{ii}(X^i)^2 \right\} \left\{ \sum_{j=2\kappa+1}^{2\kappa+2\lambda} d_{jj}(Y^j)^2 \right\} > 0, \]  

(21)

because in this case \( \langle X, Y \rangle = 0 \).

From the relation (19) we can obtain that:

\[ \sigma(X,Y)(0) = 0 \iff -\frac{A(0)}{B(0)} = 0 \iff A(0) = 0. \]  

(22)

By derivation of (19) with respect to \( t \) it holds:

\[ \sigma'_1(X,Y)(0) = -\frac{A'(0)B(0) - A(0)B'(0)}{B^2(0)}, \]  

(23)

which due to (22) becomes:

\[ \sigma'_1(X,Y)(0) = \frac{A'(0)}{B(0)}. \]  

(24)

From the formula (20) we get:

\[ A'(0) = R_{ijkl}(0)X^iX^jX^kX^l, \quad i, \rho = 1, 2, \ldots, 2\kappa, \]

\[ j, \sigma = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda. \]  

(25)

As known, the components of curvature tensor \( R_{ijkl} \) are given by:
\[ R_{\ij\ps} = \frac{1}{2} \left\{ \frac{\partial^2 d_{\ij}}{\partial u_i \partial u_j} - \frac{\partial^2 d_{\ps}}{\partial u_i \partial u_\ps} \right\} - \{ \Gamma^\tau_{j\rho} \Gamma^\rho_{i\ps} - \Gamma^\tau_{j\ps} \Gamma^\rho_{i\rho} \}, \]

\( i, \rho = 1, 2, \ldots, 2\kappa, j, \sigma = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda, \)

where \( \Gamma^\tau_{j\rho}, \Gamma^\rho_{i\ps}, \Gamma^\tau_{j\ps}, \Gamma^\rho_{i\rho} \) are the Christoffel symbols of second kind.

From the form of the Riemannian metric given by the relation (13) we obtain:

\[ d_{j\rho} = 0, d_{i\ps} = 0. \]

Hence the formula (26) becomes:

\[ R_{\ij\ps} = \frac{1}{2} \left\{ \frac{\partial^2 d_{\ij}}{\partial u_i \partial u_j} + \frac{\partial^2 d_{\ps}}{\partial u_i \partial u_\ps} \right\} - d_{rs} \{ \Gamma^\tau_{j\rho} \Gamma^\rho_{i\ps} - \Gamma^\tau_{j\ps} \Gamma^\rho_{i\rho} \}, \]

\( i, \rho = 1, 2, \ldots, 2\kappa, j, \sigma = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda. \)

It is known that the Christoffel symbols are given by the below formula:

\[ \Gamma^\beta_{\alpha\gamma} = \frac{1}{2} d^{\beta\delta} \left( \frac{\partial d_{\delta\alpha}}{\partial u_\gamma} + \frac{\partial d_{\delta\gamma}}{\partial u_\alpha} - \frac{\partial d_{\alpha\gamma}}{\partial u_\delta} \right), \]

where \( (d^{\beta\delta}) \) is the inverse matrix of \( (d_{\beta\delta}) \).

After some calculations the Christoffel symbols of the Riemannian manifold \( N^1_\lambda \times N^2_\lambda \) are given by the following formulas:

\[ \Gamma^\gamma_{j\rho} = \begin{cases} \frac{t}{2} \frac{1}{1 + t\varphi_j} \frac{\partial \varphi_j}{\partial u_\rho}, & \text{for } r = j, \\
\frac{t}{2} \frac{1}{1 + t\varphi_\rho} \frac{\partial \varphi_\rho}{\partial u_j}, & \text{for } r = \rho, \\
0 & \text{in any other case}, \end{cases} \]

\( \Gamma^\gamma_{i\sigma} = \begin{cases} \frac{t}{2} \frac{1}{1 + t\varphi_i} \frac{\partial \varphi_i}{\partial u_\sigma}, & \text{for } \tau = i, \\
\frac{t}{2} \frac{1}{1 + t\varphi_\sigma} \frac{\partial \varphi_\sigma}{\partial u_i}, & \text{for } \tau = \sigma, \\
0 & \text{in any other case}, \end{cases} \)
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\[
\Gamma_{j\sigma} = \begin{cases} 
\frac{1}{2u_{j-1}}, r \neq j - 1, r = j \in \{2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda\}, \\
\frac{1}{2u_{j-1}}, r \neq j - 1, r = \sigma \in \{2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda\}, \\
t \frac{1}{t} \partial f_i, r \in \{1, 3, \ldots, 2\kappa - 1\}, \\
r \neq \sigma = j \in \{2\kappa + 1, 2\kappa + 3, \ldots, 2\kappa + 2\lambda - 1\}, \\
t \frac{1}{t} \partial f_i, r \in \{2, 4, \ldots, 2\kappa\}, \\
r \neq \sigma = j \in \{2\kappa + 1, 2\kappa + 3, \ldots, 2\kappa + 2\lambda - 1\}, \\
\frac{1}{2} u_{j-1} (1 + \tau f_{\sigma}) \partial u_{\sigma}, r \in \{2, 4, \ldots, 2\kappa\}, \\
r \neq \sigma = j \in \{2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda\}, \\
\frac{1}{2} u_{\sigma-1} (1 + \tau f_{\sigma}) \partial u_{\sigma}, r \in \{2, 4, \ldots, 2\kappa\}, \\
r \neq \sigma = j \in \{2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda\}, \\
\frac{1}{2} u_{\sigma-1} (1 + \tau f_{\sigma}) \partial u_{\sigma}, r \in \{2, 4, \ldots, 2\kappa\}, \\
r \neq \sigma = j \in \{2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda\}, \\
0 \text{ in any other case.}
\end{cases}
\]
only in the case

We also have:

\[ R_{\alpha j j} = \begin{cases} 
    t \frac{\partial^2 f_i}{\partial^2 \varphi_j} = \frac{1}{4} t \left( \frac{1}{1 + t \varphi_j} \right)^2 + \frac{1}{1 + t f_i} \left( \frac{\partial f_i}{\partial u_j} \right)^2, \\
    i \in \{1, 3, ..., 2k - 1\}, j \in \{2k + 1, 2k + 1, ..., 2k + 2\}, \\
    t \frac{\partial^2 f_i}{\partial^2 \varphi_j} + u_{j-1} \frac{\partial^2 \varphi_j}{\partial^2 u_i} = \frac{1}{4} t \left( \frac{1}{1 + t \varphi_j} \right)^2 + \frac{1}{1 + t f_i} \left( \frac{\partial f_i}{\partial u_j} \right)^2, \\
    i \in \{1, 3, ..., 2k - 1\}, j \in \{2k + 1, 2k + 3, ..., 2k + 2\}, \\
    \frac{1}{4} \left( \frac{1}{1 + t \varphi_j} \right)^2 + t \left( \frac{1}{1 + t f_i} \right) \frac{\partial f_i}{\partial u_j}, \\
    i \in \{1, 3, ..., 2k - 1\}, j \in \{2k + 1, 2k + 3, ..., 2k + 2\}, \\
    t \frac{2}{4} \left( \frac{1}{1 + t \varphi_j} \right)^2 + u_{j-1} \frac{\partial^2 \varphi_j}{\partial^2 u_i} + u_{j-1} \left( \frac{1}{1 + t f_i} \right) \frac{\partial f_i}{\partial u_j}, \\
    i \in \{2, 4, ..., 2k\}, j \in \{2k + 1, 2k + 2, ..., 2k + 2\}, \\
    \end{cases} \]

\[ R_{\alpha j j} = \begin{cases} 
    t \frac{\partial^2 f_i}{\partial^2 \varphi_j} = \frac{1}{4} t \left( \frac{1}{1 + t \varphi_j} \right)^2 + \frac{1}{1 + t f_i} \left( \frac{\partial f_i}{\partial u_j} \right)^2, \\
    i \in \{1, 3, ..., 2k - 1\}, j \neq \sigma, j, \sigma \in \{2k + 1, 2k + 2, ..., 2k + 2\}, \\
    t \frac{2}{4} \left( \frac{1}{1 + t \varphi_j} \right)^2 + u_{j-1} \left( \frac{1}{1 + t f_i} \right) \frac{\partial f_i}{\partial u_j}, \\
    i \in \{2, 4, ..., 2k\}, j \neq \sigma, j, \sigma \in \{2k + 1, 2k + 3, ..., 2k + 2\}, \\
    \end{cases} \]

Remark 2.1 In relation (35) the components written with bold characters appear only in the case \( \sigma = j - 1 \) and the components written with bigger characters appear only in the case \( j = \sigma - 1 \).

We also have:

\[ R_{\alpha j j} = \begin{cases} 
    t \frac{\partial^2 \varphi_j}{\partial u_{j-1} u_{j-1}} = \frac{1}{4} t \left( \frac{1}{1 + t \varphi_j} \right)^2 + t \left( \frac{1}{1 + t f_i} \right) \frac{\partial f_i}{\partial u_j}, \\
    j = 2k + 1, 2k + 3, ..., 2k + 2\lambda - 1, \ i \neq \rho, i, \rho \in \{1, 2, ..., 2k\}, \\
    t \frac{2}{4} \left( \frac{1}{1 + t \varphi_j} \right)^2 + t \left( \frac{1}{1 + t f_i} \right) \frac{\partial f_i}{\partial u_j}, \\
    j = \{2k + 2, 2k + 4, ..., 2k + \lambda\}, \ i \neq \rho, i, \rho \in \{1, 2, ..., 2k\}. \\
    \end{cases} \]
Remark 2.2 In relation (36) the components written with bold characters appear only in the case \( i = \rho - 1 \) and the components written with bigger characters appear only in the case \( \rho = i - 1 \).

From the form of the Riemannian metric given by the formula (13) we get:

\[
R_{ij\rho\sigma} = 0, \ i \neq \rho, \ j \neq \sigma, \ i, \rho \in \{1, 2, ..., 2\kappa\}, \ j, \sigma = \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\} \tag{37}
\]

### 3 Special partial differential equations

We consider the functions:

\[
f_i, \ i = 1, 2, ..., 2\kappa \text{ and } \varphi_j, \ j = 2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda
\tag{38}
\]

so that they satisfy the following partial differential equations:

\[
\frac{\partial^2 f_i}{\partial u_j \partial u_\sigma} = 0, \ i = 1, 3, ..., 2\kappa - 1, \ j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}, \tag{39}
\]

\[
\frac{\partial^2 f_i}{\partial u_j \partial u_{j-1}} - \frac{1}{u_{j-1}} \frac{\partial f_i}{\partial u_j} = 0, \ i = 1, 3, ..., 2\kappa - 1, \sigma = j - 1, \ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}, \tag{40}
\]

\[
\frac{\partial^2 f_i}{\partial u_\sigma \partial u_{\sigma-1}} - \frac{1}{u_{\sigma-1}} \frac{\partial f_i}{\partial u_\sigma} = 0, \ i = 1, 3, ..., 2\kappa - 1, \ j = \sigma - 1, \ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}, \tag{41}
\]

\[
u_{i-1} \frac{\partial^2 f_i}{\partial u_j \partial u_\sigma} = 0, \ i = 2, 4, ..., 2\kappa, \ j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}, \tag{42}
\]

\[
u_{i-1} \frac{\partial^2 f_i}{\partial u_j \partial u_{j-1}} - \frac{u_{i-1}}{u_{j-1}} \frac{\partial f_i}{\partial u_j} = 0, \ i = 2, 4, ..., 2\kappa, \sigma = j - 1; \ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}, \tag{43}
\]

\[
u_{i-1} \frac{\partial^2 f_i}{\partial u_\sigma \partial u_{\sigma-1}} - \frac{u_{i-1}}{u_{\sigma-1}} \frac{\partial f_i}{\partial u_\sigma} = 0, \ i = 2, 4, ..., 2\kappa, \ j = \sigma - 1, \ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}, \tag{44}
\]

\[
\frac{\partial^2 \varphi_j}{\partial u_i \partial u_\rho}, \ j = 2\kappa + 1, 2\kappa + 3, ..., 2\kappa + 2\lambda - 1, \ i \neq \rho, \ i, \rho \in \{1, 2, ..., 2\lambda\}, \tag{45}
\]

\[
\frac{\partial^2 \varphi_j}{\partial u_\rho \partial u_{\rho-1}} - \frac{1}{u_{\rho-1}} \frac{\partial \varphi_j}{\partial u_\rho} = 0, \ j = 2\kappa + 1, 2\kappa + 3, ..., 2\kappa + 2\lambda - 1, \ i = \rho - 1, i, \rho \in \{1, 2, ..., 2\kappa\}, \tag{46}
\]
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\begin{equation}
\frac{\partial^2 \varphi_j}{\partial u_i \partial u_{i-1}} - \frac{1}{2u_{i-1}} \frac{\partial \varphi_j}{\partial u_\rho} = 0, \ j = 2\kappa + 1, 2\kappa + 3, ..., 2\kappa + 2\lambda - 1, \rho = i-1, i, \rho \in \{1, 2, ..., 2\kappa\},
\end{equation}
(47)

\begin{equation}
u_j \frac{\partial^2 \varphi_j}{\partial u_i \partial u_\rho} = 0, \ j = 2\kappa + 2, 2\kappa + 4, ..., 2\kappa + 2\lambda, \rho \neq i, \ i, \rho \in \{1, 2, ..., 2\kappa\},
\end{equation}
(48)

\begin{equation}
u_j \frac{\partial^2 \varphi_j}{\partial u_i \partial u_{i-1}} - \frac{u_{i-1}}{u_\rho} \frac{\partial \varphi_j}{\partial u_\rho} = 0, \ j = 2\kappa + 2, 2\kappa + 4, ..., 2\kappa + 2\lambda, \rho = i-1, \ i, \rho \in \{1, 2, ..., 2\kappa\},
\end{equation}
(49)

\begin{equation}
u_j \frac{\partial^2 \varphi_j}{\partial u_i \partial u_{i-1}} - \frac{u_{i-1}}{u_\rho} \frac{\partial \varphi_j}{\partial u_\rho} = 0, \ j = 2\kappa + 2, 2\kappa + 4, ..., 2\kappa + 2\lambda, \rho = i-1, \ i, \rho \in \{1, 2, ..., 2\kappa\}.
\end{equation}
(50)

A solution of the system of partial differential equations (39)-(50) is given below:

\begin{equation}
f_i = V_{i,2\kappa+1}(u_{2\kappa+1}), \ i = 1, 2, ..., 2\kappa
\end{equation}
(51)

and

\begin{equation}\varphi_j = V_{j,1}(u_1), \ j = 2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda.
\end{equation}
(52)

**Remark 3.1** We note here that if the system of partial differential equations (39)-(50) is satisfied, then the components \(R_{ij\sigma}, R_{ij\rho}\) of the curvature tensor will have the form:

\begin{equation}
R_{ij\sigma} = \begin{cases} \frac{t^2}{4} \left( 1 + \frac{t\varphi_j}{u_i} \frac{\partial f_i}{\partial u_j} \frac{\partial f_\sigma}{\partial u_\sigma} \right), & i = 1, 3, ..., 2\kappa - 1, \ j \neq \sigma, \ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\} \\
\frac{t^2}{4} \left( 1 + \frac{t\varphi_j}{u_i} \frac{\partial f_i}{\partial u_j} \frac{\partial f_\sigma}{\partial u_\sigma} \right), & i = 2, 4, ..., 2\kappa, \ j \neq \sigma, \ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}
\end{cases}
\end{equation}
(53)

and

\begin{equation}
R_{ij\rho} = \begin{cases} \frac{-t^2}{4} \left( 1 + \frac{t\varphi_j}{u_i} \frac{\partial f_i}{\partial u_j} \frac{\partial \varphi_j}{\partial u_\rho} \right), & j = 2\kappa + 1, 2\kappa + 3, ..., 2\kappa + 2\lambda - 1, \ i \neq \rho, \ i, \rho \in \{1, 2, ..., 2\kappa\} \\
\frac{-t^2}{4} \left( 1 + \frac{t\varphi_j}{u_i} \frac{\partial f_i}{\partial u_j} \frac{\partial \varphi_j}{\partial u_\rho} \right), & j = 2\kappa + 2, 2\kappa + 4, ..., 2\kappa + 2\lambda, \ i \neq \rho, \ i, \rho \in \{1, 2, ..., 2\kappa\}
\end{cases}
\end{equation}
(54)

From the formulas (53)-(54) we obtain:
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\[ R_{ij\sigma}(0) = R_{ij\rho}(0) = 0, i, \rho \in \{1, 2, ..., 2\kappa\}, \]
\[ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}. \]  

(55)

From relation (34) we get:

\[
R'_{ijij}(0) = \begin{cases} 
\frac{1}{2} \left( \frac{\partial^2 f_i}{\partial u_j^2} + \frac{\partial^2 \varphi_j}{\partial u_i^2} \right), & i \in \{1, 3, ..., 2\kappa - 1\}, \\
\frac{1}{2} \left( \frac{\partial^2 f_i}{\partial u_j^2} + u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i^2} \right) + \frac{1}{4} \frac{\partial f_i}{\partial u_{j-1}}, & i \in \{1, 3, ..., 2\kappa - 1\}, \\
\frac{1}{2} \left( \frac{\partial^2 f_i}{\partial u_j^2} + u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i^2} \right) + \frac{1}{4} \frac{\partial f_i}{\partial u_{j-1}}, & i \in \{2, 4, ..., 2\kappa\}, \\
\frac{1}{2} \left( \frac{\partial^2 f_i}{\partial u_j^2} + u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i^2} \right) + \frac{1}{4} u_{i-1} \frac{\partial \varphi_j}{\partial u_{j-1}} + \frac{1}{4} u_{i-1} \frac{\partial \varphi_j}{\partial u_{j-1}}, & i \in \{2, 4, ..., 2\kappa\}, j \in \{2\kappa + 2, 2\kappa + 4, ..., 2\kappa + 2\lambda\}. 
\end{cases}
\]  

(56)

From relation (37) it is also obvious that:

\[ R_{ij\rho\sigma}(0) = 0, i \neq \rho, j \neq \sigma, i, \rho \in \{1, 2, ..., 2\kappa\}, \]
\[ j, \sigma \in \{2\kappa + 1, 2\kappa + 2, ..., 2\kappa + 2\lambda\}. \]  

(57)

**Remark 3.2** Using the above formulas we conclude that the only non-zero components \( R'_{ijij}(0) \) are those given below:

\[
R'_{ijij}(0) = \begin{cases} 
\frac{1}{2} \left( \frac{\partial^2 f_i}{\partial u_j^2} + \frac{\partial^2 \varphi_j}{\partial u_i^2} \right), & j = 2\kappa + 1, \\
\frac{1}{2} \left( \frac{V'_{1,2\kappa+1}(u_{2\kappa+1}) + V'_{1,2\kappa+1}(u_1)}{V'_{1,2\kappa+1}(u_1)}, \right), & j = 2\kappa + 3, 2\kappa + 5, ..., 2\kappa + 2\lambda - 1, 
\end{cases}
\]  

(1)

\[
R'_{2\kappa+1:2\kappa+1}(0) = \begin{cases} 
\frac{1}{2} \left( \frac{\partial^2 f_i}{\partial u_{2\kappa+1}^2} + \frac{\partial^2 \varphi_{2\kappa+1}}{\partial u_i^2} \right), & i = 1, \\
\frac{1}{2} \left( \frac{V'_{1,2\kappa+1}(u_{2\kappa+1}) + V'_{2\kappa+1,1}(u_1)}{V'_{1,2\kappa+1}(u_{2\kappa+1})}, \right), & i = 3, 5, ..., 2\kappa - 1, 
\end{cases}
\]  

(II)
\( R_{ijij}^{0}(0) = \begin{cases} 
\frac{1}{2}u_{ij,1}V'_{j,1}(u_1), \\
i = 1, j = 2\kappa + 4, 2\kappa + 6, \ldots, 2\kappa + 2\lambda, \\
\frac{1}{2}u_{2\kappa+1}V'_{2\kappa+2,1}(u_1) + \frac{1}{4}V'_{1,2\kappa+1}(u_{2\kappa+1}), \\
i = 1, j = 2\kappa + 2, \\
\frac{1}{4}V'_{1,2\kappa+1}(u_{2\kappa+1}), \\
i = 3, 5, \ldots, 2\kappa - 1, j = 2\kappa + 2, \\
0 \text{ in any other case.} 
\end{cases} \) \hspace{1cm} (III)

\( R_{ijij}^{0}(0) = \begin{cases} 
\frac{1}{2}u_{i-1}V'_{i,2\kappa+1}(u_{2\kappa+1}), \\
j = 2\kappa + 1, i = 4, 6, \ldots, 2\kappa, \\
\frac{1}{2}u_{i}V'_{2,2\kappa+1}(u_{2\kappa+1}) + \frac{1}{4}V'_{2\kappa+1,1}(u_1), \\
i = 2, j = 2\kappa + 1, \\
\frac{1}{4}V'_{j,1}(u_1), \\
i = 2, j = 2\kappa + 3, 2\kappa + 5, \ldots, 2\kappa + 2\lambda - 1, \\
0 \text{ in any other case.} 
\end{cases} \) \hspace{1cm} (IV)

\( R_{ijij}^{0}(0) = \begin{cases} 
\frac{1}{2}u_{j-1}V'_{j,1}(u_1), \\
j = 2\kappa + 4, 2\kappa + 6, \ldots, 2\kappa + 2\lambda, i = 2, \\
\frac{1}{2}u_{i-1}V'_{i,2\kappa+1}(u_{2\kappa+1}), \\
i = 4, 6, \ldots, 2\kappa; j = 2\kappa + 2, \\
\frac{1}{4}u_{2\kappa+1}V'_{2\kappa+1,1}(u_1) + \frac{1}{4}u_{2\kappa+1}V'_{2,2\kappa+1}(u_{2\kappa+1}), \\
i = 2, j = 2\kappa + 2, \\
0 \text{ in any other case.} 
\end{cases} \) \hspace{1cm} (V)

4 Study of the sectional curvature

Using the formula (20) we have:

\[ A'(0) = R_{ijij}^{0}(0)(X^i j)^2(Y^j j')^2 < 0, \]
\[ i = 1, 2, \ldots, 2\kappa, j = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda, \]

which is equivalent to \( \sigma(X, Y)(0) > 0 \), because from (21) it holds \( B(0) > 0 \).

From relation (58) we obtain:

\[ \begin{cases} 
R_{ijij}^{0}(0) \leq 0, \forall i = 1, 2, \ldots, 2\kappa, \forall j = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda \\
\text{and} \\
R_{ijij}^{0}(0) < 0, \text{ for at least pair } (i, j) \\
\text{with } i = 1, 2, \ldots, 2\kappa, j = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda. 
\end{cases} \]

(59)

By considering cases I-V the system (59) becomes:
A family of metrics with strictly positive sectional curvature

\[\begin{align*}
\frac{1}{4} (V_{1,2\kappa+1}^\prime(u_{2\kappa+1}) + V_{2\kappa+1,1}^\prime(u_1)) & \leq 0, \\
\frac{1}{4} V_{j,1}^\prime(u_1) & \leq 0, j = 2\kappa + 3, 2\kappa + 5, \ldots, 2\kappa + 2\lambda - 1, \\
\frac{1}{4} V_{1,2\kappa+1}^\prime(u_{2\kappa+1}) & \leq 0, i = 1, 3, \ldots, 2\kappa - 1, \\
\frac{1}{4} u_{j-1} V_{j,1}^\prime(u_1) & \leq 0, j = 2\kappa + 4, 2\kappa + 6, \ldots, 2\kappa + 2\lambda, \\
\frac{1}{4} u_{2\kappa+1} V_{2\kappa+2,1}^\prime(u_1) + \frac{1}{4} V_{2\kappa+1}^\prime(u_{2\kappa+1}) & \leq 0, \\
\frac{1}{4} u_{i-1} V_{i,2\kappa+1}^\prime(u_{2\kappa+1}) & \leq 0, i = 3, 5, \ldots, 2\kappa - 1, \\
\frac{1}{4} u_{2\kappa+1} V_{2\kappa+2,1}^\prime(u_1) + \frac{1}{4} V_{2\kappa+1}^\prime(u_{2\kappa+1}) & \leq 0,
\end{align*}\]

where in one at least case the strict inequality holds.

The functions:

\[V_{i,2\kappa+1}^\prime(u_{2\kappa+1}) = 0, i = 2, 4, \ldots, 2\kappa,\]  
\[(61)\]

\[V_{i,2\kappa+1}^\prime(u_{2\kappa+1}) = \alpha_i u_{2\kappa+1} + \beta_i, \alpha_i, \beta_i \in \mathbb{R}, \alpha_i < 0,\]
\[i = 1, 3, \ldots, 2\kappa - 1,\]  
\[(62)\]

\[V_{j,1}^\prime(u_1) = 0, j = 2\kappa + 2, 2\kappa + 4, \ldots, 2\kappa + 2\lambda,\]  
\[(63)\]

\[V_{j,1}^\prime(u_1) = \alpha_j u_1 + \beta_j, \alpha_j, \beta_j \in \mathbb{R}, \alpha_j < 0,\]
\[j = 2\kappa + 1, 2\kappa + 3, \ldots, 2\kappa + 2\lambda - 1\]  
\[(64)\]

are a solution of the system (60).

Therefore the functions

\[f_i, i = 1, 2, \ldots, 2\kappa \text{ and } \varphi_j, j = 2\kappa + 1, 2\kappa + 2, \ldots, 2\kappa + 2\lambda,\]

can be the following:

\[f_i = f_i(u_{2\kappa+1}) = \begin{cases} 
\alpha_i u_{2\kappa+1} + \beta_i, & \alpha_i, \beta_i \in \mathbb{R}, \alpha_i < 0, \\
& i = 1, 3, \ldots, 2\kappa - 1, \\
0, & i = 2, 4, \ldots, 2\kappa 
\end{cases},\]  

\[(65)\]
\[ \varphi_j = \varphi_j(u_1) = \begin{cases} 
\alpha_j u_1 + \beta_j, & \alpha_j, \beta_j \in IR, \alpha_j < 0 \\
\varphi_j = 2 \kappa + 1, 2 \kappa + 3, ..., 2 \kappa + 2 \lambda - 1 \\
0, & j = 2 \kappa + 2, 2 \kappa + 4, ..., 2 \kappa + 2 \lambda 
\end{cases} \] (66)

Consequently the family of metrics can be:

\[ d(t) = \begin{cases} 
d_{ii} = 1 + t(\alpha_i u_{2 \kappa + 1} + \beta_i), & \alpha_i, \beta_i \in IR, \alpha_i < 0, i = 1, 3, ..., 2 \kappa - 1, \\
d_{ii} = u_{i-1}, & i = 1, 3, ..., 2 \kappa - 1, \\
d_{jj} = 1 + t(\alpha_j u_1 + \beta_j), & \alpha_j, \beta_j \in IR, \alpha_j < 0, \\
& j = 2 \kappa + 1, 2 \kappa + 3, ..., 2 \kappa + 2 \lambda - 1, \\
d_{ij} = u_{j-1}, & j = 2 \kappa + 2, 2 \kappa + 4, ..., 2 \kappa + 2 \lambda, \\
d_{ij} = 0, & i \neq j, i, j \in \{1, 2, ..., 2 \kappa + 2 \lambda\},
\end{cases} \] (67)

which is an answer to our problem.

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References


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