Abstract

Let $\mathbb{R}^n$ be a vector space on $\mathbb{R}$. From this space we form a new vector space $\mathbb{R}^n$ with new structures, that is: isovectorial, isotopological, isoaffine and isometric.

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1 Basic Definitions

We consider the isofield $(\mathbb{R}, +, *)$ and we form a new isospace $(\mathbb{R}^2, +, *)$ as follows:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \left\{ (\tilde{a}_1, \tilde{a}_2) / \tilde{a}_i = a_i \hat{I}, a_i \in \mathbb{R}, i = 1, 2 \right\}$$

This new isospace $\mathbb{R}^2$ is called Cartesian isospace.

Generally, the isospace:

$$\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R} = \left\{ (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) / \tilde{a}_i = a_i \hat{I}, a_i \in \mathbb{R}, i = 1, 2, \ldots, n \right\}$$

where $\mathbb{R} \times \ldots \times \mathbb{R}$ is taken by $n$ times, is the Cartesian product of isofield $\mathbb{R}$ $n$ times and it is called real Cartesian isospace. On this isospace we define the following isostructures:

2 Isovector and Isoaffine Spaces

From the vector space $V^n(\mathbb{R})$, by means of the isofield $\mathbb{R}$, we obtain an isovector space $V''(\mathbb{R})$, which is called real Cartesian isovector space.

The vectors:

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$$
that constitute the canonical base of $V^n(\mathbb{R})$, are a base of $V^n(\hat{\mathbb{R}})$, because every isovector $v \in V^n$ can be written:

$$v = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n, \; \lambda_i \in \hat{\mathbb{R}}, \; i = 1, \ldots, n$$

(4)

We can associate to the isovector space $V^n(\hat{\mathbb{R}})$ the Affine space $A^n(\hat{\mathbb{R}})$ which as a set identified with $\hat{\mathbb{R}}^n$. Then $A^n(\hat{\mathbb{R}})$ is called real Cartesian isoaffine space of dimension $n$.

Let:

$$P_0 = (0,0,\ldots,0), \ P_1 = (0,1,0,\ldots,0), \ldots, P_n = (0,0,\ldots,1)$$

(5)

be $n+1$ points of $A^n(\hat{\mathbb{R}})$. These points form an Affine base $A^n(\hat{\mathbb{R}})$, because the vectors:

$$P_0P_1, \ P_0P_2, \ldots, P_0P_n$$

(6)

constitute of a base of $V^n(\hat{\mathbb{R}})$, which is called fundamental Affine base of $V^n(\hat{\mathbb{R}})$.

If $P$ is a point of $A^n(\hat{\mathbb{R}})$, then $P_0P \in V^n(\hat{\mathbb{R}})$ can be written:

$$P_0P = \beta_1 P_0P_1 + \beta_2 P_0P_2 + \ldots + \beta_n P_0P_n$$

(7)

where $\beta_i \in \hat{\mathbb{R}}$ are called isoaffine coordinates of point $P$ in connection with fundamental Affine base of $V^n(\hat{\mathbb{R}})$.

On the Cartesian isoaffine space $A^n(\hat{\mathbb{R}})$ we consider the functions $\hat{x}_i$, which are defined as follows:

$$\hat{x}_i : A^n(\hat{\mathbb{R}}) \rightarrow \hat{\mathbb{R}}$$

(8)

$$\hat{x}_i : P = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n) \mapsto \hat{x}_i(P) = p_i, \ i = 1, \ldots, n$$

(9)

The above functions $\hat{x}_i$ are called natural isoaffine coordinates.

### 3 Isotopological Space

On the set $\mathbb{R}^n$ we consider a topology $T$:

$$T = \{\emptyset, \mathbb{R}^n, \bigcup_{i \in I} B_i\}$$

(10)

where $B_i$ is subset of $\mathbb{R}^n$ defined as follows:

$$B_i = \{P = (p_1, p_2, \ldots, p_n) / a_i < p_i < b_i, a_i, b_i \in \mathbb{R}, i = 1, \ldots, n\}$$

(11)

and it is called open rectangle of $\mathbb{R}^n$.

The topology $T$ can be considered as the Cartesian product of topology of the open intervals of straight line n times. Topological space $(\mathbb{R}^n, T)$ is symbolized $T^n(\mathbb{R})$ and is called real Cartesian topological space.

We consider the same topology on the set $\hat{\mathbb{R}}^n$, which coincides with set $\mathbb{R}^n$. The set $\hat{\mathbb{R}}^n = \mathbb{R}^n$ with topology $T$ is called real Cartesian isotopological space and is symbolized $T^n(\hat{\mathbb{R}})$. It’s evident that $T^n(\mathbb{R}) = T^n(\hat{\mathbb{R}})$. 


Let $\mathbb{R}^n = \{(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)/\hat{x}_i = \hat{R}, i = 1, 2, \ldots, n\}$ be a real isocartesian space of dimension $n$. On this we obtain the previous structures, that is, the isovector structure, the isoaframe structure and the isotopological structure. We consider an isomapping, that is a mapping between two isosets:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f : P \mapsto f(P) = P, \quad \forall P \in \mathbb{R}^n$$

(12)

which is the identity mapping on $\mathbb{R}^n$. The set $\mathbb{R}^n$ with three above structures and isomapping $f$ is called Cartesian isomanifold of dimension $n$.

The isosubset $\tilde{U}$ of $\mathbb{R}^n$ defined as follows:

$$\tilde{U} = \{P = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)/a_i < p_i < b_i, a_i, b_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$$

(13)

is called open rectangle of $\mathbb{R}^n$.

The isosuccessor $\tilde{U}$ of $\mathbb{R}^n$ is defined as follows:

$$\tilde{U} = \{P = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)/a_i < p_i < b_i, a_i, b_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$$

(14)

of $\mathbb{R}^n$, where $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, n$ is called closet rectangle of $\mathbb{R}^n$.

The isosubset $\tilde{U}$ of $\mathbb{R}^n$ is closed, if and only if, for every point $P \in \tilde{U}$ there is an open rectangle $A$ of $\mathbb{R}^n$, such that $P \in A \subset \tilde{U}$.

The isosubset $\tilde{U}$ of $\mathbb{R}^n$ is called closed, if and only if:

$$\mathbb{R}^n - \tilde{U} = \tilde{U}^c$$

(15)

is open.

If $A$ is an isosubset of $\mathbb{R}^n$, then a point $P \in \mathbb{R}^n$ is called interior point of $\hat{A}$, if and only if, $P$ belong to open isoset $\hat{B}$ such that $\hat{B} \subset \hat{A}$.

The isoset of interior points of $\hat{A}$ symbolized $\hat{A}$ or $Int(\hat{A}) = \hat{Int}(A)$, is called interior of $\hat{A}$.

Exterior of isosubset $\hat{A}$ of $\mathbb{R}^n$, written $Ext(\hat{A})$, is the interior of the complement of $\hat{A}$, that is:

$$Ext(\hat{A}) = \hat{Ext}(A) = Int(\hat{A}^c) = \hat{Int}(A^c)$$

(16)

The isoset of points which do not belong to the interior or the exterior of $\hat{A} \subseteq \mathbb{R}^n$ is called boundary of $\hat{A}$, written $\partial \hat{A} = \hat{\partial} A$.

A collection $\hat{O} = \{\hat{A}_i\}_{i \in I}$ of open isosubsets of $\mathbb{R}^n$, is called open cover or, simply cover of $\hat{U}$, if $\tilde{U} \subseteq \mathbb{R}^n$, if and only if, $\tilde{U} = \bigcup_{i \in I} \hat{A}_i$, or equivalently, if and only if:

$$(\forall P \in \tilde{U})(\exists \hat{A}_i \in \hat{O} = \{\hat{A}_i\}_{i \in I})[P \in \hat{A}_i]$$

(17)

The isosubset $\tilde{U}$ of $\mathbb{R}^n$ is called compact, if and if, each open cover $\hat{O}$ of $\tilde{U}$ contains a finite subcollection of open isosets, which constitutes a cover of $\tilde{U}$.

If $\hat{A}$ is an isosubset of isotopological space $\hat{F}$, then intersection of closed isosubsets, which contain the isosubset $\hat{A}$, is called closure of $\hat{A}$, written $\hat{A}$.
4 Isometric Space

On the isoset $\mathbb{R}^n$ we consider isofunction $\hat{d}$:

$$\hat{d}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$$


(18)

$$\hat{d}: (X = (x_1, \ldots, x_n)), Y = (y_1, \ldots, y_n)) \rightarrow \hat{d}(X, Y) \overset{def}{=} T \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$


(19)

Isofunction $\hat{d}$, following properties:

i) $\hat{d}(X, Y) > 0 \iff X \neq Y$

ii) $\hat{d}(X, Y) = 0 \iff X = Y$

iii) $\hat{d}(X, Y) = \hat{d}(Y, X)$

iv) $\hat{d}(X, Y) \leq \hat{d}(X, Z) + \hat{d}(Z, Y)$, $\forall X, Y, Z \in \mathbb{R}^n$

is called isodistance and $\mathbb{R}^n$ with isodistance $\hat{d}$ is called Cartesian real metric isospace.

The isosubset $\hat{S}(P_0, a)$ of $\mathbb{R}^n$:

$$\hat{S}(P_0, a) = \{ P = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathbb{R}^n / \hat{d}(P, P_0) < a, a \in \mathbb{R}_+ \}$$

(24)

is called open isosphere with center $P_0 = (p_0^1, \ldots, p_0^n)$ and radius $a$.

5 Isorthogonal Space

On $\mathbb{R}^n$, equipped with isovector structure, we consider an isoinner product defined as follows:

$$<\cdot,\cdot>: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$


(25)

$$<\cdot,\cdot>: (X = (\hat{x}_1, \ldots, \hat{x}_n)), Y = (\hat{y}_1, \ldots, \hat{y}_n)) \rightarrow <X, Y> \overset{def}{=} T(\hat{x}_1\hat{y}_1 + \ldots + \hat{x}_n\hat{y}_n) = x_1y_1 + \ldots + x_ny_n$$


(26)

In this case, the attached real Cartesian Affine space $A^n(\mathbb{R})$ is called isoeuclidean real orthogonal isospace of dimension $n$ and it is symbolized $O^n$.

If $X = Y$, then we have:

$$<X : Y> = x_1^2 + \ldots + x_n^2$$


(27)

Therefore, isoinner product on $\mathbb{R}^n$ defines an isonorm $\|\cdot\|$ on $\mathbb{R}^n$, that is:

$$\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}_+$$


(28)
\[ \| \| : X = (\hat{x}_1, \ldots, \hat{x}_n) \mapsto \| \| \overset{\text{def}}{=} \sqrt{\langle X, X \rangle} = \sqrt{x_1^2 + \ldots + x_n^2} = |X| \] (29)

We accept that \( \hat{\mathbb{R}}^n \) has the isovector structure, the isoaffine structure and the isomultiplier product \( \langle \cdot, \cdot \rangle \), from which it is clear the isometric structure on \( \hat{\mathbb{R}}^n \). On this \( \hat{\mathbb{R}}^n \) we will study isofunctions of several variables.

The points \( Q_1, Q_2, \ldots, Q_n \) of iso euclidean orthogonal real isospace \( \hat{\mathbb{O}}^n = A^n(\hat{\mathbb{R}}) \) constitute an orthogonal base of \( \hat{\mathbb{O}}^n \), if and only if, the vectors \( Q_0 Q_1, Q_0 Q_2, \ldots, Q_0 Q_n \) form an orthogonal base of \( V^n(\hat{\mathbb{R}}) \).

Fundamental affine base \( \{P_0, P_1, \ldots, P_n\} \) of \( A^n(\hat{\mathbb{R}}^n) \) is an orthogonal base of \( \hat{\mathbb{O}}^n \).

6 Isoeuclidean Manifold

On the isoset \( \hat{\mathbb{R}}^n \), equipped with all the above isostructures, we consider isomapping \( f \):

\[ f : \hat{\mathbb{R}}^n \rightarrow \hat{\mathbb{R}}^n \] (30)
\[ f : P = (\hat{p}_1, \ldots, \hat{p}_n) \mapsto f(P) = P, \forall P \in \hat{\mathbb{R}}^n \] (31)

The pair \( (\hat{\mathbb{R}}^n, f) \) is called linear isomanifold of dimension \( n \) or Euclidean real isomanifold of dimension \( n \). In short the \( (\hat{\mathbb{R}}^n, f) \) we note \( \hat{\mathbb{R}}^n \).

Let \( \hat{\mathbb{R}}^n \) be the Euclidean real isomanifold its point \( P = (\hat{p}_1, \ldots, \hat{p}_n) \) we consider the following special functions \( \hat{x}_1, \ldots, \hat{x}_n \) on \( \hat{\mathbb{R}}^n \), which are defined as follows:

\[ \hat{x}_1 : \hat{\mathbb{R}}^n \rightarrow \mathbb{R}, \] (32)
\[ \hat{x}_1 : P = (\hat{p}_1, \ldots, \hat{p}_n) \mapsto \hat{x}_1(P) = \hat{x}_1(\hat{p}_1, \ldots, \hat{p}_n) \overset{\text{def}}{=} x_1(p_1, \ldots, p_n) = p_1 \]

\[ \hat{x}_n : \hat{\mathbb{R}}^n \rightarrow \mathbb{R}, \] (33)
\[ \hat{x}_n : P = (\hat{p}_1, \ldots, \hat{p}_n) \mapsto \hat{x}_n(P) = \hat{x}_n(\hat{p}_1, \ldots, \hat{p}_n) \overset{\text{def}}{=} x_n(p_1, \ldots, p_n) = p_n \]

The above functions \( \hat{x}_1, \ldots, \hat{x}_n \) are called natural cordinates isofunctions on \( \hat{\mathbb{R}}^n \), or simply, cordinates isofunctions on \( \hat{\mathbb{R}}^n \).

7 Some Basic Sets of \( \hat{\mathbb{R}}^n \)

Let \( \hat{\mathbb{R}}^n \) be the Euclidean real isomanifold and its point \( P_0 = (p_0^1, \ldots, p_0^n) \).

Its open isosubset \( \hat{\Delta} (P_0, a) \), or \( \hat{S}_a \), or \( \hat{S}(P_0, a) \) defined as follows:

\[ \hat{\Delta}(P_0, a) = \{(\hat{x}_1, \ldots, \hat{x}_n) \in \hat{\mathbb{R}}^n / (\hat{x}_1 - p_1^0)^2 + \ldots + (\hat{x}_n - p_n^0)^2 < a^2, a \in \mathbb{R}_+ \} \] (34)

is called open isosphere with center \( P_0 = (\hat{p}_1^0, \ldots, \hat{p}_n^0) \) and radius \( a \).
Elements of isoanalysis on $\tilde{\mathbb{R}}^n$

The isosubset $\tilde{\Delta} (P_0, a)$ of $\tilde{\mathbb{R}}^n$ defined as follows:

$$\tilde{\Delta} (P_0, a) = \{(\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{\mathbb{R}}^n / (\tilde{x}_1 - \tilde{p}_1^0)^2 + \ldots + (\tilde{x}_n - \tilde{p}_n^0)^2 < \tilde{a}^2, a \in \mathbb{R}_+ \}$$

is called closed isosphere with center $P_0 = (\tilde{p}_1^0, \ldots, \tilde{p}_n^0)$ and radius $a$.

The isosubset $\tilde{S}^{n-1}$ of $\mathbb{R}^n$ defined as follows:

$$\tilde{S}^{n-1} = \tilde{\Delta} (P_0, a) - \tilde{\Delta} (P_0, a) = \{(\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{\mathbb{R}}^n / (\tilde{x}_1 - \tilde{p}_1^0)^2 + \ldots + (\tilde{x}_n - \tilde{p}_n^0)^2 = \tilde{a}^2, a \in \mathbb{R}_+ \}$$

is called isosphere of dimension $n - 1$ with center $P_0 = (\tilde{p}_1^0, \ldots, \tilde{p}_n^0)$ and radius $a$.

References


Authors’ address:

Gr.Tsagas and Ath. Tsonis

Division of Mathematics
Department of Mathematics and Physics
Aristotle University of Thessaloniki
Thessaloniki 54006, Greece