NILPOTENT LIE ALGEBRAS OF MAXIMAL RANK AND OF TYPE $F_4$ AS AN ASSOCIATED GENERALIZED CARTAN MATRIX

Gr. Tsagas

Abstract

The aim of the present paper is to determined all Nilpotent Lie algebras of the maximal rank and rank $F_4$. The number of such algebras is 43.

Key words: Nilpotent Lie algebras Kac-Moody Algebra, Generalized Cartan Matrix

1 INTRODUCTION

Let $A = (A_{ij})$, $i, j = 1, \ldots, n$ be a Generalized Cartan Matrix denoted briefly by G.C.M. From this matrix and a given root system we can construct Nilpotent Lie algebras of maximal rank having $A = (A_{ij})$, $i, j = 1, \ldots, n$ as a G.C.M. In order to obtain these we consider the positive part $L^+(A)$ of the Kac-Moody Lie algebras $L(A)$ taken by the G.C.M., $A = (A_{ij})$ and the given root system $\Delta$.

The aim of the present paper is to obtain all the Nilpotent Lie algebras of maximal rank whose G.C.M. is the Cartan matrix of the exceptional Lie algebras $F_4$. Each of them is called of type $F_4$. The cases $A_n$, $B_n$, $C_n$, $D_n$ and $G_2$ have been studied in ([9]) and ([21]). The cases for $E_6$, $E_7$ and $E_8$ are studied in ([24]), ([25]) and ([26]).

The whole paper contains seven paragraphs each of them is analyzed as follows. The second paragraph gives the general theory of Kac-Moody Lie algebras. The basic elements and properties of Nilpotent Lie algebras are given in the fourth paragraph. The relation between Kac-Moody Lie algebras and Nilpotent Lie algebras is given in the fourth paragraph. The fifth paragraph contains estimates and constructions of Nilpotent Lie algebras of maximal rank and of type $F_4$. The sixth paragraph includes the determination of the ideals. The structure constants and some other properties of the fortythree Nilpotent Lie algebra of maximal rank and of type $F_4$ are included in the last paragraph.


©2001 Balkan Society of Geometers, Geometry Balkan Press
2 Kac-Moody Lie algebra

Let $A = (A_{ij})$, $i, j = 1, \ldots, n$, be a square matrix of order $n$ with entries in $\mathbb{Z}$ satisfying:

(i) $A_{ii} = 2, i = 1, \ldots, n$;

(ii) $A_{ij} \leq 0$, if $i \neq j, i, j = 1, \ldots, n$;

(iii) if $A_{ij} = 0, i \neq j$, then $A_{ij} = 0$.

$A = (A_{ij})$ is called Generalized Cartan Matrix denoted briefly by G. C. M.

All through this paper the G. C. M. will be of order $n$.

Two G. C. M. $A$ and $D$ are called equivalent if there exists $\sigma \in G_n$, where $G_n$ is the group of permutations of $\{1, \ldots, n\}$, such that:

$$B_{ij} = A_{\sigma i \sigma j}, \forall i, j = 1, \ldots, n.$$ 

We consider the Lie algebra $L(A)$, associated to the G. C. M. $A(A_{ij})$, generated by the set $\{e_1, \ldots, e_n, h_1, \ldots, h_n, f_1, \ldots, f_n\}$ satisfying:

$$[h_i, h_j] = 0, [e_i, f_j] = \delta_{ij}h_i, i, j = 1, \ldots, n, \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$[h_i, e_j] = A_{ij}e_j, [h_i, f_j] = -A_{ij}f_j, i, j = 1, \ldots, n$$

$$(ad e_i)^{-A_{ij} + 1}e_j = 0, (ad f_i)^{-A_{ij} + 1}f_j = 0, i, j = 1, \ldots, n, i \neq j.$$ 

Let $\{a_1, \ldots, a_n\}$ be the canonical base of $\mathbb{Z}^n$. If $a \in \mathbb{N}^n - \{0\}$, then $a = \sum_{i=1}^n d_i a_i$, where $d_i \in \mathbb{N}$ and at last one of them is different zero. We denote by $L_a$ (resp. $L_{-a}$) the subvector space of $L(A)$ generated by the elements $\{[e_i, \ldots, e_i]\}$ (resp. $f_i$) appears $d_i$ times and the meaning between the two brackets is the following:

$$([X_1, \ldots, X_k]) = (X_1[X_2, \ldots,])$$

We assume that if $a = \sum d_i a_i \in \mathbb{Z}^n$ and all the $d_i$s are not the same sign, then $L_a = (0)$. We denote:

$$L_0 = H = Lh_1 \oplus Kh_2 \oplus \ldots \oplus Kh_n$$

The root system of $L(A)$, denoted briefly $\Delta$, is defined by:

$$\Delta = \{a \in \mathbb{Z}^n / a \neq 0, L_a \not\equiv (0)\}$$

The Lie algebra $L(A)$, by means of $\Delta \cup \{0\}$, is grade, that means:

$$L(A) = \bigoplus_{a \in \Delta \cup \{0\}} L_a[a, L_b] \subset L_{a+b}, \forall a, b \in \Delta \cup \{0\}.$$ 

The positive root system is defined as follows:

$$\Delta_+ = \{a \in \mathbb{N}^n / a \neq 0, L_a \not\equiv (0)\}$$
The negative roots, denoted by $\Delta$, are defined by:

$$\Delta_- = -\Delta_+ = \{-a/a \in N^n, a \neq 0, \text{L}_a = (0)\}$$

It is obvious that:

$$\Delta = \Delta_- \cup \{0\} \cup \Delta_+.$$  

From the above we conclude that the Lie algebra $L(A)$ can be written:

$$L(A) = L_-(A) \oplus H \oplus L_+(A)$$

where:

$$L_-(A) = \bigoplus_{a \in \Delta_+} \text{L}_a, L_+(A) = \bigoplus_{a \in \Delta_-} \text{L}_a$$

The Lie algebra $L(A)$, associated to G. C. M., generated by:

$$\{e_1, \ldots, e_n, h_1, \ldots, h_n, f_1, \ldots, f_n\}$$

and defined above, is called Kac-Moody Lie algebra.

**Remark 1** If $a = \sum d_i a_i$, then we denote by $|a| = \sum d_i$ which is called height of $a$.

We denote by:

$$\Delta^n = \{a \in \Delta_+/|a| = k\}, \Delta_p = \{a \in \Delta_+/|a| \leq p\}$$

and therefore we have $\Delta^n_+ = \{a_1, \ldots, a_n\}$.

### 3 Nilpotent Lie algebras.

Let $g$ be a Nilpotent Lie algebra of dimension $m$ over the algebraically closed field $K$ of characteristic zero. We denote by Der$g$ and Aut$g$ the derivation Lie algebra and automorphism group of $g$ respectively.

A torus $T$ on $g$ is a commutative subalgebra of Der$g$ consisting of semi-simple endomorphisms. A torus $T$ is called maximal, if it is not contained strictly in any other torus. A torus defines a representation in $g$, that means:

$$Txg \rightarrow g, (t, x) \rightarrow tx$$

From the fact that $T$ is a commutative family of semi-simple endomorphisms and the properties of $K$, we conclude that the elements of $T$ can be diagonalized simultaneously. Therefore $g$ is decomposed into a direct sum of root spaces, that means:

$$g = \bigoplus_{\beta \in T^*} g^\beta$$

where $T^*$ is the dual of the vector space $T$ and:

$$g^\beta = \{x \in g/tx = \beta(t)x, \forall t \in T\}.$$
The root system of $g$ associated to $T$, denoted by $R(T)$, is defined by:

$$R(T) = \{ \beta \in T^*/g^\beta \neq (0) \}.$$ 

From now on we assume that $g$ is a Nilpotent Lie algebra. We suppose that $T$ is a maximal torus on $g$ and $\dim T = k$. Let $\{\beta_1, ..., \beta_k\}$ be a base of $T^*$ whose dual base of $T$ is $\{t_1, ..., t_n\}$, that means:

$$\beta_i(t_j) = \delta_{ij}.$$ 

The vectors $\{x_1, ..., x_k\}$ of $g$ with the property:

$$t_i(x_j) = \delta_{ij}x_j$$

is called $T$ minimal system of generators or briefly $T-mg$. Hence we have:

$$g^{\beta_i} = Kx_i$$

and therefore $\{x_1, ..., x_k\}$ are root vectors for $T$. $\beta_i$, $i = 1, ..., k$, is called root of $x_i$, $i = 1, ..., k$, respectively.

We have the following propositions ([21]).

**Proposition 1** If $g$ is a Nilpotent Lie algebra, then the following two statements are equivalent:

1. $\{x_1, ..., x_k\}$ is a minimal system of generators;
2. $\{x_1 + c^2g, ..., x_k + c^2g\}$ is a base for the vector space $g/c^2g$, where $c^2g = [g, g]$.

The type of $g$ is defined the dimension of $g/c^2g$.

**Proposition 2** Let $g$ be a Nilpotent Lie algebra of type (1). Let $T$ be a maximal torus on $g$, $\{x_1, ..., x_k\}$ $T - mg$ system, $\beta_i$ the root of $x_i$. The dimension of $T$ is equal to the rank of $\{\beta_1, ..., \beta_k\}$.

**Proposition 3** Let $g$ be a Nilpotent Lie algebra of type $s$. The dimension of the maximal torus $T$ on $g$ is called rank of $g$. If $k$ is the rank, then we have $k \leq s$.

### 4 Connection between Nilpotent Lie algebras and Kac-Moody Lie algebras

Let $g$ be a Nilpotent Lie algebra of type $n$ which is the dimension of the Lie algebra $g/c^2g$. If the rank of $g$ is $n$, then $g$ is called maximal rank.

Let $g$ be a Nilpotent Lie algebra. Let $T$ be a maximal torus on $g$. Let $(x_1, ..., x_n)$ be a $T-mg$, for those elements we have:

$$(adx_i)^{-A_{ij}+1} x_j = 0, \text{ then } A_{ij} \in \mathbb{Z}_+ \cup \{0\}.$$ 

If we put $A_{ij} = 2$, then using $A_{ij}$ $i = 1j$ from the above we have the matrix:

$$A = (A_{ij})$$
with the properties:

1. $A_{ij} = 2$, $i = 1, \ldots, n$

2. $A_{ij} \leq 0$, $i, j = 1, \ldots, n$, $i \neq j$.

3. If $A_{ij} = 0$, then $A_{ij} = 0$, when $i \neq j, i, j = 1, \ldots, n$.

This is the G. C. M. associated to $g$.

Let $A = A_{ij}$ be a G. C. M. we assume that the of positive roots $\Delta_+$ are given, whose number is finite, that means:

$$\Delta_+ = \left\{ a^v = \sum_{i=1}^{n} d_i^v a_i / d_i^v \in \mathbb{N} \right\}, \{a_1, \ldots, a_n\} \text{ base of } \mathbb{Z}^n$$

the number of roots $a^v$, $v = 1, \ldots, m$, is finite.

We denote by $L_+(A)$ the positive part of the Kac- Moody Lie algebra $L(A)$ assoicated to $A$ and $\Delta_+$. Therefore $L_+(A)$ is a Lie algebra generated by $\{e_1, \ldots, e_n\}$ satisfying only the relations:

1. $(ade_i)^{-A_{ij}+1} e_j = 0$, $i \neq j, i = 1, \ldots, n$

2. $L_+(A)$ is grated by: $L_+(A) = \oplus_{a \in \Delta_+} L_a, [L_a, L_\beta] \subset L_{a+\beta}, \forall a, \beta \in \Delta_+$

We refer the following propositions ([21]):

**Proposition 4** Let $L_+(A)$ be the positive part of the Kac- Moody Lie algebra $L(A)$ associated to G. C. M. $A$. Then we have:

$$C^m L_+(A) = \oplus_{|a| \geq m} L_a,$$

where $C^m L_+(A)$ is the nth. term of central descending series.

**Proposition 5** Let $\Delta_+$ be the set of positive roots of the Kac- Moody Lie algebra $L(A)$ associated to the G. C. M. $A = (A_{ij})$, $i, j = 1, \ldots, n$. Then for all $a \in \Delta_+ - \{a_1, \ldots, a_n\}$, there exists $i \in \{1, \ldots, n\}$ such that $a - a_i \in \Delta_+$.

**Proposition 6** Let $\Delta_+$ be the set of positive roots of the Kac- Moody Lie algebra $L(A)$ associated to the G. C. M. $A = (A_{ij})$, $i, j = 1, \ldots, n$. If $\Delta^p_+ = \{a \in \Delta_+ / |a| = p\} = \emptyset$ for some $p \in \mathbb{N}^*$, then $\Delta^{p+m}_+ = \emptyset$ for all $m \in \mathbb{N}^*$.

**Proposition 7** Let $L(A)$ be the Kac- Moody Lie algebra associated to the G. C. M $A = (A_{ij})$, $i, j = 1, \ldots, n$, then we have:

$$L(A) = \oplus_{a \in \Delta_+ [0]} L_a.$$

If $\{a_1, \ldots, a_n\}$ is the natural base of $\mathbb{Z}^n$, then we have:

$$L_{a+i a_i} = K(a a_i) a_j$$

We consider the conditions $H_1$ and $H_2$ for the $p \in \mathbb{N}^*$. 
In order to construct the Nilpotent Lie algebras \( m_p(A) \) from the positive part \( L_+(A) \) of the Kac-Moody Lie algebra \( L(A) \) associated to the G. C. M. \( A = (A_{ij}), \) \( i, j = 1, \ldots, n, \) then the \( p \in N^* \) satisfies the inequalities:

\[
H_1: \text{The number } p \leq p_A, \text{ where } p_A \text{ is the height of the highest root of } L_+(A)
\]

\[
H_2: p \geq \text{Sup}\{-A_{ij} + 1/i, j = 1, \ldots, n\}
\]

Now, we can state the basic theorem ([21]).

**Theorem 8** We consider the quotient Lie algebra:

\[
m = m_p(A) = L_+(A)/C^{p+1}L_+(A), \quad p > 1
\]

\[
\mu : L_+(A) \rightarrow m_p(A), \mu : x \rightarrow \mu(x) = \overline{x}, \text{ where } \mu \text{ is the canonical map.}
\]

The following are valid:

1. The restriction of \( \mu \) to the vector space \( L_a, \) such that \( |a| \leq p, \) is an isomorphism from \( L_a \) into \( I_a \) and \( m_p(A) \) is graded by:

\[
\{ a \in \Delta_+ : |a| \leq p \} : m_p(A) = \bigoplus_{|a| \leq p} I_a, \quad [I_a, I_\beta] \subset I_{a+\beta}
\]

2. The Lie algebra \( m_p(A) \) is Nilpotent and its \( \rho \) is obtained from hypothesis \( H_1. \)

3. The set \( \{\overline{a_1}, \ldots, \overline{a_n}\} \) is a minimal system of generators of \( m_p(A). \)

4. Let \( t_i \in \text{Der}(m_p(A)), i = 1, \ldots, n, \) be \( n \) derivations on \( m_p(A) \) defined by:

\[
t_i\overline{a}_j = \delta_{ij}\overline{a}_n, \text{ then } T = \bigoplus_{i=1} t_i Kt_1
\]

is a maximal torus on \( m_p(A) \) and the Nilpotent Lie algebra \( m_p(A) \) is of maximal rank. Furthermore \( \{\overline{a}_1, \ldots, \overline{a}_n\} \) is a \( T - \text{msg}. \)

5. Let \( (t_1^{-1}, \ldots, t_n^{-1}) \) be the dual basis of \( (t_1, \ldots, t_1). \) If we identify \( t^a \) and \( a_i, \)

\[
i = 1, \ldots, 1, \text{ then the root space decomposition of } m_p(A) \text{ with respect to } T \text{ is identical to the decomposition}
\]

\[
m_p(A) = \bigoplus_{a \in \Delta_+ \atop |a| \leq p} I_a.
\]

6. Under the hypothesis \( H_2: A = (A_{ij}), i, j = 1, \ldots, 1, \) is a G.C.M. associated to \( m_p(A) \) and \( (\overline{a}_1, \ldots, \overline{a}_n) \) is order relative to \( A = (A_{ij}). \)

Now, we can obtain from \( L_+(A) \) the following Nilpotent Lie algebras

\[
m_p(A) = L_+(A)/C^{p+1}L_+(A)
\]

where \( p_0 = \{\text{Sup } - A_{ij} + 1/i, j = 1, \ldots, n\} \leq p \leq p_A, p_A \text{ the height of the highest root.} \)

The number of these Nilpotent Lie algebras of maximal rank of Nilpotent \( p \) and type \( n, \) that means having \( A = (A_{ij}) \) a G.C.M., is:

\[
p_A - p_0 + 1.
\]
These Nilpotency Lie algebras are the following:
\[ L_+(A)/C^{p_0+1}(L_+(A)), L_+(A)/C^{p_0+2}(L_+(A)), \ldots, L_+(A)/C^{p_{A+1}}L_+(A) \].

It can be easily proved the following proposition.

**Proposition 9** Let \( g \) be a Nilpotent Lie algebra of finite dimension over an algebraically closed field \( K \) of characteristic zero. If \( v \) is an ideal of \( g \), then the quotient Lie algebra \( g/v \) is a Nilpotent.

Let \( \beta \) be an ideal of the Nilpotent Lie algebras \( m_p(A) \), where \( p \) satisfies the conditions \( H_1 \) and \( H_2 \). We consider the Lie algebra:
\[ g = m_p(A)/\beta \] and \( \pi : m_p(A) \to g \),
where \( \pi \) is the canonical map and the Nilpotency of \( gm \) is less than \( p \).

We have the following propositions [{2}].

**Proposition 10** Let \( \beta \) be an ideal of \( m_p(A) \). From these we obtain the Nilpotent Lie algebra
\[ g = m_p(A)/\beta \] and \( \pi : m_p(A) \to g \). The following statements are equivalent:

- (I) \( \beta \subset C^2m_p(A) \)
- (II) \((\pi \tau_1, \ldots, \pi \tau_n)\) is a minimal system of generators of \( g \).

**Proposition 11** Let \( \beta \) be an ideal of \( m_p(A) = L_+(A)/C^{p+1}L_+(A) \) contained in \( C^2m_p(A) \). Let \( T \) be the maximal torus on \( m_p(A) \). Then we have:

- (I) For any \( t \in T \) there exists \( \pi t \in Derg \) such that:
  \[ \pi o t = \pi o \pi(t) : m_p(A) \to g \] : Comutive diagram,
  where \( \pi : Der m_p(A) \to Der(g) \).
- (II) The Nilpotent Lie algebra \( g \) is of maximal rank with \( \pi(T) \) as maximal torus and \((\pi \tau_1, \ldots, \pi \tau_n)\) is a \( \pi(T) - msg \).
- (III) If \((y_1, \ldots, y_n)\) is a \( T - msg \) of \( g \), then there exists a unique \( T - msg \ (x_1, \ldots, x_n) \) of \( m_p(A) \) such that \( \pi x_i = y_i, i = 1, \ldots, n \). We must notice that \( \beta \) is called maximal ideal of \( m_p(A) \), if and only if, is \( T - invariant \), that is:
  \[ t \in T \] 

**Proposition 12** Let \( \beta \) be the homogeneous ideal of \( m_p(A) \). Then \( g = m_p(A)/\beta \) is a maximal rank and having \( A = (A_{ij}) \), \( i, j = 1, \ldots, n \), as the G.C.M. , if:
\[ (ade_i)_{-A_{ij}e_j} \notin b \forall i, j = 1, \ldots, n \ text{ and } i = 1, j \]

**Proposition 13** Let \( g = m_p(A)/\beta \) be the quotient Lie algebra, where \( \beta \) is a maximal ideal of \( m_p(A) \). Then the following statements are equivalent:

- (I) \( g \) is Nilpotency \( p \)
- (II) \( C^p m_p(A) \notin \beta \).
Nilpotent Lie algebras of maximal rank

Let $A = (A_{ij})$, $i, j = 1, \ldots, n$ be a G.C.M. The group

$$G_n(A) = \{ \sigma \in G_n / A_{\sigma_1 \sigma j} = A_{ij}, \forall i, j = 1, \ldots, n \} \quad (4.1)$$

is called automorphism group of $A = (A_{ij})$.

**Proposition 14** Let $m_p(A)$ be the Nilpotent Lie algebra defined above. There exists $\sigma \in \text{Aut } m_p(A)$ with the property $\sigma e_i = e_{\sigma i}$, $\forall i = 1, \ldots, n$, if, and only if, $\sigma \in G_1(A)$.

Now, we define:

$$G_1(A) = \{ \sigma \in \text{Aut } m_p(A) / \sigma e_i = e_{\sigma i}, \forall i = 1, \ldots, n, \exists \sigma \in G_1(A) \} \quad (4.2)$$

We also define:

$$J = J_p(A) = \{ \beta \text{ homogeneous ideas of } m_p(A)/C^p m_p(A) \not\subset \beta, (\text{ad } e_i)^{-A_{ij}} e_j \notin \beta, \forall i, j = 1, \ldots, n, i \neq j \} \quad (4.3)$$

**Proposition 15** The set $J_p(A)$ is stable under the action of $G_1(A)$.

**Proposition 16** Let $m_p(A)$ be the Nilpotent Lie algebra defined above. Let $g$ be a Nilpotent Lie algebra of maximal rank, of Nilpotency $p$ such that $A = (A_{ij})$, $i, j = 1, \ldots, n$, is an associated G.C.M. Then we have:

I) There exists $\beta \in J$ such that $g \cong m_p(A) \beta$.

II) If $\beta' \in J$ such that $g \cong m_p(A) \beta'$, then there exists $\sigma \in G_n(A)$ such that $\sigma \beta = \beta'$.

**Theorem 17** Let $m_p(A)/C^{p+1} L_+(A)$ be the Nilpotent Lie algebra given in theorem 4.5 $J_p(A)$ is the set of homogeneous ideals of $m_p(A)$ defined by (4.3). Then the isomorphism classes of Nilpotent Lie algebras of maximal rank, of Nilpotency $p$ such that $A = (A_{ij})$, $i, j = 1, \ldots, n$ is an associated G.C.M. are in bijection with the orbits of $J_p(A)$ under the action of $G_n(A)$ defined by (3.2).

From this theorem we conclude that the construction of Nilpotent Lie algebras of maximal rank, of Nilpotency $p$ and such that $A = (A_{ij})$, $i, j = 1, \ldots, n$, is a G.C.M., is the following:

We determine all the ideals of $m_p(A)$ which are stable under the action of $G_n(A)$. If the number of these ideals is equal to $\lambda(p)$, then we obtain $\lambda(p)$ Nilpotent Lie algebras with these properties. Since there exist:

$$p_0, p_0 + 1, \ldots, p_0 + (p_A - p_0) = p_A,$$

we conclude that the number of Nilpotent Lie algebras of maximal rank and of type:

$$g \text{ is } \sum_{p=p_0}^{p_A} \lambda(p)$$

Because the determination of the ideas of \( m_p(A) \), with the properties defined in (4.3), is difficult for this reason we reduce this problem to study a similar notion in
\[
\Delta_p = \{ a \in \Delta \in + \mid |a| \leq p \}.
\]

Now, we explain this theory.

Let \( \beta \) be a homogeneous ideal of \( m_p(A) \). Then we have:
\[
\beta = \bigoplus_{a \in \Delta_p} \beta \cap I_a \beta \cap I_a
\]
(4.4)

Since we have:
\[
\beta \cap I_a = \begin{cases} (0) \\ I_a \end{cases}
\]
(4.5)

we conclude that:
\[
\beta = \bigoplus_{a \in \Delta_p(\beta)} I_a
\]
(4.6)

where:
\[
\Delta_p(\beta) = \{ a \in \Delta / I_a \neq (0) \}
\]
(4.7)

we have the following:

(1) \( C^p m_p(A) \not\subset \beta \iff \Delta^p_+ \subset \Delta_p(\beta) \)  
(II) \( (ad a_i)^{-A_i} a_j \not\subset \beta \iff a_j - A_{ij} a_i \in \Delta_p(\beta) \)  

Let \( E \) be a subset of \( \Delta_p \). \( E \) is called ideal of \( \Delta_p \), if for all \( a \in E \) and some \( a_i, i = 1, \ldots, n \), such that \( a + a_i \in \Delta_p \) we have \( a + a_i \in E \). Therefore, \( \beta \) is an ideal of \( m_p(A) \), if, and only if, \( \Delta_p(\beta) \) is an ideal of \( \Delta_p \).

Now, we define:
\[
\Delta_p(A) = \{ E \text{ ideal of } \Delta_p / \Delta^p_+ \subset E \text{ and } a_j - A_{ij} a_i \not\subset E \} 
\]
(4.10)

From (4.3) and (4.10) we obtain the mapping:
\[
\psi : J_p(A) \to \Delta_p(A), \psi : \beta \to \Delta_p(\beta)
\]
(4.11)

which is a bijection with inverse:
\[
\psi^{-1} : \Delta_p(A) \to J_p(A), \psi^{-1} : E \to \beta_E = \bigoplus_{a \in E} I_a
\]
(4.12)

The group \( G_n(A) \) operates on \( \Delta_p \) by:
\[
\sigma(\Sigma a_i) = \Sigma a_i \sigma_i
\]
(4.13)

We define the following sets:
\[
\mathcal{J}_p(A) = \{ \text{set of orbits from the action } G_n(A) \text{ on } J_p(A) \}
\]
(4.14)

\[
\mathcal{N}_p(A) = \{ \text{isomorphism classes of Nilpotent Lie algebras of maximal rank, of Nilpotency} \ p \text{ such that} \ A = (A_{ij}), i, j = 1, \ldots, n, \text{is an associated G.C.M.} \}
\]
(4.15)

From the above we have the theorem ([2.1]).
Nilpotent Lie algebras of maximal rank

Theorem 18 If \( A = (A_{ij}) \), \( i, j = 1, \ldots, n \), is a G.C.M. and if \( p \) satisfies \( H_1 \) and \( H_2 \), then the \( G_n(A) \)-orbits of \( \mathcal{J}_p(A) \) classify canonically the elements of \( \mathcal{N}_p(A) \). More precisely, the map

\[
\Phi : \mathcal{N}_p(A) \to \mathcal{J}_p(A), \quad \Phi : \mathcal{J}_p(A) \to G_n(A) \quad \Delta_p(\beta)
\]

is bijection and

\[
\Phi^{-1} : \mathcal{J}_p(A) \to \mathcal{N}_p(A), \quad \Phi^{-1} : G_n(A) \to m_p(A)\beta_E
\]

is a inverse.

Therefore in order to find the Nilpotent Lie algebras of maximal rank, of Nilpotency \( p \) such that \( A = (A_{ij}) \), \( i, j = 1, \ldots, n \), is an associated G.C.M. we estimate the elements of \( \mathcal{J}_p(A) \).

5 Constructions of Lie algebras by means of \( F_4 \)

Now, we consider the Cartan matrix \( F_4 \) of the exceptional Lie algebras denoted also \( F_4 \). Therefore \( F_4 \) has the form

\[
F_4 = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\]

The positive root system \( \Delta_+ \) of \( F_4 \) is following:

\[
\Delta_+ = \{ a_1, a_2, a_3, a_4, a_1 + a_2, a_2 + a_3, a_3 + a_4, a_1 + a_2 + a_3, a_2 + 2a_3, a_1 + 2a_2 + a_4, a_1 + 2a_2 + 2a_3 + a_4, a_1 + a_2 + a_3 + 2a_4, a_1 + a_2 + a_3 + 2a_4 + a_4, a_1 + 2a_2 + 2a_3 + 2a_4 + a_4, \}
\]

These roots, if we use the canonical base \( \{ a_1 = (1, 0, 0, 0), a_2 = (0, 1, 0, 0), a_3 = (0, 0, 1, 0), a_4 = (0, 0, 0, 1) \} \) of \( Z^4 \), can be written

\[
a_1 = X_1 = (1, 0, 0, 0), \quad a_2 = X_2 = (1, 0, 0, 0), \quad a_3 = X_3 = (0, 1, 0, 0), \quad a_4 = X_4 = (1, 0, 0, 0), \quad a_1 + a_2 = X_5 = (1, 1, 0, 0), \quad a_2 + a_3 = X_6 = (0, 1, 1, 0),
\]
In order to construct the ideals of $m_p(F_4)$, where $p$ satisfies the conditions $H_1$ and $H_2$ or to construct the ideals of $\Delta_p$ and having the properties of (4.10) we must write the root system explicitly using the relation $\beta = a_i$, it means, that there exists one element $a_i$, $i = 1, 2, 3, 4$, of the base $\{a_1, a_2, a_3, a_4\}$ of $Z^4$ such that

$$\beta = a_i,$$

where $i$ one of $\{1, 2, 3, 4\}$

**Proposition 19** Let $\Delta_4$ be the root system defined by (5.2) or equivalently by (5.3). Write $\Delta_4$ using the relation (5.5).

**Proof:** From the form of the root system of $\Delta_+$ we obtain the following diagram:

![Diagram](Figure 5.1)
This figure allows us to construct the ideals of \( \Delta_p \), where \( p \) takes the values defined by the conditions \( H_1 \) and \( H_2 \). It is known:

\[
H_1 : \text{that } p \leq p_{F_4} \\
H_2 : \text{Sup}(\mathbf{A}_{ij} + 1/i, j = 1, \ldots, 1) \leq p_{F_4}
\]

(5.6)

Where \( p_{F_4} \) the height of the highest root. From (5.1) and (5.3) we conclude that:

\[
3 \leq p \leq 11
\]

(5.7)

Therefore we can construct nine Nilpotent Lie algebras \( m_p(F_4) \), which are:

\[
m_3(F_4), m_4(F_4), m_5(F_4), m_6(F_4), m_7(F_4), m_8(F_4), m_9(F_4), m_{10}(F_4) \text{ and } m_{11}(F_4),
\]

(5.8)

**Proposition 20** Let \( F_4 \) be the G.C.M. defined by (5.1). Describe the Nilpotent Lie algebras \( m_p(F_4) \), \( p = 3, \ldots, 11 \).

**Proof:** The Nilpotent Lie algebra \( m_3(F_4) \) is given by

\[
m_3(F_4) = L_+(F_4)/C^4(L_+(F_4))
\]

and by means of the theorem 4.5 takes the form:

\[
m_3(F_4) = \bigoplus_{|a| \leq 3} I_a = KX_1 \oplus KX_2 \oplus KX_3 \oplus KX_4 \oplus KX_5 \oplus KX_6 \oplus KX_7 \oplus KX_8 \oplus KX_9 \oplus KX_{10} = \bigoplus_{i=1}^{10} K_{X_i} \text{ whose dimension is 10, that is } \dim m_3(F_4) = 10.
\]

The Nilpotent Lie algebra \( m_4(F_4) \) is described as follows

\[
m_4(A) = \bigoplus_{i=1}^{13} K_{X_i}
\]

whose dimension is 13.

The other Nilpotent Lie algebras \( m_5(F_4), m_6(F_4), m_7(F_4), m_8(F_4), m_9(F_4), m_{10}(F_4) \) and \( m_{11}(F_4) \) have the form

\[
m_5(F_4) = \bigoplus_{i=1}^{16} K_{X_i}, m_6(F_4) = \bigoplus_{i=1}^{13=8} K_{X_i}, m_7(F_4) = \bigoplus_{i=1}^{20} K_{X_i}, m_8(F_4) = \bigoplus_{i=1}^{21} K_{X_i}, m_9(F_4) = \bigoplus_{i=1}^{22} K_{X_i}, m_{10}(F_4) = \bigoplus_{i=1}^{23} K_{X_i}, m_{11}(F_4) = \bigoplus_{i=1}^{24} K_{X_i}.
\]

The dimensions of these Nilpotent Lie algebras are:

\[
\dim(m_5(F_4)) = 16, \dim(m_6(F_4)) = 19, \dim(m_7(F_4)) = 20, \\
\dim(m_8(F_4)) = 21, \dim(m_9(F_4)) = 22, \dim(m_{10}(F_4)) = 23, \\
\dim(m_{11}(F_4)) = 24.
\]
Let \( m_\nu(F_4) \), \( v = 3, 4, \ldots, 11 \), be the Nilpotent Lie algebras. The homogeneous ideas \( \beta \) of \( m_\nu(F_4) \), \( v = 3, 4, \ldots, 11 \), which the properties described in (4.5), will give the Nilpotent Lie algebras:

\[
\{m_\nu(F_4)/\beta \text{ having properties described in (4.5) of maximal rank with } F_4 \text{ as an associated G.C.M.} \}.
\]

This problem is equivalent to determine the ideals \( E \) of

\[
\Delta_p, \ p = 3, 4, \ldots, 11,
\]

with the properties \( \Delta^p_+ \not\in E, a_j - A_{ij}a_i \not\in E. \)

**Problem 5.5** Determine the ideal \( E \) of \( \Delta_p, \ p = 3, 4, \ldots, 11 \), with the properties \( \Delta^p_+ \not\in E \) and \( a_j - A_{ij}a_i \not\in E. \)

**Solution.** Firstly, we define the basic chain \( N \) for \( F_4 \), which has the form

\[
a_j - A_{ij}a_i, \ i \neq j, \ i, j = 1, 2, 3, 4
\]

where \( A_{ij} \) are the entries of the matrix \( F_4 \) given by (5.1). After some estimates we have:

\[
N = \{a_1, a_2, a_3, a_4, a_1 + a_2, a_2 + a_3, a_3 + a_4, a_2 + 2a_3\}.
\]

Now, we consider \( \Delta_p, p = 3, 4, \ldots, 11 \)

\[
\Delta_3 = N \cup \{a_1 + a_2 + a_3, a_2 + a_3 + a_2\} = N \cup \Delta^3_+ - \{a_2 + 2a_3\} = N \cup T_3
\]

where

\[
T = \{a_1 + a_2 + a_3, a_2 + a_3 + a_4\}
\]

\[
\Delta_4 = N \cup T \cup \{a_1 + a_2 + 2a_3, a_1 + a_2 + a_3 + a_4\} = N \cup T_3 \cup \Delta^3_+ \\
\Delta_5 = N \cup T_3 \cup \Delta^4_+ \cup \{a_1 + 2a_2 + 2a_3, a_1 + a_2 + 2a_3 + a_1 + a_2 + 2a_3 + 2a_4\} = \\
N \cup T_3 \cup \Delta^4_+ \cup \Delta^5_+ \\
\Delta_6 = N \cup T_3 \cup \Delta^4_+ \cup \Delta^5_+ \cup \{a_1 + 2a_2 + 2a_3 + a_4, a_1 + a_2 + 2a_3 + 2a_4\} = \\
N \cup T_3 \cup \Delta^4_+ \cup \Delta^5_+ \cup \Delta^6_+ \\
\Delta_7 = N \cup T_3 \cup \Delta^4_+ \cup \Delta^5_+ \cup \Delta^6_+ \cup \{a_1 + 2a_2 + 3a_3 + a_4, a_1 + a_2 + 3a_3 + a_4\} = \\
a_1 + 2a_2 + 3a_3 + a_4\} = N \cup T_3 \cup \Delta^6_+
\]

\[
\Delta_8 = N \cup T_3 \cup \Delta^7_+ \cup \{a_1 + 2a_2 + 3a_3 + a_4\} = N \cup T_3 \cup \Delta^7_+ \\
\Delta_9 = N \cup T_3 \cup \Delta^8_+ \cup \{a_1 + 2a_2 + 3a_3 + a_4\} = N \cup T_3 \cup \Delta^8_+ \\
\Delta_10 = N \cup T_3 \cup \Delta^9_+ \cup \{a_1 + 2a_2 + 4a_3 + 2a_4\} = N \cup T_3 \cup \Delta^9_+ \\
\Delta_11 = N \cup T_3 \cup \Delta^{10}_+ \cup \{2a_2 + 3a_3 + 4a_3 + 2a_3\} = N \cup T_3 \cup \Delta^{10}_+. \\
\]

Gr. Tsagas
6 Calculations of the ideals

We calculate the ideals of $\Delta_v$, $v = 3, 4, \ldots, 11$, and using the notation of (5.3) we obtain

$$E_1 = \{X_9, X_{10}\}, E_2 = \{X_9\}, E_3 = \{X_{10}\}, E_4 = \{\emptyset\} \text{ for } \Delta_3.$$

We can proceed with the same method as for $\Delta_3$ for calculations the ideals of the others $\Delta_v$, $v = 4, \ldots, 11$ and take under the consideration some of these ideals, used for the quotient Lie algebras, give the same Lie algebras, then we represent only these ideals which give the non-isomorphic 43 Nilpotent Lie algebras of maximal rank and of type $F_4$.

We list now the ideals and the generators of the corresponding Lie algebra respectively.

\[
\begin{align*}
\{X_8 = (1, 1, 1, 0)\}, \\
X_{10} = (0, 1, 1, 1) &\quad \{X_1, \ldots, X_7\} F_4^1
\end{align*}
\]

\[
\begin{align*}
\{X_{10} = (0, 1, 1, 1)\} &\quad \{X_1, \ldots, X_9\} F_4^2
\end{align*}
\]

\[
\begin{align*}
\{X_8 = (1, 1, 1, 0)\} &\quad \{X_1, \ldots, X_7, X_9, X_{10}\} F_4^3
\end{align*}
\]
\{X_8 = (1,1,1,0), \\
X_{11} = (1,1,2,0), \\
X_{12} = (1,1,1,1)\} \\
\{X_1, \ldots, X_9, X_{10}, X_{13} \} F_4^4

\{X_{10} = (0,1,1,1), \\
X_{12} = (1,1,1,1), \\
X_{13} = (0,1,2,1)\} \\
\{X_1, \ldots, X_9, X_{11} \} F_4^5

\{X_{11} = (1,1,2,0), \\
X_{12} = (1,1,1,1), \\
X_{13} = (0,1,2,1)\} \\
\{X_1, \ldots, X_{10} \} F_4^6

\{X_{11} = (1,1,2,0), \\
X_{12} = (1,1,1,1)\} \\
\{X_1, \ldots, X_{10}, X_{13} \} F_4^7
\begin{align*}
\{X_{11} &= (1,1,2,0), \\
X_{13} &= (0,1,2,1)\} & \quad \{X_1, \ldots, X_{10}, X_{12}\} F^9_4 \\
\{X_{10} &= (0,1,1,1), \\
X_{11} &= (1,1,2,0), \\
X_{13} &= (1,1,1,1), \\
X_{14} &= (0,2,2,0), \\
X_{16} &= (0,1,2,2)\} & \quad \{X_1, \ldots, X_{9}, X_{12}, X_{15}\} F^9_4 \\
\{X_9 &= (1,1,1,0), \\
X_{11} &= (1,1,2,0), \\
X_{12} &= (0,1,2,1), \\
X_{15} &= (1,1,2,1), \\
X_{16} &= (0,1,2,2)\} & \quad \{X_1, \ldots, X_9, X_{12}, X_{15}\} F^{10}_4 \\
\{X_{12} &= (0,1,2,1), \\
X_{13} &= (1,1,1,1)\} & \quad \{X_1, \ldots, X_{11}, X_{14}, \ldots, X_{16}\} F^{11}_4 
\end{align*}
\( \{X_{11} = (1, 1, 2, 0), \) 
\( X_{12} = (0, 1, 2, 1), \) 
\( X_{14} = (1, 2, 2, 0), \) 
\( X_{15} = (0, 1, 2, 1) \} \)

\( \{X_{11} = (1, 1, 2, 0), \) 
\( X_{12} = (0, 1, 2, 1), \) 
\( X_{14} = (1, 2, 2, 0), \) 
\( X_{15} = (1, 1, 2, 1), \) 
\( X_{16} = (0, 1, 2, 2) \} \)

\( \{X_{12} = (1, 1, 1, 1), \) 
\( X_{13} = (0, 1, 2, 1), \) 
\( X_{15} = (1, 1, 2, 1), \) 
\( X_{16} = (0, 1, 2, 2) \} \)

\( \{X_{13} = (0, 1, 2, 1), \) 
\( X_{15} = (1, 1, 2, 1), \) 
\( X_{16} = (0, 1, 2, 2) \} \)
\begin{align*}
\{X_{13} = (0, 1, 2, 1)\} & \quad \{X_1, \ldots, X_{12}, X_{14}, \ldots, X_{16}\} F_4^{16} \\
\{X_{11} = (1, 1, 2, 0), \\
X_{14} = (1, 2, 2, 0), \\
X_{15} = (1, 1, 2, 1)\} & \quad \{X_1, \ldots, X_{10}, X_{12}, X_{13}, X_{16}\} F_4^{17} \\
\{X_{12} = (1, 1, 1, 1), \\
X_{15} = (1, 1, 2, 1), \\
X_{16} = (0, 1, 2, 2)\} & \quad \{X_1, \ldots, X_{11}, X_{13}, X_{14}\} F_4^{18} \\
\{X_{12} = (1, 1, 1, 1), \\
X_{14} = (1, 2, 2, 0), \\
X_{15} = (1, 1, 2, 1)\} & \quad \{X_1, \ldots, X_{11}, X_{13}, X_{16}\} F_4^{19}
\end{align*}
\[ \{X_{13} = (0, 1, 2, 1), \]
\[ X_{15} = (1, 1, 2, 1), \]
\[ X_{16} = (0, 1, 2, 2) \} \]

\[ \{X_{14} = (1, 2, 2, 0), \]
\[ X_{15} = (1, 1, 2, 1), \]
\[ X_{16} = (0, 1, 2, 2) \} \]

\[ \{X_{12} = (1, 1, 1, 1), \]
\[ X_{15} = (1, 1, 2, 1) \} \]

\[ \{X_{14} = (1, 2, 2, 0), \]
\[ X_{15} = (1, 1, 2, 1) \} \]

\[ \{X_{1} \ldots X_{12}, X_{14} \} F^2_{20} \]

\[ \{X_{1} \ldots X_{13} \} F^2_{21} \]

\[ \{X_{1} \ldots X_{11}, X_{13}, X_{14}, X_{16} \} F^2_{22} \]

\[ \{X_{1} \ldots X_{13}, X_{16} \} F^2_{23} \]
$\{X_{14} = (1, 2, 2, 0), \quad X_{16} = (1, 1, 2, 1)\}$

$\{X_{15} = (1, 1, 2, 1), \quad X_{16} = (0, 1, 2, 2)\}$

$\{X_{14} = (1, 2, 2, 0)\}$

$\{X_{15} = (1, 1, 2, 1)\}$

$\{X_{16} = (0, 1, 2, 2)\}$
\[ X_{14} = (1, 2, 2, 0), \quad X_{17} = (1, 2, 2, 1) \]

\[ X_{16} = (0, 1, 2, 2), \quad X_{18} = (1, 2, 2, 2) \]

\[ X_{17} = (1, 2, 2, 1), \quad X_{18} = (1, 1, 2, 2) \]

\[ X_{16} = (0, 1, 2, 2), \quad X_{18} = (1, 1, 2, 2), \quad X_{20} = (1, 2, 2, 2) \]

\[ X_{17} = (1, 2, 2, 1) \]
Nilpotent Lie algebras of maximal rank

\[ X_{18} = (1, 1, 2, 2) \]
\[ X_{20} = (1, 2, 2, 2) \]
\[ \{ X_{18} = (1, 1, 2, 2), \]
\[ X_{20} = (1, 2, 2, 2) \} \]
\[ \{ X_{19} = (1, 2, 3, 1), \]
\[ X_{20} = (1, 2, 2, 2) \} \]
\[ \{ X_{19} = (1, 2, 3, 1) \} \]
\[ \{ X_{20} = (1, 2, 2, 2) \} \]
\[ \{ X_{1}, \ldots, X_{17}, X_{19}, X_{20} \} F_{34}^{34} \]
\[ \{ X_{1}, \ldots, X_{17}, X_{19} \} F_{35}^{35} \]
\[ \{ X_{1}, \ldots, X_{18}, X_{20} \} F_{36}^{36} \]
\[ \{ X_{1}, \ldots, X_{19} \} F_{38}^{38} \]
\[ \{X_{21} = (1, 2, 3, 2)\} \]
\[ \{X_{22} = (1, 2, 4, 2)\} \]
\[ \{X_{23} = (1, 3, 4, 2)\} \]
\[ \{X_{24} = (2, 3, 4, 2)\} \]
\[ \{\emptyset\} \]
7 Elements of the Lie algebras $F_4^\lambda$

We have estimated the above 43 Nilpotent Lie algebras of maximal rank and of type $F_4$. In this section we determine the structure constants of these Lie algebras and give some other elements, which are the following.

(I) We write each $F_4^\lambda$, $\lambda = 1, \ldots, 43$, as a quotient Lie algebras that means

$$F_4^\lambda = m_a(A)/\beta$$

where $a = 3, \ldots, 11$ and $\beta$ an ideal of $m_a(a)$.

(II) We give the dimension each of the Lie algebras $F_4^\lambda$, $\lambda = 1, \ldots, 43$.

(III) We compute the sequence $v^\lambda_1, v^\lambda_2, \ldots, v^\lambda_p$

where $v^\lambda_i = \text{dim}(C_{i+1}F_4^\lambda)$ $i = 1, \ldots, p$

$p + 1$ is the nilpotency of $F_4^\lambda$, $\lambda = 1, \ldots, 43$.

(IV) Briefly we note the Lie brackets by form $[t]$ where $t$ is a positive integer which runs from 1 to 60. Therefore for $F_4$, whose Lie brackets are

$$[1] = [X_1, X_2] = -X_5, [2] = [X_2, X_3] = -X_6, [3] = [X_3, X_4] = -X_7,$$

$$[6] = [X_4, X_5] = -2X_9$$

we have its representations by

$$F_4^1 : \{[1], [3], [6]\}.$$

The Lie brackets of $F_4^1$ are valid for $F_4^2$ having two new non-zero Lie brackets denoted by

$$[7] = [X_2, X_7] = -X_{10}, [8] = [X_1, X_9] = X_{10}.$$  

Therefore $F_4^2$ is characterized by

$$F_4^2 : \{[1], \ldots, [8]\}.$$  

For thee Lie algebra $F_4^3$ some of the previous Lie brackets do not appear however for this new Lie brackets appear:


Hence $F_4^3$ is characterized by

$$F_4^3 : \{[1], \ldots, [6]\}.$$  

Therefore for each Lie algebra $F_4^\lambda$, $\lambda = 1, \ldots, 43$, we write the Lie brackets in the form:

$$\{[1], [2], \ldots\}. $$
Now, we give the list of $F^\lambda_4$, $\lambda = 1, \ldots, 43$, with all the elements which have been referred above.

$F^1_4 : F^1_4 = m_3(F_4)/L_{X_8} \oplus L_{X_{10}},$ (8, 4, 1), dim $F^1_4 = 8.$

$F^2_4 : F^2_4 = m_4(F_4)/L_{X_8}$, (9, 5, 2), dim $F^2_4 = 9$

$F^2_4 : F^2_4 = m_4(F_4)/L_{X_8}$, (9, 5, 2), dim $F^2_4 = 9$

$F^3_4 : F^3_4 = m_3(F_4)/L_{X_{10}},$ (9, 5, 2), dim $F^3_4 = 9$

$F^4_4 : F^4_4 = m_4(F_4)/L_{X_8}$, (9, 5, 2), dim $F^4_4 = 9$

$F^5_4 : F^5_4 = m_4(F_4)/L_{X_8}$, (9, 5, 2), dim $F^5_4 = 9$

$F^6_4 : F^6_4 = m_4(F_4)/L_{X_8}$, (9, 5, 2), dim $F^6_4 = 9$

$F^7_4 : F^7_4 = m_4(F_4)/L_{X_8}$, (9, 5, 2), dim $F^7_4 = 9$

$F^8_4 : F^8_4 = m_4(F_4)/L_{X_8}$, (9, 5, 2), dim $F^8_4 = 9$
Nilpotent Lie algebras of maximal rank

\[ F_{4}^{11} : F_{4}^{11} = m_{5}(F_{4})/ \oplus L_{X_{12}} \oplus L_{X_{13}}, \ (11, 7, 4, 1), \ \text{dim} \ F_{4}^{11} = 11 \]
\[-[X_{5}, X_{7}] = -[X_{4}, X_{9}] = [X_{1}, X_{10}] = X_{11} \]
\[ F_{4}^{11} : \{[1], [2], \ldots, [10] \} \]

\[ F_{4}^{12} : F_{4}^{12} = m_{5}(F_{4})/L_{X_{11}} \oplus L_{X_{12}} \oplus L_{X_{14}} \oplus L_{X_{15}}, \ (12, 8, 5, 2, 1), \ \text{dim} \ F_{4}^{12} = 12 \]
\[[24] = [X_{4}, X_{13}] = -2X_{16}, [25] = [X_{7}, X_{10}] = 2X_{16} \]
\[ F_{4}^{12} : \{[1], \ldots, [8], [14], [15], [24], [25] \} \]

\[ F_{4}^{13} : F_{4}^{13} = m_{5}(F_{4})/L_{X_{11}}, \ (12, 8, 5, 2, 1), \ \text{dim} \ F_{4}^{13} = 12 \]
\[ F_{4}^{13} : \{[1], \ldots, [8], [11], \ldots, [16] \} \]

\[ F_{4}^{14} : F_{4}^{14} = m_{5}(F_{4})/L_{X_{12}}, \ (12, 8, 5, 2), \ \text{dim} \ F_{4}^{14} = 12 \]
\[ F_{4}^{14} : \{[1], \ldots, [10], [14], \ldots, [16] \} \]

\[ F_{4}^{15} : F_{4}^{15} = m_{5}(F_{4})/L_{X_{11}} \oplus L_{X_{13}} \oplus L_{X_{15}} \oplus L_{X_{16}}, \ (12, 8, 5, 2), \ \text{dim} \ F_{4}^{15} = 12 \]
\[ F_{4}^{15} : \{[1], [10], [17], \ldots, [19] \} \]

\[ F_{4}^{16} : F_{4}^{16} = m_{5}(F_{4})/L_{X_{13}}, \ (12, 8, 5, 2), \ \text{dim} \ F_{4}^{16} = 12 \]
\[ F_{4}^{16} : \{[1], \ldots, [13] \} \]

\[ F_{4}^{17} : F_{4}^{17} = m_{5}(F_{4})/L_{X_{11}} \oplus L_{X_{14}} \oplus L_{X_{15}}, \ (13, 9, 6, 3, 1), \ \text{dim} \ F_{4}^{17} = 13 \]
\[ F_{4}^{17} : \{[1], \ldots, [8], [11], \ldots, [16], [24], [25] \} \]

\[ F_{4}^{18} : F_{4}^{18} = m_{5}(F_{4})/L_{X_{12}} \oplus L_{X_{13}} \oplus L_{X_{16}}, \ (13, 9, 6, 3, 1), \ \text{dim} \ F_{4}^{18} = 13 \]
\[ F_{4}^{18} : \{[1], \ldots, [10], [14], \ldots, [19] \} \]

\[ F_{4}^{19} : F_{4}^{19} = m_{5}(F_{4})/L_{X_{11}} \oplus L_{X_{13}} \oplus L_{X_{15}}, \ (13, 9, 6, 3, 1), \ \text{dim} \ F_{4}^{19} = 13 \]
\[ F_{4}^{19} : \{[1], [10], [14], [16], [24], [25] \} \]

\[ F_{4}^{20} : F_{4}^{20} = m_{5}(F_{4})/L_{X_{12}} \oplus L_{X_{15}} \oplus L_{X_{16}}, \ (13, 9, 6, 3, 1), \ \text{dim} \ F_{4}^{20} = 13 \]
\[ F_{4}^{20} : \{[1], \ldots, [13], [17], \ldots, [19] \} \]

\[ F_{4}^{21} : F_{4}^{21} = m_{5}(F_{4})/L_{X_{14}} \oplus L_{X_{15}} \oplus L_{X_{16}}, \ (13, 9, 6, 3), \ \text{dim} \ F_{4}^{21} = 13 \]
\[ F_{4}^{21} : \{[1], \ldots, [16] \} \]

\[ F_{4}^{22} : F_{4}^{22} = m_{5}(F_{4})/L_{X_{12}} \oplus L_{X_{13}}, \ (14, 10, 7, 4, 2), \ \text{dim} \ F_{4}^{22} = 14 \]
\[ F_{4}^{22} : \{[1], \ldots, [10], [14], \ldots, [16] \} \]

\[ F_{4}^{23} : F_{4}^{23} = m_{5}(F_{4})/L_{X_{14}} \oplus L_{X_{15}}, \ (14, 10, 7, 4, 1), \ \text{dim} \ F_{4}^{23} = 14 \]
\[ F_{4}^{23} : \{[1], \ldots, [16], [24], [25] \} \]

\[ F_{4}^{24} : F_{4}^{24} = m_{5}(F_{4})/L_{X_{14}} \oplus L_{X_{16}}, \ (14, 10, 7, 4, 1), \ \text{dim} \ F_{4}^{24} = 14 \]
\[20\] \quad [X_1, X_{13}] = -X_{15}, \quad [21] = [X_3, X_{12}] = -X_{15}, \quad [22] = [X_4, X_{11}] = X_{15},
\[23\] \quad [X_7, X_8] = X_{15}
\[F_{24}^0 : \{[1], \ldots, [16], [21], \ldots, [23]\}\]
\[F_{25}^4 : F_{25}^4 = m_5(F_4) / L_{X_{15}} \oplus L_{X_{16}}, \quad (14, 10, 7, 4, 1), \quad \dim F_{25}^4 = 15\]
\[F_{25}^5 : \{[1], \ldots, [17]\}\]
\[F_{26}^4 : F_{26}^4 = m_5(F_4) / L_{X_{14}}, \quad (15, 11, 8, 5, 2, 1), \quad \dim F_{26}^4 = 15\]
\[F_{26}^5 : \{[1], \ldots, [16], [20], \ldots, [25]\}\]
\[F_{27}^4 : F_{27}^4 = m_5(F_4) / L_{X_{15}}, \quad (15, 11, 8, 5, 2, 1), \quad \dim F_{27}^4 = 15\]
\[F_{27}^5 : \{[1], \ldots, [19], \ldots, [24], [25]\}\]
\[F_{28}^4 : F_{28}^4 = m_5(F_4) / L_{X_{16}}, \quad (15, 11, 8, 5, 2), \quad \dim F_{28}^4 = 15\]
\[F_{28}^5 : \{[1], \ldots, [23]\}\]
\[F_{29}^4 : F_{29}^4 = m_6(F_4) / L_{X_{15}} \oplus L_{X_{17}}, \quad (16, 12, 9, 6, 3, 1), \quad \dim F_{29}^4 = 16\]
\[F_{30}^4 : F_{30}^4 = m_6(F_4) / L_{X_{16}} \oplus L_{X_{18}}, \quad (16, 12, 9, 6, 2), \quad \dim F_{30}^4 = 16\]
\[F_{31}^4 : F_{31}^4 = m_6(F_4) / L_{X_{17}} \oplus L_{X_{18}}, \quad (16, 12, 9, 6, 3), \quad \dim F_{31}^4 = 16\]
\[F_{32}^4 : \{[1], \ldots, [25]\}\]
\[F_{32}^4 : F_{32}^4 = m_6(F_4) / L_{X_{16}} \oplus L_{X_{18}} \oplus L_{X_{20}}, \quad (17, 13, 10, 7, 4, 3, 1), \quad \dim F_{32}^4 = 17\]
\[F_{33}^4 : F_{33}^4 = m_7(F_4) / L_{X_{17}}, \quad (17, 13, 10, 7, 4, 2, 1), \quad \dim F_{33}^4 = 17\]
\[F_{34}^4 : F_{34}^4 = m_7(F_4) / L_{X_{18}}, \quad (17, 13, 10, 7, 4, 2), \quad \dim F_{34}^4 = 17\]
\[F_{35}^4 : F_{35}^4 = m_7(F_4) / L_{X_{18}} \oplus L_{X_{20}}, \quad (18, 14, 11, 8, 5, 3, 1), \quad \dim F_{35}^4 = 18\]
\[F_{36}^4 : \{[1], \ldots, [30], [34], \ldots, [37]\}\]
Nilpotent Lie algebras of maximal rank

\[ F_4^{36} : F_4^{36} = m_7(F_4)/LX_{19} \oplus LX_{20}, \quad (18,14,11,8,5,3), \quad \dim F_4^{36} = 18 \]

\[ F_4^{37} : F_4^{37} = m_7(F_4)/LX_{19}, \quad (19,12,9,6,4,2,1), \quad \dim F_4^{37} = 19 \]

\[ [X_2, X_{18}] = -X_{20}, \quad [X_4, X_{17}] = -2X_{20}, \quad [X_5, X_{16}] = X_{20} \]

\[ [X_{10}, X_{12}] = 2X_{20} \]

\[ F_4^{38} : F_4^{38} = m_7(F_4)/LX_{21}, \quad (19,15,12,9,6,4,2), \quad \dim F_4^{38} = 19 \]

\[ F_4^{39} : F_4^{39} = m_8(F_4)/LX_{21}, \quad (20,16,13,10,7,4,2), \quad \dim F_4^{39} = 20 \]

\[ F_4^{40} : F_4^{40} = m_9(F_4)/LX_{22}, \quad (21,17,14,11,8,5,3,1), \quad \dim F_4^{40} = 20 \]

\[ [X_3, X_{20}] = X_{21}, \quad [X_4, X_{19}] = X_{21}, \quad [X_6, X_{18}] = X_{21} \]

\[ [X_7, X_{17}] = -X_{21}, \quad [X_8, X_{16}] = -X_{21}, \quad [X_{10}, X_{15}] = -X_{21} \]

\[ [X_{12}, X_{13}] = -X_{21} \]

\[ F_4^{41} : F_4^{41} = m_{10}(F_4)/LX_{23}, \quad (23,18,15,12,9,6,4,2,1), \quad \dim F_4^{41} = 22 \]

\[ [X_4, X_{21}] = -2X_{22}, \quad [X_7, X_{19}] = -2X_{22}, \quad [X_9, X_{18}] = X_{22} \]

\[ [X_{11}, X_{16}] = -X_{22}, \quad [X_{13}, X_{15}] = X_{22} \]

\[ F_4^{42} : F_4^{42} = m_{11}(F_4)/LX_{24}, \quad (23,19,16,13,10,7,5,3,2,1), \quad \dim F_4^{42} = 23 \]

\[ [X_2, X_{22}] = -2X_{23}, \quad [X_6, X_{21}] = -2X_{23}, \quad [X_9, X_{20}] = X_{23} \]

\[ [X_{10}, X_{19}] = -2X_{23}, \quad [X_{13}, X_{17}] = -2X_{23}, \quad [X_{14}, X_{16}] = -X_{23} \]

\[ F_4^{43} : F_4^{43} = m_{11}(F_4)/L\{0\}, \quad (24,20,17,14,11,8,6,4,3,2,1), \quad \dim F_4^{43} = 24 \]

\[ [X_{11}, X_{23}] = -X_{24}, \quad [X_5, X_{22}] = -X_{24}, \quad [X_6, X_{21}] = -2X_{24} \]

\[ [X_{11}, X_{20}] = X_{24}, \quad [X_{12}, X_{19}] = -2X_{24}, \quad [X_{14}, X_{18}] = -X_{24} \]

\[ [X_{14}, X_{18}] = 2X_{24} \]

\[ F_4^{43} : \{1\}, \ldots, \{66\} \]

From the above we have the following theorem.

**Theorem 21** Up to isomorphism, \( F_4^v, v = 1, \ldots, 43, \) defined above, are the only Nilpotent Lie Algebras of maximal rank with \( F_4 \) as an associated G.C.M.
References


Author’s address:

Gr. Tsagas
Division of Mathematics
Department of Mathematics and Physics
Aristotle University of Thessaloniki
Thessaloniki 54006, Greece