COMPARISON THEOREM FOR GENERALIZED HEISENBERG GROUPS

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Abstract

We establish new inequalities for the sectional curvature of generalized Heisenberg groups. Then we prove that these Lie groups are completely characterized (in the class of nilpotent ones) by the sectional curvature of 2-planes which contain one central direction.

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1 Introduction

The generalized Heisenberg groups play a role of "standard spaces" between the Lie groups endowed with invariant metrics, similar to that of symmetric spaces in global Riemannian geometry: they have large isometry groups and their geodesic symmetries preserve the volume form ([7]); Eberlein shown they are the 2-nilpotent groups characterized by the fact that any unitary geodesic is contained in at least one three-dimensional totally geodesic submanifold ([4]). As isometry groups of symmetric spaces with negative curvature, the Heisenberg groups operate transitively on the horospheres ([8]). Moreover, these groups are the elementary "bricks" for constructing solvable Lie groups with Einstein invariant metrics ([9], [12]).

On another hand, the Lie groups with invariant metrics, having sectional curvature with constant sign, are involved in remarquable comparison theorems ([1], [2], [5], [6], [9], [10]). A natural extension of the framework is to characterize the invariant geometries on Lie groups from *partial* informations concerning their sectional curvature sign; for example, J. Milnor proved ([10]) that if X is a central element in the Lie algebra of a Lie group endowed with a left invariant Riemannian metric, then the sectional curvature of 2-planes containing X is non-negative.

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In this paper, we study boundaries for the sectional curvature of generalized Heisenberg groups. The main result characterizes these Lie groups, in terms of certain central directions:

THEOREM. *Let G a nilpotent Lie group, endowed with a left invariant Riemannian metric. Suppose the sectional curvature of the 2-planes, containing one central and one non-central orthogonal directions, is constant.*

Then G is a generalized Heisenberg group.

2 Preliminaries

Let G a real Lie group, $L(G)$ its Lie algebra and g a left invariant Riemannian metric on G (i.e. the left translations are isometries).

We say G is 2-step nilpotent if $[L(G),[L(G),L(G)]]=0$; in this case, it has non-trivial center, denoted by ζ. We denote by ν the orthogonal complement of ζ. In what follows, we establish the notation: $X,Y\in\nu$; $Z,Z^*\in\zeta$. For every central element Z, we define an endomorphism J_Z of the subspace ν, by

$$g(J_ZX,Y)=g([X,Y],Z)$$

For 2-step nilpotent groups, the geometry of (G,g) is completely determined by the family of all endomorphisms J_Z ([4]).

A 2-step nilpotent group is called generalized Heisenberg group if there exists a non-null constant a such that

$$J_Z^2=-a^2\|Z\|^2 Id \tag{1}$$

If $a^2=1$, we say G is a *standard* generalized Heisenberg group.

We denote by $K(U,V)$ the sectional curvature of the 2-plane spanned by the left invariant vector fields $U,V\in L(G)$.

1. PROPOSITION - *Let (G,g) a standard generalized Heisenberg group. Then the sectional curvature is bounded between $-3/4$ and $1/2$. In particular:*

(i) $K(Z,Z^*)=0$; (ii) $K(X,Z^*)=\frac{1}{4}$; (iii) - $\frac{3}{4}\le K(X,Y)\le 0$;
(iv) - $\frac{3}{4}\le K(X,Y+Z^*)\le\frac{1}{4}$; (v) $0\le K(X+Z,Z^*)\le\frac{1}{4}$.

Proof. - Let Π a 2-plane spanned by the orthonormal system $\{X+Z,Y+Z^*\}$. Then

$$\|X\|^2+\|Z\|^2=1 \quad , \quad \|Y\|^2+\|Z^*\|^2=1 \quad , \quad g(X,Y)+g(Z,Z^*)=0$$

By direct computation, we derive ([3]) the sectional curvature of Π as

$$K(X+Z,Y+Z^*)=\frac{1}{4}(\|X\|^2\|Z^*\|^2+\|Y\|^2\|Z\|^2)+g(X,Y)g(Z,Z^*)-$$

$$-\frac{3}{4}\|[X,Y]\|^2 - \frac{3}{2}g(J_Z X, J_{Z^*} Y)$$

Using the Cauchy-Schwartz inequality and relation (2), we obtain

$$\|[X,Y]\|^2 = g(J_{[X,Y]}X, Y) \leq \|[X,Y]\| \ \|X\| \ \|Y\|$$

Denote $p = \|X\|^2$, $q = \|Y\|^2$;

$$\beta(p,q) = \frac{1}{4}(p + q - 2pq) + \frac{3}{2}\sqrt{pq(1-p)(1-q)}$$

$$\alpha(p,q) = \frac{1}{4}(4 - 3p - 3q - pq) - \frac{3}{2}\sqrt{pq(1-p)(1-q)}$$

Using all previous relations we obtain the inequalities

$$\alpha(p,q) \leq K(X + Z, Y + Z^*) \leq \beta(p,q)$$

We study the variation of functions α and β on the closed square $[0,1]^2$; we remark that α has $(-3/4)$ as minimum value, and β has $1/2$ as maximum value. So, the first part of the proposition is proved.

Properties (i) and (ii) result by replacing $X = Y = 0$ and $Y = 0, Z = 0$ respectivelly in the formula of sectional curvature. For (iii) we have

$$K(X,Y) = -\frac{3}{4}\|[X,Y]\|^2$$

and, as previously,

$$\|[X,Y]\| \leq \|X\| \ \|Y\|$$

On another hand, there exist linearly independent $X, Y \in \nu$, such that $[X,Y] = 0$. By continuity, the sectional curvature takes all values from the interval $[-\frac{3}{4}, 0]$.

The same method leads to inequalities (iv) and (v).□

3 Proof of the Theorem

Step I. Let ν the orthogonal complement of ζ in $L(G)$. By hypothesis, there exists a real constant a, such that for any orthonormal $Z \in \zeta$ and $X \in \nu$, we have $K(X,Z) = a$. By the previously quoted result of Milnor, we have $a \geq 0$.

The Levi-Civita connection of g is

$$(2) \qquad \nabla_P Q = \frac{1}{2}\{[P,Q] - (adP)^* Q - (adQ)^* P\}$$

where $(adP)^*$ is the adjoint of adP, and P, Q are arbitrary elements of $L(G)$. We denote

$$S(Z,X) = \frac{1}{2}(\nabla_Z X + \nabla_X Z)$$

A direct computation gives the sectional curvature

$$K(Z,X) = \frac{1}{2}\{g([[Z,X],Z],X) + g(Z,[X,[Z,X]])\} -$$

$$-\frac{3}{4}\|[Z,X]\|^2 + \|S(Z,X)\|^2 - g(S(Z,Z),S(X,X))$$

Taking into account that Z is central, we derive

$$K(Z,X) = \|S(Z,X)\|^2 - g(S(Z,Z),S(X,X)) \tag{3}$$

Step II. From (2) we remark that, for the central element Z, we obtain $\nabla_Z Z = 0$, so $S(Z,Z) = 0$. In the same manner we derive $\nabla_Z X = \nabla_X Z$, so $S(Z,X) = \nabla_Z X$. Then, the relation (3) implies $\|\nabla_Z X\|^2 = a$. By linearity, we deduce that, for all $Z \in \zeta$ and $X \in \nu$,

$$\|\nabla_Z X\|^2 = a\|Z\|^2 \ \ \|X\|^2 \tag{4}$$

Step III. We polarise (4), we use (2) and we obtain the relation (1) (with a constant equal to $4a$). As a by product, we have also

$$\|ad^*_X Z\|^2 = a \tag{5}$$

for unitary X, Z as above. If $a = 0$, then ν is a nilpotent ideal of $L(G)$ and $L(G)$ is an orthogonal sum of ν with ζ. But the center of ν is non-trivial, hence the center of $L(G)$ strictly contains ζ, which contradicts the choice of ζ. We deduce $a > 0$. If the group G is 2-step nilpotent, then the relation (1) ensures us that G is even a generalized Heisenberg group.

Suppose now G is k-nilpotent, with $k > 2$. If $dim\zeta \geq 2$, then there exist $X, Y \in \nu$ and $Z \in \zeta$ such that $[X,Y] \perp Z$. Then (5) leads to $a = 0$, contradiction.

If $dim\zeta = 1$, using the nilpotency of G we find non-vanishing $X, Y \in \nu$ with $[X,Y] = 0$. Again from (5) we derive $a = 0$, contradiction. □

4 Comments and further results

(i) In (iii), (iv) and (v) of Proposition 1, the intervals are fully filled, due to the continuity of the curvature. We don't know if the upper bound $1/2$ is effectively attained by the sectional curvature, or this estimation may be improved (somewhere toward $1/4$).

(ii) The condition " G nilpotent" is necessary in the hypothesis of the theorem. Indeed, consider a 3-dimensional resoluble Lie group with a left invariant Riemannian metric and an orthonormal basis of left invariant vector fields X, Y, Z, such that

$[X, Y] = X + Z$, $[X, Z] = [Y, Z] = 0$. Then the center is spanned by Z, we have $J_Z^2 = -I$ and $K(U, Z) = \frac{1}{4}$ for every $U \perp Z$. Our theorem do not apply, because G is not nilpotent.

(iii) The theorem remains no longer valid for pinched curvature; indeed, let a positive constant a and $\epsilon \in (0, a)$. As in [4], I, p. 649, we can construct 2-step nilpotents groups, which are not generalized Heisenberg groups, and which have (non-constant !) sectional curvature $K(X, Z) \in (a - \epsilon, a + \epsilon)$, for every $Z \in \zeta$ and $X \in \nu$.

(iv) Let G a Lie group with $dim\zeta \geq 2$. Then, for every $Z, Z^* \in \zeta$, we have $K(Z, Z^*) = 0$; so, the behaviour of the sectional curvature for "central" 2-planes leads not to a result similar to our theorem. In exchange, from relation (4) we derive the

COROLLARY. *Let G a 2-step nilpotent Lie group with $dim\zeta \geq 2$ and let a fixed $Z^* \in L(G)$. If $K(Z, Z^*) = 0$, for every $Z \in \zeta$, then Z^* is a central element.*

Counterexamples exist which show that none of the other properties (iii)-(v) of Proposition 1 characterizes (alone) the generalized Heisenberg groups.

(v) A variation of the proof of the Theorem leads to a partial refinement for the caracterization of Lie groups with flat left invariant Riemannian metric ([5], [10], [2]):

PROPOSITION. *Let G a Lie group with non-trivial center, endowed with a left invariant Riemannian metric. Suppose that the sectional curvature of any 2-plane, spanned by one central and one non-central orthogonal directions, vanishes. Then G is a semi-direct product, with one factor a distinguished abelian subgroup.*

(vi) If in the hypothesis of the Theorem we impose the constant be $a = \frac{1}{4}$, then G becomes a standard generalized Heisenberg group. The important role played by this constant (see also Proposition 1) is similar to that from the "sphere" (comparison) theorems. This unexpected link strengthens the suspicion that the behaviour of the sectional curvature is directed by some "universal" constants.

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