ELIMINATION OF CONCEALED ISOMORPHIC LIE ALGEBRAS FROM NILPOTENT LIE ALGEBRAS OF DIMENSION EIGHT WHOSE MAXIMUM ABELIAN IDEAL IS OF DIMENSION FIVE

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Abstract

The classification of Nilpotent Lie Algebras of dimension eight whose maximum abelian ideal is of dimension five has been achieved, resulting in 63 basic Lie Algebras non-isomorphic each to other. These Lie Algebras have a various number of parameters. All the possible combinations of values of these parameters for each Lie Algebra end up with 6860 different Nilpotent Lie Algebras. After applying a proper algorithm, it results that 403 of them have a maximum abelian ideal greater than five and, among the remaining 6457, there are 35 that are isomorphic each to other. Therefore, we conclude that the non-isomorphic Nilpotent Lie Algebras of dimension eight whose maximum abelian ideal is of dimension five are 6422.

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Key words: Solvable, nilpotent algebras, Algebraic manipulation, Classification of Nilpotent Lie Algebras of dimension nine, Abelian ideal and the classification problem.

1 Introduction

The main problem about Lie algebras is their classification. According to Levi's theorem there are three main categories of Lie algebras, that is: simple, solvable and semidirect sums of solvable and semisimple algebras.

E.Cartan has achieved the classification of simple Lie Algebras. The classification of solvable Lie algebras depends on the classification of Nilpotent Lie algebras, which has been completed so far up to dimension eight.

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The classification of Abelian Nilpotent Lie Algebras of dimension eight whose maximum abelian ideal is of dimension five consists of 63 basic Lie Algebras. All of them have a various number of parameters. To obtain all the relative Lie Algebras, each parameter in turn becomes zero or remains unchanged. Therefore, from a basic Lie Algebra with k parameters, 2^k different Lie Algebras are formed. In the present case the final amount of different Abelian Nilpotent Lie Algebras is 6860. In order to achieve all these combinations of values, several original computer programs have been created. The aim of this paper is to present the algorithm used in order to examine if there are any isomorphic Nilpotent Lie Algebras left among them.

2 Nilpotent Lie Algebras

Let us now consider g to be a Nilpotent Lie Algebra over a field K of characteristic zero whose dimension is eight. Let also g_0 be the maximum abelian ideal of g whose dimension is five. We assume $\{e_1, e_2, \ldots, e_8\}$ to be a basis of g such that $\{e_4, e_5, \ldots, e_8\}$ is a basis of g_0.

In the case under consideration the process results in 63 non-isomorphic Nilpotent Lie Algebras of this type. It turns out that:

There is 1 non-isomorphic Nilpotent Lie algebra with 11 parameters different than 1.

There is 1 non-isomorphic Nilpotent Lie algebra with 9 parameters different than 1.

There are 5 non-isomorphic Nilpotent Lie algebras with 8 parameters different than 1.

There are 9 non-isomorphic Nilpotent Lie algebras with 7 parameters different than 1.

There are 19 non-isomorphic Nilpotent Lie algebras with 6 parameters different than 1.

There are 15 non-isomorphic Nilpotent Lie algebras with 5 parameters different than 1.

There are 9 non-isomorphic Nilpotent Lie algebras with 4 parameters different than 1.

There are 3 non-isomorphic Nilpotent Lie algebras with 3 parameters different than 1.

There is 1 non-isomorphic Nilpotent Lie algebra with 2 parameters different than 1.

3 Algorithm

Since isomorphic Lie Algebras are considered as a single case, the algorithm used at this stage eliminates isomorphic cases.

ALGORITHMIC DESCRIPTION OF THE PROCEDURE IN MATHEMATICS

comment Consider the set Q of all Nilpotent Lie Algebras of dimension eight, whose maximum abelian ideal is of dimension five.

procedure SPLIT;
  begin
    while not end of **Q do**
      begin
1. Read all non-zero Lie brackets $[e_i, e_j]$ (i= 1,2,3 and j =2,3,...,8 that form a Nilpotent Lie Algebra.
2. **if** the maximum abelian ideal of it has dimension five
   then begin
3. Consider all the non-zero parameters contained in this algebra.
4. Assume that their amount is "k" and set a counter $\leftarrow$ 1.
   while *counter* $\leq 2^k$ **do**
     begin
5. Perform a new combination of values of these parameters, giving to them the value zero or keeping them unchanged.
6. Count the amount of the remaining non-zero Lie brackets
7. **if** there are already Lie Algebras with the same amount of non-zero Lie brackets
   then begin
8. Call the procedure CHECK_ISO to examine if the current Lie Algebra is isomorphic to any previous one belonging in the same group
9. **if** it is isomorphic **then** it is rejected **else** it is appended to the same group
   end
10. **else** put this Lie Algebra in a new file.
11. Set *counter* $\leftarrow$ *counter* +1
    end (* while *counter* $\leq 2^k$ *)
  end (* if the maximum ...*)
12. **else** Reject the Nilpotent Lie Algebra under consideration
  end (* while not end of Q do*)
end (* procedure SPLIT *)

To examine whether the Lie Algebra in question is isomorphic to any previous one of the same group, a special technique is applied, the main steps of which are described in the procedure CHECK_ISO. The non-zero Lie brackets $[e_i, e_j]$ (i= 1,2,3 and j =2,3,...,8, which form a Nilpotent Lie Algebra have the general form:

$$[e_i, e_j] = ae_4 + be_5 + ce_6 + de_7 + fe_8 \tag{3.1}$$

Assuming that, so far, the number of non-isomorphic Lie Algebras belonging to the same group is $P >= 1$ and a counter, namely *count*, is used, the procedure CHECK_ISO is as follows:

procedure CHECK_ISO;
  begin
1. Set *count* $\leftarrow$ 1
   while *count* <= P **do**

begin
2. Read the algebra with serial number "*count*" from the group
3. Calculate the amount of occurrences of e_i, $(\forall) i = 1, 2, \ldots, 7$ inside the Lie brackets $[e_i, e_j]$ of relations such as (3.1).
4. Calculate the amount of occurrences of e_i ,(∀ i=4,...,8 at the right side of relations such as (3.1)
5. Sort in descending order the above two sequences of numbers.
6. Perform steps 3,4 and 5 to the algebra in question.
7. **if** the two sequences of numbers that have derived from an algebra with serial number "*count*" and from the current algebra are NOT the same

then begin
8. it means that this algebra is non-isomorphic to the previous one
9. Set *count* ← *count* +1

end

else begin
10. Mark second algebra as "similar" to the algebra with serial number "*count*"
11. Calculate the amount of occurrences of e_i, $\forall i = 1, 2, \ldots 7$, on both sides of relations such as (3.1).
12. **if** among e_i, $\forall i = 1, 2, 3$ or among e_i, $\forall i = 4, 5, 6, 7$ there are two or more e_i's, which have exactly the same amount of occurrences, these are permuted, in all possible ways, in the second algebra.
13. After each permutation this algebra is examined to find whether it is isomorphic to the first one.
14. **if** these two algebras turn out to be isomorphic,
15. **then** the new one is rejected
16. **else** the new one is added to the same group
17. Set *count* ← *count* +1

end (* if .. *)

end (* while *count* <= P *)

end (* **procedure** CHECK_ISO *)

4 APPENDIX WITH EXAMPLES

LIE ALGEBRA number 3

[e1,e2] = +(+a22-a33)*e4+a12*e5+a13*e6+a15*e8
[e1,e3] = +a21*e4+a22*e5+a23*e6+a25*e8
[e1,e6] = +e7
[e2,e3] = +a31*e4+a32*e5+a33*e6+a35*e8
[e2,e5] = +e7
[e3,e4] = +e7
H= 1 > 2 * a22 |H= 2 > 2 * a33 |H= 3 > 1 * a12 |H= 4 > 1 * a13 |H= 5 > 1 * a15 |

H= 6 > 1 * a21 |**H= 7 > 1 * a23** |**H= 8 > 1 * a25** |**H= 9 > 1 * a31** |**H=10 > 1 * a32** |
H=11 > 1 * a35 |

THIS IS STEP 19 of values for case 3
[e1,e3] = +e6
[e1,e6] =+e7
[e2,e3] = +a32*e5+a35*e8
[e2,e5] = +e7
[e3,e4] = +e7
****
1 3 6 # 1 6 7 # 2 3 5 8 # 2 5 7 # 3 4 7 #| |
e1:2, e2:2, e3:3, e4:1, e5:1, e6:1, e7:0, =ˆ= e4:0, e5:1, e6:1, e7:3, e8:1,
e1:2+0,e2:2+0,e3:3+0, =ˆ= e4:1+0, e5:1+1, e6:1+1, e7:0+3,
3, 2, 2, 1, 1, 1, 0, =ˆ= 3, 1, 1, 1, 0,
>Similar to step 14 of case 3 <
COMPARING THE CASE 14/3WITH THE CASE 19/3="TR" 4 TIMES
====================

THIS IS STEP 21 of values for case 3
[e1,e3] = +e6
[e1,e6] = +e7
[e2,e3] = +a31*e4+a35*e8
[e2,e5] = +e7
[e3,e4] = +e7
****
1 3 6 # 1 6 7 # 2 3 4 8 # 2 5 7 # 3 4 7 # | |
e1:2, e2:2, e3:3, e4:1, e5:1, e6:1, e7:0, =ˆ= e4:1, e5:0, e6:1, e7:3, e8:1,
e1:2+0,e2:2+0,e3:3+0, =ˆ= e4:1+1, e5:1+0, e6:1+1, e7:0+3,
3, 2, 2, 1, 1, 1, 0, =ˆ= 3, 1, 1, 1, 0,
>Similar to step 14 of case 3 <
COMPARING THE CASE 14/3 WITH THE CASE 21/3="TR" 4 TIMES
====================
>Similar to step 19 of case 3<
COMPARING THE CASE 19/3 WITH THE CASE 21/3="TR" 4 TIMES
====================

THIS IS STEP 26 of values for case 3
[e1,e3] = +e6+a25*e8
[e1,e6] = +e7
[e2,e3] = +a32*e5
[e2,e5] = +e7
[e3,e4] = +e7
****

1 3 6 8 # 1 6 7 # 2 3 5 # 2 5 7 # 3 4 7 # | |
e1:2, e2:2, e3:3, e4:1, e5:1, e6:1, e7:0, =ˆ= e4:0, e5:1, e6:1, e7:3, e8:1,
e1:2+0,e2:2+0,e3:3+0, =ˆ= e4:1+0, e5:1+1, e6:1+1, e7:0+3,
3, 2, 2, 1, 1, 1, 0, =ˆ= 3, 1, 1, 1, 0,
>Similar to step 14 of case 3 <
COMPARING THE CASE 14/3WITH THE CASE 26/3="TR" 4 TIMES
====================
>Similar to step 19 of case 3 <
COMPARING THE CASE 19/3WITH THE CASE 26/3="TR" 4 TIMES
>ISOMORPHIC to step 19 of case 3 <
2 1 | | | | 2 1 | | | |
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