IRREDUCIBLE BECCHI-ROUET-STORA-TYUTIN (BRST) SYMMETRY FOR REDUCIBLE HAMILTONIAN SYSTEMS

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Abstract

Quantization of reducible Hamiltonian first-class theories in an irreducible manner is accomplished in the framework of the Becchi-Rouet-Stora-Tyutin symmetry based on a (co)homological approach.

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1 BRST symmetry— geometric ingredients

Let P be a manifold (regarded as the phase-space of a dynamical system) endowed with a symplectic structure (the Poisson bracket) and let Σ be a surface embedded in P, described by the equations

$$G_{a_0}\left(z^A\right) \approx 0, \tag{1}$$

with z^A the phase-space co-ordinates. We suppose that

$$\left[G_{a_0}, G_{b_0}\right] = C^{c_0}{}_{a_0 b_0} G_{c_0}, \tag{2}$$

where $[,]$ denotes the Poisson bracket and $C^{c_0}{}_{a_0 b_0}$ are some functions on P. By defining the vector fields

$$X_{a_0} \bullet = \left[\bullet, G_{a_0}\right], \tag{3}$$

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it follows that they are tangent to Σ and, moreover, close on Σ, i.e.,

$$[X_{a_0}, X_{b_0}]_{\mathcal{L}} \approx C^{c_0}{}_{a_0 b_0} X_{c_0}, \tag{4}$$

where the notation $[,]_{\mathcal{L}}$ stands for the Lie bracket, while *the weak equality "$\approx$" means an equality modulo equations (1). The integral submanifolds corresponding to the vector fields (3) are named gauge orbits.* Whenever these two ingredients are present, namely, a surface Σ embedded in a manifold P and a set of vector fields tangent to Σ that close on Σ (and, hence, define gauge orbits), one can construct a nilpotent odd derivation s, usually known like the BRST differential (or, equivalently, BRST symmetry) [1]–[3]. The nilpotency of s is expressed by the equation

$$s^2 = 0. \tag{5}$$

The BRST differential acts on a graded algebra containing $C^\infty(P)$ and is such that *its zeroth order cohomological class* $H^0(s)$ satisfies

$$H^0(s) \simeq \{\text{functions of } C^\infty(\Sigma) \text{ constant along the gauge orbits}\}. \tag{6}$$

In theoretical physics language, *relations (1–2) define a set of so-called first-class constraints*, while the right hand-side of (6) represents the class of physical observables. We recall that *any physical observable F satisfies $[F, G_{a_0}] \approx 0$.*

An interesting situation is where the functions G_{a_0} from (1) are not all independent, i.e., there exist some non-vanishing functions $Z^{a_0}{}_{a_1}$ such that

$$Z^{a_0}{}_{a_1} G_{a_0} = 0. \tag{7}$$

One then says that the constraints are reducible and that one is in the reducible case. In the opposite situation, where G_{a_0} are independent, one is in the irreducible case. In the reducible case we can assume that the functions $Z^{a_0}{}_{a_1}$ are not all independent, so there can in principle exist a tower of reducibility relations, of the type

$$Z^{a_1}{}_{a_2} Z^{a_0}{}_{a_1} = 0,\ Z^{a_2}{}_{a_3} Z^{a_1}{}_{a_2} = 0, \cdots, Z^{a_{L-2}}{}_{a_{L-1}} Z^{a_{L-1}}{}_{a_L} = 0, \tag{8}$$

where $a_k = 1, \cdots, M_k$ for any $k = 0, \cdots, L$. *If (7–8) are present, one says that the constraint set (1) is L-stage reducible.* The reducible situation is frequently met in theoretical physics in connection with many models, like, for instance, p-form gauge theories or superstring theory.

2 Derivation of an irreducible BRST symmetry associated with a reducible one

2.1 Setting the problem

It is well-known [1]–[3] that the BRST differential s_R associated with a reducible theory contains two crucial operators

$$s_R = \delta_R + D_R + \cdots, \tag{9}$$

where δ_R is a true differential (Koszul-Tate differential), while D_R is *an odd derivation that anticommutes with* δ_R *and is nilpotent up to* δ_R*-exact terms* (D_R is a *differential modulo* δ_R, named longitudinal derivative along the gauge orbits). The remaining pieces in s_R play no role in the cohomology of s_R, being required in order to ensure the nilpotency of s_R. The Koszul-Tate differential acts on polynomials in some generators (antighosts) to be introduced below with coefficients from $C^{\infty}(P)$, and *realizes an homological resolution of* $C^{\infty}(\Sigma)$*, i.e., its homological classes are restricted to be*

$$H_0(\delta_R) = C^{\infty}(\Sigma), \; H_k(\delta_R) = 0, \; k \neq 0. \tag{10}$$

The degree of δ_R *is called the antighost number (antigh)*, and is given by *antigh* $(\delta_R) = -1$. The longitudinal derivative is initially defined on Σ like acting on polynomials in other generators (ghosts) with coefficients from $C^{\infty}(\Sigma)$. It can be shown that D_R is a true differential on Σ and, moreover, that its zeroth order cohomological class is isomorphic to the class of physical observables. *The degree of* D_R *is named pure ghost number (pgh)*, and *pgh* $(D_R) = 1$. Further, D_R can be "lifted" to P and also extended to the antighosts in such a way that on the one hand it anticommutes with δ_R, and, on the other hand, its square vanishes up to δ_R-exact terms (α *is said to be* δ_R*-exact if* $\alpha = \delta_R\beta$*, for some* β). The degree of s_R is called ghost number (gh). *The ghost number of an object* A *is defined by* $gh(A) = pgh(A) - antigh(A)$, while the ghost number of s_R is taken to be $gh(s_R) = 1$. Under these considerations, the homological perturbation theory [2]–[3] ensures the existence of the BRST differential s_R, whose zeroth order cohomological class satisfies (6) with s replaced by s_R. Actually, relations (5–6) represent the main equations underlying the BRST formalism.

For a first-stage reducible theory ($L = 1$), the construction of the Koszul-Tate differential is based on the definitions

$$\delta_R z^A = 0, \; \delta_R \mathcal{P}_{a_0} = -G_{a_0}, \tag{11}$$

$$\delta_R P_{a_1} = -Z^{a_0}_{\;\;a_1} \mathcal{P}_{a_0}, \tag{12}$$

where the antighosts $\mathcal{P}_{a_0}$ and P_{a_1} have the antighost number one, respectively, two. The antighosts P_{a_1} are required in order to "kill" the non-trivial co-cycles

$$\mu_{a_1} = Z^{a_0}_{\;\;a_1} \mathcal{P}_{a_0}, \tag{13}$$

in the first order homological class of δ_R. (*An object* m *is said to be non-trivial co-cycle of* δ_R *if* $\delta_R m = 0$ *and* $m \neq \delta_R n$*, for any* n*.*) In the irreducible case, the definitions (11) are sufficient for obtaining (10). On the contrary, in a reducible case with $L > 1$ there are necessary more antighosts in order to recover (10). Consequently, the definitions (11–12) must be supplemented with some new appropriate ones. The additional antighosts and corresponding definitions of δ_R acting on them are due to the extra reducibility relations (8).

In the sequel we investigate the problem of obtaining an irreducible BRST differential s_I associated with a starting reducible one, s_R. In this light, our main idea is to redefine the antighosts $\mathcal{P}_{a_0}$ in such a manner that the non-trivial co-cycles of the type (13) vanish identically. If we succeed in accomplishing this purpose, then the antighosts P_{a_1} are no longer necessary, therefore the definitions (12) together with the supplementary ones due to the higher-order reducibility, will be all discarded. In consequence, there will be present only some definitions of the type (11), that will describe an irreducible theory based on some new irreducible first-class constraints. In turn, with the help of these irreducible first-class constraints we can construct an irreducible longitudinal derivative along the gauge orbits D_I, and thus an irreducible BRST symmetry. The link between the reducible and irreducible BRST differentials is expressed by the isomorphism between their zeroth order cohomological classes (the classes of physical observables corresponding to the reducible and irreducible theories can be shown to coincide).

2.2 Main results

Acting along the line discussed in the above we enlarge the initial phase-space and infer the irreducible first-class constraints associated with (1) of the form [4]

$$\gamma_{a_0} \equiv G_{a_0} + A_{a_0}^{\ \ a_1} \pi_{a_1} \approx 0, \tag{14}$$

$$\gamma_{a_{2k}} \equiv Z^{a_{2k-1}}_{\ \ a_{2k}} \pi_{a_{2k-1}} + A_{a_{2k}}^{\ \ a_{2k+1}} \pi_{a_{2k+1}} \approx 0,\ k = 1, \cdots, a, \tag{15}$$

with $\left(y^{a_{2k+1}}, \pi_{a_{2k+1}}\right)_{k=0,\cdots,b}$ denoting the new canonical pairs extending the phase-space and a, b defined by $a = L/2$ for L even, $a = (L-1)/2$ for L odd, respectively, $b = L/2 - 1$ for L even, $b = (L-1)/2$ for L odd. The functions $A_{a_{k-1}}^{\ \ a_k}$ involved with (14–15) depend only on z^A's and are taken to fulfill

$$rank\left(Z^{a_{k-1}}_{\ \ a_k} A_{a_{k-1}}^{\ \ b_k}\right) = \sum_{i=k}^{L} (-)^{k+i} M_i. \tag{16}$$

Moreover, we can show [4] that

$$G_{a_0} = m_{a_0}^{\ \ b_{2k}} \gamma_{b_{2k}},\ \pi_{a_{2k+1}} = m_{a_{2k+1}}^{\ \ b_{2i}} \gamma_{b_{2i}},\ k = 0, \cdots, b. \tag{17}$$

On account of (17), it is easy to prove [4] that the class of physical observables associated with the constraints (1) coincides with that corresponding to (14–15).

At this point, we are in the following situation. We can construct an irreducible BRST differential s_I starting with the constraints (14–15). The above mentioned equality between the classes of physical observables means that

$$H^0\left(s_R\right) \simeq H^0\left(s_I\right). \tag{18}$$

Relation (18) together with the nilpotency of s_R and s_I,

$$s_R^2 = 0 = s_I^2, \tag{19}$$

enable us to **substitute (from the point of view of the basic equations (5–6) underlying the BRST formalism) the reducible BRST symmetry s_R with the corresponding irreducible one, s_I.** This is the main result of this talk.

This result can be important in theoretical physics at the BRST quantization of p-form gauge theories and superstring theory. In fact, this treatment has already been applied to some theories with p-forms [5]–[6], but yet not to superstring theory. This is mainly because superstrings possess, besides first-class constraints, also reducible second-class ones, which are difficult to deal with in a "covariant manner". Thus, if it were possible to develop a general treatment that associates some irreducible second-class constraints with some original reducible ones (without afflicting the theory), then our method would be an effective tool at the covariant quantization of superstrings.

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