REGULARIZED HILBERT SPACE LAPLACIAN AND LONGITUDE OF HILBERT SPACE

Akira Asada and Nobuhiko Tanabe

Abstract

Let H be a real Hilbert space equipped with a non-degenerate symmetric positive Schatten class operator G whose zeta function $Z(G,s) = trG^s$ is holomorphic at $s = 0$. By using spectres of G, a regularization $:\Delta:$ of the Laplacian Δ of H is proposed. To study $:\Delta:$, polar coordinate of H is useful. Polar coordinate of H lacks longitude and adding longitude, we get an extended space H_{l_g} of H on which $:\Delta:$ is definded. $:\Delta:$ on H_{l_g} induces a family of spherical Laplacians $\Lambda_c,\ 0 \le c < 1,\ \Lambda_0$ is the spherical Laplacian of H induced by $:\Delta:$. Spectres of Λ_c are the same as Λ_0, proper functions of Λ_c and they are expressed by Gegenbauer polynomials (including negative weights) and most of them shrinks on H.

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1 Introduction

Let H be a real Hilbert space with the coordinates $x = \sum x_n e_n,\ \{e_n\}$ an O.N.-bases of H. Then the Laplacian Δ of H is given by $\sum \dfrac{\partial^2}{\partial x_n^2}$. But even the metric function $r(x) = ||x||$, $\Delta(r(x))^p$ deverges unless $p = 0$. So some regularization of Δ is needed.

In this paper, we propose a zeta-regularization of Δ. To do this, similar to Connes' spectre triple [4] we equipped a non-degenerate symmetric positive Schatten class oprator G on H such that whose zeta function $Z(G,s) = \mathrm{tr}G^s = \sum \lambda_n^s$ is continued holomorphically to $s = 0$, with H(cf. [2],[4]). We take the proper functions $\{e_n\}$ of G to be the O.N.-basis of H and introduce the operator

$$\Delta(s) = \sum \lambda_n^{2s} \frac{\partial^2}{\partial x_n^2},\ Ge_n = \lambda_n e_n. \tag{1}$$

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In concrete examples, $\Delta(s)$ gives the Laplacian of a Sobolev space. The regularized Laplacian $:\Delta:$ is defined by

$$:\Delta: f = \Delta(s)f|_{s=0}, \text{ if } \Delta(s)f \text{ exists for Re}s \text{ large and continued to } s=0. \quad (2)$$

For example, we have:

$$:\Delta: r(x)^p = p(p+\nu-2)r(x)^{p-2}, \quad \nu = Z(G,0). \quad (3)$$

This shows $:\Delta:$ is not elliptic if $\nu < 0$ unless ν is an even integer.

To study $:\Delta:$, we introduce the polar coordinate of H by

$$x_1 = r\cos\theta_1,\ x_2 = r\sin\theta_1\cos\theta_2, \dots, x_n = r\sin\theta_1\cdots\sin\theta_{n-1}\cos\theta_n, \dots, \quad (4)$$
$$0 \le \theta_n \le \pi,\ \theta_m = 0 \text{ if } \theta_n = 0 \text{ and } m > n. \quad (5)$$

This polar coordinate has only latitudes and lacks longitude. Since we have$\sum x_n^2 = r^2(1-(\lim\sin\theta_1\sin\theta_2\cdots\sin\theta_n)^2)$, we introduce the longitude x_∞ by

$$x_\infty = rc, \quad c = \lim\sin\theta_1\cdots\sin\theta_n. \quad (6)$$

If $x = \sum x_n e_n$ is an element of H and $r = ||x||, \theta_1, \theta_2, \dots$ need to satisfy the constraint $x_\infty = 0$, that is

$$\lim\sin\theta_1\sin\theta_2\cdots\sin\theta_n = 0. \quad (7)$$

The polar coordinate expression of $:\Delta:$ depends only on ν. Denoting this operator by $\Delta[\nu]$, we have:

$$\Delta[\nu] = \frac{\partial^2}{\partial r^2} + \frac{\nu-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda[\nu], \quad (8)$$
$$\Lambda[\nu] = \sum_{n=1}^{\infty}\frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{n-1}}\left(\frac{\partial^2}{\partial\theta_n^2} + (\nu-n-1)\frac{\cos\theta_n}{\sin\theta_n}\frac{\partial}{\partial\theta_n}\right). \quad (9)$$

These operators are defined on the extended space $H_{l_g} = \{(x, x_\infty)|\ x \in H\}$.

From (8) and (9), we have $\Delta[\mu] = \Delta[\nu] + \dfrac{\mu-\nu}{r}K$, where

$$K = \frac{\partial}{\partial r} + \sum\frac{\cos\theta_n}{r\sin^2\theta_1\cdots\sin^2\theta_{n-1}\sin\theta_n}\frac{\partial}{\partial\theta_n}. \quad (10)$$

$:\Delta: f$ does not depend on regularization if and only if $Kf = 0$. Since the characteristic curve of K starting from $(x,0) \in H \subset H_{l_g}$, is given by

$$x(t) = x,\ x_\infty(t) = \sqrt{(||x||+t)^2 - ||x||^2},\ t \ge 0, \quad (11)$$

$:\Delta: f$ does not depend on regularization (as a function on H_{l_g}), if and only if f is constant in x_∞-direction. This suggests significance of the longitude to the study of the Laplacian on H. We also ask: is there any relation between this longitude and the central charge in the definition of the Dirac-Romond operator (cf. [7])?

Formal treaties of radical and spherical part of $\Delta[\nu]$ are similar to the finite dimensional case ([5],[12]). But since most of $\{\nu - n - 1\}$ are negative, Λ is not definite if ν is negative. Negative weight Gegenbauer polynomials appear as the components of proper functions of $\Lambda[\nu]$. We ask: is there any relation between this result and negative dimensional integration methods [11]?

Since $:\Delta:$ is defined on H_{l_g}, $\Lambda[\nu]$ induces an operator on $\{(x,c)|\ ||x|| = r\},\ 0 \le c < r$. We denote this operator by $\Lambda_c (= \Lambda[\nu]_c)$. Λ_0 is the original spherical Laplacian induced from $:\Delta:$. Since there are infinitely many independent proper functions of $\Lambda[\nu]$ of the form

$$\lim_{N\to\infty} (\sin\theta_1 \cdots \sin\theta_N)^l f(\theta_1, \theta_2, \cdots),\ f \text{ finite on } 0 \le \theta_i \le \pi,\ l = 1, 2, \cdots, \tag{12}$$

there are infinitely many independent 1-parameter family of proper functions of Λ_c which degenarate as the proper functions of Λ_0.

2 Regularized Laplacian. Definition and examples

Let H be a real Hilbert space equipped with a non-degenarate positive symmetric Schatten class operator G on H such that whose zeta function $Z(G,s) = \mathrm{tr} G^s$ is continued holomorphically to $s = 0$ (cf.[4]). Then taking the proper functions $\{e_n\}$ of G as O.N.-basis of H, we define the operator $\Delta(s)$ by

$$\Delta(s) = \sum \lambda_n^{2s} \frac{\partial^2}{\partial x_n^2},\ Ge_n = \lambda_n e_n. \tag{13}$$

Example 1. Let H be $L^2(X)$, the Hilbert space of square integrable sections of a symmtric vector bundle E over X, a compact Riemannian manifold, D a non-degenerate selfadjoint elliptic (pseudo)differential operator of order m acting on the sections of E. Then we can take as above the Green operator of D to be G. By definitions, we have:

$$Z(G,s) = \zeta(D, -s). \tag{14}$$

Hence $Z(G,s)$ is holomorphic at $s = 0$ (see [6]). Since mk-th Sobolev norm $||f||_k$ for the sections of E can be fixed by

$$||f||_k = ||D^k f||, \tag{15}$$

$\{\lambda_n^k e_n\}$ gives an O.N.-basis of $W^{mk}(X)$, the mk-th Sobolev space of sections of E : $De_n = \lambda_n^{-1} e_n$. Hence $\Delta(k)$ is the Laplacian of $W^k(X)$.

Definition 2.1 *If $\Delta(s)f$ exists for* $\mathrm{Re}s$ *large and continued holomorphically to $s = 0$, we define the regularized Laplacian $:\Delta:$ by*

$$:\Delta: f = \Delta(s) f|_{s=0}. \tag{16}$$

Example 2. Since $\dfrac{\partial^2}{\partial x_n^2}(r(x))^p = pr(x)^{p-2} + p(p-2)r(x)^{p-4}x_n^2$, we have:

$$\Delta(s)(r(x))^p = Z(G,2s)pr(x)^{p-2} + \sum \lambda_n^{2s} p(p-2)r(x)^{p-4}x_n^2. \tag{17}$$

Since $Z(G,0) = \nu$ is finite by assumption, we have:

$$:\Delta : r(x)^p = p(p+\nu-2)r(x)^{p-2}. \tag{18}$$

Using (18) , $:\Delta : r(x)^{2-\nu}$ is equal to 0. If $\nu < 0$, $r(x)^{2-\nu}$ is C^2-class on H, but not smooth unless ν is an even integer. So $:\Delta:$ is not elliptic if $\nu < 0$ unless ν is an even integer.

We have defined the regularized dimension of H (equipped with G) by $\nu = Z(G,0)$(see [1]). To consider Grassmann algebra or Clifford algebra over H with $(\infty - p)$-forms or ∞-spinors, ν needs to be an integer. Examples show that ν may be negative.

Example 3. Since $\sum x_n \dfrac{\partial h}{\partial x_n} = ph$ holds for homogeneous functions of degree p on H, if $:\Delta : h$ is defined, we have:

$$:\Delta : r^m h = m(m+\nu-2+2p)r^{m-2}h + r^m :\Delta : h. \tag{19}$$

Using (19) , similar to the finite dimensional case (see [12]), denote by $C^m(H)$ the module of homogeneous functions of degree m such that $:\Delta:^p$ is defined for $1 \le p \le [m/2]$, $by N^m(H)$ the module of homogeneous functions of degree m vanished by $:\Delta:$; we have

$$C^{2m}(H) = \sum_{p=0}^{m} r^{2p}N^{2(m-p)}, \text{ if } \nu + 2p \neq 0,\ 0 \le p \le m, \tag{20}$$

$$C^{2m+1}(H) = \sum_{p=0}^{m} r^{2p}N^{2(m-p)+1}, \text{ if } \nu + 2p + 1 \neq 0,\ 0 \le p \le m. \tag{21}$$

3 Polar coordinate of H and longitude of H

To set $r = ||x||$, the polar coordinate of $x \in H$ is given by:

$$x_1 = r\cos\theta_1,\ x_2 = r\sin\theta_1\cos\theta_2, \ldots, x_n = r\sin\theta_1\cdots\sin\theta_{n-1}\cos\theta_n, \ldots,\ 0 \le \theta_n \le \pi. \tag{22}$$

$\{\theta_1, \theta_2, \cdots\}$ is uniquely determined by x under the assumption

$$\theta_m = 0 \text{ if } \theta_n = 0 \text{ and } m > n. \tag{23}$$

Since $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2(1 - \sin^2\theta_1\cdots\sin^2\theta_n)$, $\theta_1, \theta_2, \ldots$ must satisfy the constraint

$$\lim_{n\to\infty} \sin\theta_1\cdots\sin\theta_n = 0. \tag{24}$$

In general, if $\theta_1, \theta_2, \ldots$ are independent variables ($0 \le \theta_n \le \pi$), $\lim \sin\theta_1\cdots\sin\theta_n = c$ always exists and $0 \le c \le 1$. From (23), we have:

Lemma 3.1

$$\lim_{N\to\infty} \sin\theta_n \sin\theta_{n+1}\cdots\sin\theta_N = 0 \tag{25}$$

for some n and if (23) holds, (25) holds for any n.

Definition 3.1 Considering $\theta_1, \theta_2, \ldots$, to be independent variables, we set

$$x_\infty = rc, \quad c = \lim \sin\theta_1 \cdots \sin\theta_n. \tag{26}$$

We call x_∞ *the longitude* of H.

By definition, we have $\sum x_n^2 + x_\infty^2 = r^2$. So the set $\{(x, x_\infty)|\ x \in H\}$ is contained in the Hilbert space $H \oplus R$. Since $0 \le x_\infty \le ||x||$ by (26), we set

$$H_{l_g} = \{(x, c)|\ x \in H,\ 0 \le c \le ||x||\} \subset H \oplus R. \tag{27}$$

Similar to the finite dimensional case, setting $r_k = \sqrt{\sum_{n\ge k} x_n^2}$, $r_1 = r$, we have:

$$\sin\theta_k = \frac{r_{k+1}}{r_k},\ \cos\theta_k = \frac{x_k}{r_k},\ r_k = r\sin\theta_1\cdots\sin\theta_{k-1}. \tag{28}$$

From (28) and the definition of $:\Delta:$, we obtain the following result.

Proposition 3.1 *Polar coordinate expression of $:\Delta:$ depends only on $\nu = Z(G, 0)$. Denoting this operator by $\Delta[\nu]$ and its spherical part by $\Lambda[\nu]$, we have:*

$$\Delta[\nu] = \frac{\partial^2}{\partial r^2} + \frac{\nu-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda[\nu], \tag{29}$$

$$\Lambda[\nu] = \sum_{n=1}^{\infty} \frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{n-1}} \left(\frac{\partial^2}{\partial\theta_n^2} + (\nu-n-1)\frac{\cos\theta_n}{\sin\theta_n}\frac{\partial}{\partial\theta_n}\right). \tag{30}$$

Corollary 3.1 *We have:*

$$\Delta[\mu] = \Delta[\nu] + \frac{\mu-\nu}{r}K, \tag{31}$$

$$K = \frac{\partial}{\partial r} + \frac{1}{r}\sum_{n=1}^{\infty} \frac{\cos\theta_n}{\sin^2\theta_1\cdots\sin^2\theta_{n-1}\sin\theta_n}\frac{\partial}{\partial\theta_n}. \tag{32}$$

From (32), $\Delta[\mu]f = \Delta[\nu]f$ if and only if $Kf = 0$ and if $Kf = 0$, $\Delta[\nu]f$ does not depend on ν, that is, $:\Delta: f$ does not depend on the regularization.

K is a 1-st order linear partial differential equation. So its solution is constant along the characteristic curves. Since the characteristic equation of K is

$$\frac{dr}{dt} = 1,\ \frac{d\theta_n}{dt} = \frac{\cos\theta_n}{r\sin^2\theta_1\cdots\sin^2\theta_{n-1}\sin\theta_n},\ n = 1, 2, \ldots, \tag{33}$$

its solution is given by:

$$r = t + c,\ \cos\theta_1 = \frac{c_1}{t+c}, \ldots, \cos\theta_n = \frac{c_n}{\sqrt{(t+c)^2 - (\sum_{k=1}^{n-1} c_k^2)}}. \quad (34)$$

From (34), we get:

$$\sin\theta_n = \sqrt{\frac{(t+c)^2 - (\sum_{k=1}^{n} c_k^2)}{(t+c)^2 - (\sum_{k=1}^{n-1} c_k^2)}}. \quad (35)$$

From (34) and (35), we have:

$$x_n = c_n,\ n = 1, 2, \ldots,\ x_\infty = \sqrt{(t + ||x||)^2 - ||x||^2},\ t \geq 0. \quad (36)$$

Hence, considering K to be an equation on H_{l_g}, the characteristic curve of K starting from $x = (x_1, x_2, \ldots) \in H$, is given by:

$$x(t) = x,\ x \in H,\ x_\infty = \sqrt{(t + ||x||)^2 - ||x||^2},\ t \geq 0. \quad (37)$$

4 Proper functions of $\Lambda[\nu]$

Let $\Theta(\theta_1, \theta_2, \ldots)$ be a proper function of $\Lambda[\nu]$ belonging to the proper value μ. We assume Θ is the infinite product $T_1(\theta_1)T_2(\theta_2)\cdots$. Then similar to the finite dimensional case, we have the equations:

$$\sin^{-\nu+n+1}\theta_n \frac{d}{d\theta_n}\left(\sin^{\nu-n-1}\theta_n \frac{dT_n}{d\theta_n}\right) + \left(a_{n-1} - \frac{a_n}{\sin^2\theta_n}\right) T_n = 0,\ n = 1, 2, \ldots,\ a_0 = \mu. \quad (38)$$

Replacing $\omega_n = \cos\theta_n$, (38) is changed to

$$(1 - \omega_n^2)\frac{d^2T_n}{d\omega_n^2} - (\nu - n)\omega_n \frac{dT_n}{d\omega_n} + \left(a_{n-1} - \frac{a_n}{1 - \omega_n^2}\right) T_n = 0. \quad (39)$$

The equation (39) needs to have a continuous solution at $\omega_n = \pm 1$. For this, assuming ν to be an integer, it is sufficient to take

$$a_n = l_n(l_n + \nu - n - 2),\ l_0 \geq l_1 \geq \ldots \geq 0,\ l_0, l_1, \ldots, \text{are integers.} \quad (40)$$

From (40), the series $\{l_0, l_1, \ldots\}$ satisfy

$$l_n = l_{n+1} = \ldots = l_\infty \geq 0,\ \text{for } n \text{ enough large.} \quad (41)$$

In order to solve the equations (39) under the assumption (40), we consider two cases. For a finite dimensional spherical Laplacian, case 2 provides only constant solution.

But in our case, case 2 provides infinitely many independent solutions and causes the phase transition phenomenons stated in the Introduction.

Case 1: $l_{n-1} \neq l_n$. This case occurs only finite times.
In this case, the solutions of the equations (39) are given using the Gegenbauer polynomials $C_l^{\mu}(x)$ defined by:

$$\frac{1}{(1-2xt+t^2)^{\mu}} = \sum_{l=0}^{\infty} C_l^{\mu}(x)t^l. \tag{42}$$

The general solution is:

$$T_n(\omega_n) = C_1(1-\omega_n^2)^{l_n/2} C_{l_{n-1}-l_n}^{l_n+(\nu-n-1)/2}(\omega_n) + C_2(1-\omega_n^2)^{l_n/2} C_{n+1-l_{n-1}-l_n-\nu}^{l_n+(\nu-n-1)/2}(\omega_n). \tag{43}$$

Notice that the weight $l_n + (\nu - n - 1)/2$ may be smaller than -1. But we still have

$$C_l^{\mu}(x) = \frac{(-1)^l}{2^l l!} \frac{\Gamma(\mu+1/2)}{\Gamma(l+\mu+1/2)} (2\mu+l-1)\cdots 2\mu \cdot (1-x^2)^{\frac{1}{2}-\mu} \frac{d^l}{dx^l}(1-x^2)^{l+\mu-\frac{1}{2}}, \tag{44}$$

even $\mu < -1$. Here $\dfrac{\Gamma(\mu+1/2)}{\Gamma(-l+\mu+1/2)}$ means $(\mu-1/2)\cdots(\mu-l-1/2)$ if μ is a negative half integer.

Case 2: $l_{n-1} = l_n$. From (41), taking $l_n = l_\infty$, the equation (39) belongs to this case if n is large.
In this case, it is convenient to solve the original equations (38). Setting $T_n(\theta_n) = \sin^{l_n}\theta_n \cdot S_n(\theta_n)$, the equations become:

$$\frac{d^2 S_n}{d\theta_n^2} + (2l_n + \nu - n - 1)\frac{\cos\theta_n}{\sin\theta_n}\frac{dS_n}{d\theta_n} = 0. \tag{45}$$

Hence, if $n + 1 - \nu - 2l_n \geq 0$, the general solution of (38) is:

$$T_n(\theta_n) = \sin^{l_n}\theta_n \left(c_1 + c_2 \int_0^{\theta_n} (\sin x)^{n+1-\nu-2l_n} dx \right). \tag{46}$$

To take infinite product $T_1(\theta_1)T_2(\theta_2)\cdots$, we need only to consider infinite product of the functions of the form (46). In this case, since $\int_0^{\pi}(\sin x)^{n+1-\nu-2l_n}dx = B((n+1-\nu)/2 - l_n, 1/2) = O(1/\sqrt{n})$, the infinite product $\prod_{n\geq N}(1 + a_n \int_0^{\theta_n}(\sin x)^{n+1-\nu-2l_n}dx)$ converges if

$$\sum \left| \frac{a_n}{\sqrt{n}} \right| < \infty. \tag{47}$$

Summarizing, we have the following result.

Proposition 4.1 *The operator $\Lambda[\nu]$ considered on $\{(\theta_1, \theta_2, \ldots)|\ 0 \le \theta_n \le \pi\}$ has the proper values $-l(l+\nu-2),\ l = 0, 1, 2, \ldots$, with infinitely many independent proper functions of the form*

$$\Theta(\theta_1, \theta_2, \ldots) = F(\theta_1, \theta_2, \ldots, \theta_{N-1}) \prod_{n \ge N} (\sin\theta_n)^{l_\infty} \left(1 + a_n \int_0^{\theta_n} (\sin)^{n+1-\nu-2l_\infty} dx\right), \tag{48}$$

where l_∞ is an integer satisfing $l \ge l_\infty \ge 0$, $\{a_n\}$ and $\{b_n\}$ satisfy (47).

Corollary 4.1 *$\Lambda[\nu]$ is not defined if $\nu < 1$.*

Taking $r = 1$, $\{(\theta_1, \theta_2, \ldots)|\ 0 \le \theta_n \le \pi\}$ is mapped to $\{(x, x_\infty)|\ ||x|| = 1,\ 0 \le x_\infty \le 1\} \subset H_{l_g}$. We set $S_c^\infty = \{(x, c)|\ ||x||^2 = 1 - c^2\} \subset H_{l_g},\ 0 \le c < \sqrt{2}/2$. Then $\Lambda[\nu]$ induces an operator $\Lambda_c = \Lambda[\nu]_c$ on S_c^∞, Λ_0 is the original spherical Laplacian. Using Lemma 3.1 and Proposition 3.1, we have:

Theorem 4.1 *Each Λ_c has common proper values $-l(l+\nu-2),\ l = 0, 1, 2, \ldots$. Each proper value has infinitely many independent 1-parameter family of proper functions $\Theta_c(\theta_1, \theta_2, \ldots)$; $\Lambda_c \cdot \Theta_c = l(l+\nu-2)\Theta_c,\ c \ge 0$, and $\Theta_c \ne 0$. If $l \ge 1$, the proper value $l(l+\nu-2)$ has infinitely many independent 1-parameter family of proper functions Φ_c such that:*

$$\Lambda_c \Phi_c = l(l+\nu-2)\Phi_c,\ \Phi_c \ne 0,\ c \ne 0,\ \Phi_0 = 0. \tag{49}$$

If ν is an integer and $\nu \le 1$, there are infinitely many independent 1-parameter families of functions Ψ_c such that

$$\Lambda_c \Psi_c = 0,\ \Psi_c \ne 0,\ c \ne 0,\ \Psi_0 = 0. \tag{50}$$

There is another choice of l_n which provides continuous solution at $\omega_n = \pm 1$ of (38) such that $l_n \ge l_{n-1} + 1$. Taking $l_n = l_{n-1} + 1$ for n enough large, we again observe phase transition phenomenon sililar to Theorem4.1.

5 Supplementary remarks

1. As for radical part, let $R(r)\Theta(\theta_1, \theta_2, \ldots)$ be a proper function of $\Delta[\nu]$ belonging to λ, where $\Theta(\theta_1, \theta_2, \ldots)$ is a proper function of $\Delta[\nu]$ belonging to $p(p+\nu-2)$, then R satisfies the equation

$$\frac{d^2R(r)}{dr^2} + \frac{\nu-1}{r}\frac{dR(r)}{dr} - \left(\lambda + \frac{p(p+\nu-2)}{r^2}\right)R(r) = 0. \tag{51}$$

The solution of this equation is given by:

$$R(r) = C_1 r^p + C_2 r^{2-p-\nu}, \quad \lambda = 0, \tag{52}$$

$$R(r) = C_1 r^{1-\nu/2} J_\mu(\frac{\lambda^2}{4} r) + C_2 r^{1-\nu/2} J_{-\mu}(\frac{\lambda^2}{4} r),\ \mu = p + \nu/2 - 1, \tag{53}$$

where λ is a negative real number. Notice that since ν may be negative, $2-p-\nu$ may be positive and $r^{1-\nu/2}J_{-\mu}(\frac{\lambda^2}{4}r)$ may be continuous (or smooth) in $r=0$.

From (53), phase transition phenomenon similar to Theorem4.1 holds for $\Delta[\nu]$ considered on $\{(x,x_\infty)|||x||^2+|x_\infty|^2 \le a^2\}$ with the Dirichlet or Neumann boundary condition at $\{(x,x_\infty)|||x||^2+|x_\infty|^2 = a^2\}$, regarding the longitude variable x_∞ as a parameter.

2. Considering $\Lambda[\nu]$ an infinite dimensional spherical symmetric hamiltonian without interaction, one of the authors (NT) defined angular momentum operators of $\Lambda[\nu]$, using Jordan algebra constructed by the inner product of H (see [9]). This Jordan algebra is an infinite dimensional flat space version of Turtoi's Jordan algebra (cf. [10]), so the angular momentum operations are closely related to Petroşanu's Dirac kind operator (cf. [8]).

3. Computation of proper values and functions of $:\Delta:$ for the periodic boundary condition such as

$$u\big|_{x_n=-\lambda_n^{-d/2}} = u\big|_{x_n=\lambda_n^{d/2}},$$
$$\left.\frac{\partial u}{\partial x_n}\right|_{x_n=-\lambda_n^{-d/2}} = \left.\frac{\partial u}{\partial x_n}\right|_{x_n=\lambda_n^{d/2}}, \tag{54}$$

also provides an extra-dimension to H. This new dimension can be interpreted as the determinant bundle constructed from the Ray-Singer determinant of D (cf. [1],[2]). For the details, see [3].

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Authors' address:
Akira Asada and Nobuhiko Tanabe
Department of Mathematical Science,
Faculty of Science,
Shinshu University, Japan
email: asada@math.shinshu-u.ac.jp and tanabe@math.shinshu-u.ac.jp