DYNAMICS INDUCED BY SECOND-ORDER OBJECTS

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Abstract

§1 defines the second-order objects. §2 (§3) describes a nonclassical electric (magnetic) dynamics produced by a "clever second-order Lagrangian", via the extremals of the energy functional. §4 generalize this dynamics, having in mind possible applications for dynamical systems coming from Biomathematics, Economical Mathematics, Industrial Mathematics, etc.

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1 Introduction

This paper will concentrate on certain geometric ideas that are very important in the physical applications. To avoid too much repetition, M will denote a differentiable manifold of dimension n, and all the functions are of class C^∞.

Let $x^a = x^a(x^\alpha)$, $a, \alpha = 1, \ldots, n$ be a changing of coordinates on M. Then we introduce the symbols

$$D^a_\alpha = \frac{\partial x^a}{\partial x^\alpha}, \quad D^a_{\alpha\beta} = \frac{\partial^2 x^a}{\partial x^\alpha x^\beta}.$$

For a differentiable function $f : M \to R$ we use the simplified coordinate expression $f(x^\alpha) = f(x^a(x^\alpha))$, the first order derivatives $f_{,\alpha}, f_{,a}$ and the second order derivatives $f_{,\alpha\beta}, f_{,ab}$. These are connected by the rule

$$(*) \qquad (f_{,\alpha}, f_{,\alpha\beta}) = (f_{,a}, f_{,ab}) \begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}.$$

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The pair (first-order partial derivatives, second-order partial derivatives) possesses the "tensorial" change law that the second derivative, by itself, lacks. This pair was used in the classical works like "contact element" or like "jet".

If a curve is given by the parametric equations $x^a = x^a(t)$, $t \in I$, then the preceding diffeomorphism modifies the pair $(\ddot{x}, \dot{x} \otimes \dot{x})^T$ as follows

$$\dot{x}^\alpha = \dot{x}^a D^\alpha_a, \; \ddot{x}^\alpha = \ddot{x}^a D^\alpha_a + \dot{x}^a \dot{x}^b D^\alpha_{ab},$$

$$\begin{pmatrix} \ddot{x}^\alpha \\ \dot{x}^\alpha \dot{x}^\beta \end{pmatrix} = \begin{pmatrix} D^\alpha_a & D^\alpha_{ab} \\ 0 & D^\alpha_a D^\beta_b \end{pmatrix} \begin{pmatrix} \ddot{x}^a \\ \dot{x}^a \dot{x}^b \end{pmatrix}.$$

The pair (acceleration, "square of velocity") is suggested by the equations of geodesics $\ddot{x}^c + \Gamma^c_{ab} \dot{x}^a \dot{x}^b = 0$.

Consequently, when first and second derivatives come into play together, then matrices of blocks such as

$$K = \begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}, \quad K^{-1} = \begin{pmatrix} D^\alpha_a & D^\alpha_{ab} \\ 0 & D^\alpha_a D^\beta_b \end{pmatrix}$$

are useful.

Definition. Let (ω_a) be an 1-form and (ω_{ab}) be a mathematical object with two indices. Any pair (ω_a, ω_{ab}) admitting the changing law $(*)$ is called *second-order object.*

The theory of superior-order objects was initiated by Foster [3], suggesting new point of view about the fields theory.

We use these ideas to study the extremals of some energy functionals produced by particular second-order Lagrangians associated to $n + n^2$ or $n + \dfrac{n(n+1)}{2}$ potentials.

Let $U \subset R^3 = M$ be a domain of linear homogeneous isotropic media. In terms of differential forms, *Maxwell equations* on $U \times R$ can be expressed as

$$dD = \rho, \qquad dH = J + \partial_t D$$

$$dB = 0, \qquad dE = -\partial_t B,$$

where the magnetic induction B, the electric displacement D, and the electric current density J are all 2-forms; the magnetic field H, and the electric field E are 1-forms; the electric charge density ρ is a 3-form. The operator d is the exterior derivatives and the operator ∂_t is the time derivative.

The constitutive relations are

$$D = \varepsilon * E, \; B = \mu * H,$$

where the star operator $*$ is the Hodge operator, ε is the permitivity, and μ is the scalar permeability.

The local components E_i, $i = 1, 2, 3$, of E are called *electric potentials*, and the local components H_i, $i = 1, 2, 3$, of H are called *magnetic potentials.* Since the electric field E, and the magnetic field H are 1-forms [1], we combine our ideas [4], [5] with the ideas of Foster, creating a nonclassical electric or magnetic dynamics. Finally, we generalize the results to nonclassical dynamics induced by a second-order object.

2 Nonclassical electric dynamics

Let E_i be the electric potentials. The usual derivative $E_{i,j}$ may be decomposed into skew-symmetric and symmetric parts,

$$E_{i,j} = \frac{1}{2}(E_{i,j} - E_{j,i}) + \frac{1}{2}(E_{i,j} + E_{j,i}).$$

The skew-symmetric part

$$m_{ij} = \frac{1}{2}(E_{i,j} - E_{j,i})$$

is called *Maxwell tensor field* giving the opposite of the time derivative of magnetic induction. The symmetric part

$$\frac{1}{2}(E_{i,j} + E_{j,i})$$

is not an ordinary tensor, but the pair

$$(E_i, \frac{1}{2}(E_{i,j} + E_{j,i}))$$

is a second-order object, whereas the pair $(E_i, 0)$ is not [3].

Let ω_{ij} be a general object such that (E_i, ω_{ij}) is a second-order object. The difference

$$(0, \omega_{ij} - \frac{1}{2}(E_{i,j} + E_{j,i}))$$

is a second-order object, having the form $(0, g_{ij})$ of a Riemann (or semi-Riemann) metric if ω_{ij} is symmetric, g_{ij} is a (0,2) tensor field and $\det(g_{ij}) \neq 0$. In this context the equality

$$(E_i, \omega_{ij}) = (E_i, \frac{1}{2}(E_{i,j} + E_{j,i})) + (0, g_{ij})$$

shows that the valuable objects

$$\frac{1}{2}(E_{i,j} + E_{j,i})$$

come from electricity and mate with gravitational potentials g_{ij}. Consequently they have to be *electrogravitic potentials.*

The preceding potentials determine an *electric energy element* (square of the *electric arc element*),

$$d\eta^2 = E_i\, d^2x^i + \frac{1}{2}(E_{i,j} + E_{j,i})dx^i dx^j.$$

If the *most general energy element* (square of *the arc element*) is given by

$$d\sigma^2 = E_i\, d^2x^i + \omega_{ij}dx^i dx^j,$$

and the *gravitational energy element* (square of the *pure gravitational arc element*) is

$$ds^2 = g_{ij}dx^i dx^j,$$

then

$$d\sigma^2 = d\eta^2 + ds^2.$$

The electric energy element is zero along integral curves of the distribution generated by the electric 1-form E. The general energy element determines the *energy functional*

$$\int \left(E_i \frac{d^2x^i}{dt^2} + \omega_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} \right) dt, \tag{1}$$

which is not independent of the parameter t.

Theorem. *The extremals of the energy functional (1) are described by the DEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + \left[\omega_{kji} - \frac{1}{2}(E_{k,ij} + E_{j,ik} - E_{i,jk}) \right] \frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

Proof. Using the second-order Lagrangian

$$L = E_i \frac{d^2x^i}{dt^2} + \omega_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt},$$

the Euler-Lagrange equations

$$L_{x^k} - \frac{d}{dt} L_{\frac{dx^k}{dt}} + \frac{d^2}{dt^2} L_{\frac{d^2x^k}{dt^2}} = 0$$

transcribe like the equations in the theorem.

To an arc element there may correspond a field theory. Consequently we obtain a field theory having as basis the general electrogravitic potentials. The pure gravitational potentials are given by

$$g_{ij} = \omega_{ij} - \frac{1}{2}(E_{i,j} + E_{j,i}).$$

We define

$$\Gamma_{ijk} = \omega_{ijk} - \frac{1}{2}(E_{k,ij} + E_{j,ik} - E_{i,jk}),$$

where

$$\omega_{ijk} = \frac{1}{2}(\omega_{ij,k} + \omega_{ik,j} - \omega_{kj,i})$$

are the Christoffel symbols of ω_{ij}. It is verified that

$$\Gamma_{ijk} = g_{ijk} + m_{ijk},$$

where g_{ijk} are the Christoffel symbols of g_{ij}, and

$$m_{ijk} = m_{ij,k} + m_{ik,j}$$

is the symmetrized derivative of the Maxwell tensor

$$m_{ij} = \frac{1}{2}(E_{i,j} - E_{j,i}).$$

Corollary. *The extremals of the energy functional (1) are described by the DEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (g_{kji} + m_{kji})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

3 Nonclassical magnetic dynamics

Let H_i be the magnetic potentials. The usual derivative $H_{i,j}$ may be decomposed into skew-symmetric and symmetric parts,

$$H_{i,j} = \frac{1}{2}(H_{i,j} - H_{j,i}) + \frac{1}{2}(H_{i,j} + H_{j,i}),$$

where

$$M_{ij} = \frac{1}{2}(H_{i,j} - H_{j,i})$$

is the *Maxwell tensor field* giving the sum between the electric current density and the time derivative of the electric displacement.

The pair

$$\left(H_i, \frac{1}{2}(H_{i,j} + H_{j,i})\right)$$

is a second-order object. If (H_i, ω_{ij}) is a general second-order object, then the difference

$$g_{ij} = \omega_{ij} - \frac{1}{2}(H_{i,j} + H_{j,i})$$

represents the gravitational potentials (a metric) provided that ω_{ij} is symmetric, g_{ij} is a (0,2) tensor field and $\det(g_{ij}) \neq 0$. Consequently the valuable objects $\frac{1}{2}(H_{i,j}+H_{j,i})$, which come from magnetism and mate, have to be *magnetogravitic potentials*.

The preceding potentials produce the following energy elements:

1) *magnetic energy element*,

$$d\lambda^2 = H_i\, d^2x^i + \frac{1}{2}(H_{i,j} + H_{j,i})dx^i dx^j;$$

2) *gravitational energy element*,

$$ds^2 = g_{ij}dx^i dx^j;$$

3) *general energy element*,

$$d\mu^2 = H_i\, d^2x^i + \omega_{ij}dx^i dx^j,$$

which satisfy the relation

$$d\mu^2 = d\lambda^2 + ds^2.$$

The magnetic energy element is zero along integral curves of the distribution generated by the magnetic 1-form H.

Let us consider the *energy functional*

$$\text{(2)} \qquad \int \left(H_i \frac{d^2x^i}{dt^2} + \omega_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) dt$$

determined by a particular second-order Lagrangian.

Theorem. *The extremals of the energy functional (2) are solutions of the DEs*

$$g_{ki}\frac{d^2x^i}{dt^2}+\left[\omega_{kji}-\frac{1}{2}(H_{k,ij}+H_{j,ik}-H_{j,ik})\right]\frac{dx^i}{dt}\frac{dx^j}{dt}=0.$$

To an energy element (arc element) there may correspond a field theory. Consequently we obtain a field theory having as basis the general magnetogravitic potentials. The pure gravitational potentials are (components of a Riemann or semi-Riemann metric)

$$g_{ij}=\omega_{ij}-\frac{1}{2}(H_{i,j}+H_{j,i}).$$

We introduce

$$\Gamma_{ijk}=\omega_{ijk}-\frac{1}{2}(H_{k,ij}+H_{j,ik}-H_{i,jk}).$$

It is verified the relation

$$\Gamma_{ijk}=g_{ijk}+M_{ijk},$$

where g_{ijk} are the Christoffel symbols of g_{ij}, and

$$M_{ijk}=M_{ij,k}+M_{ik,j}$$

is the symmetrized derivative of the Maxwell tensor

$$M_{ij}=\frac{1}{2}(H_{i,j}-H_{j,i}).$$

Corollary. *The extremals of the energy functional (2) are solutions of the DEs*

$$g_{ki}\frac{d^2x^i}{dt^2}+(g_{kji}+M_{kji})\frac{dx^i}{dt}\frac{dx^j}{dt}=0.$$

4 Nonclassical dynamics induced by a second-order object

Now we want to generalize the preceding explanations since they can be applied to dynamical systems coming from Biomathematics, Economical Mathematics, Industrial Mathematics etc.

Let ω_i be given potentials. The usual derivative $\omega_{i,j}$ may be decomposed as

$$\omega_{i,j}=\frac{1}{2}(\omega_{i,j}-\omega_{j,i})+\frac{1}{2}(\omega_{i,j}+\omega_{j,i}),$$

where

$$\mathcal{M}_{ij}=\frac{1}{2}(\omega_{i,j}-\omega_{j,i})$$

is a *Maxwell tensor field*. The pair

$$\left(\omega_i,\frac{1}{2}(\omega_{i,j}+\omega_{j,i})\right)$$

is a second-order object. If (ω_i, ω_{ij}) is a general second-order object, then we suppose that the difference

$$g_{ij} = \omega_{ij} - \frac{1}{2}(\omega_{i,j} + \omega_{j,i})$$

represents the components of a metric, i.e., ω_{ij} is symmetric, g_{ij} is a (0,2) tensor field and $\det(g_{ij}) \neq 0$.

The preceding potentials produce the following energy elements:

1) *potential-produced energy element,*

$$d\alpha^2 = \omega_i\, d^2x^i + \frac{1}{2}(\omega_{i,j} + \omega_{j,i})dx^i dx^j;$$

2) *gravitational energy element,*

$$ds^2 = g_{ij}dx^i dx^j;$$

3) *general energy element,*

$$d\beta^2 = \omega_i\, d^2x^i + \omega_{ij}dx^i dx^j,$$

which verify

$$d\beta^2 = d\alpha^2 + ds^2.$$

The Pfaff equation $\omega_i dx^i = 0$, $i = 1, \dots, n$ defines a distribution on M. The valuable objects

$$\frac{1}{2}(\omega_{i,j} + \omega_{j,i})$$

are the components of the second fundamental form of that distribution [6].

The potential-produced energy element is zero along integral curves of the distribution generated by the given 1-form $\omega = (\omega_i)$.

The general energy element produces the *energy functional*

$$\int \left(\omega_i \frac{d^2x^i}{dt^2} + \omega_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \right) dt, \tag{3}$$

which depends on t. This energy functional is associated to a particular Lagrangian L of order two.

Theorem. *The extremals of the energy functional (3) are solutions of the DEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + \left[\omega_{kji} - \frac{1}{2}(\omega_{k,ij} + \omega_{j,ik} - \omega_{j,ik})\right]\frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

To an energy element there may correspond a field theory. We introduce

$$\Gamma_{ijk} = \omega_{ijk} - \frac{1}{2}(\omega_{k,ij} + \omega_{j,ik} - \omega_{i,jk}),$$

where ω_{ijk} are the Christoffel symbols of ω_{ij}. After some computations we find

$$\Gamma_{ijk} = g_{ijk} + \mathrm{m}_{ijk},$$

where g_{ijk} are the Christoffel symbols of g_{ij}, and

$$\mathrm{m}_{ijk} = \mathcal{M}_{ij,k} + \mathcal{M}_{ik,j}$$

is the symmetrized derivative of the Maxwell tensor field $\mathcal{M}$.
Corollary. *The extremals of the energy functional (3) are solutions of the DEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (g_{kji} + \mathrm{m}_{kji})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

Open problem. 1) Find the linear connections [2] associated to the preceding second-order DEs.

2) Analyse the second variations of the preceding energy functionals.

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