THE BIANCHI IDENTITY FOR WEAK CONNECTIONS

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Abstract

The aim of this article is to present a generalised version of the Bianchi identity in the case of Sobolev completions of the spaces of sections and of the space of connections. The definitions of the Sobolev norms and of the Sobolev completions and the proofs of the theorems of continuous inclusions, of continuous extensions of the differential operators and of continuous multiplication can be found in [1], [2], [5], [6] and [7]. The definitions related to vector bundles and connections can be found in [2], [3], [4], [5] and [7].

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1 Introduction

Let (X, g) be a Riemannnian compact manifold of dimension n and E a hermitian vector bundle over X of rank r. We shall assume that the associated vector bundles like $\Lambda^q T^*(X)$ or $End(E)$ are endowed with the canonically induced respective metrics. We shall use the following

Notations:

$ad(E) = P \times_{ad_{U(r)}} u(r)$, where P is the principal bundle of the unitary frames in E and $u(r)$ is the Lie algebra of the unitary group $U(r)$.

$A^0(E) = \{s : X \longrightarrow E;\ s \text{ is a smooth section in } E\}$.

$A^q(X) = A^0(\Lambda^q(T^*(X))$.

$A^q(E) = A^0(\Lambda^q T^*(X) \otimes E)$.

$\mathcal{A}(E) = \{a : A^0(T(X)) \times A^0(E) \longrightarrow A^0(E);\ a \text{ unitary connection in } E\}$.

$L^p_k(A^0(E)) =$ the completion of $A^0(E)$ with respect to a L^p_k–Sobolev norm.

We make use of the following:

Theorem 1.1 *Let X be a compact manifold and $p, k \in N$ such that $1 \leq p < \infty$. Then $L^p_k(A^0(E)) \subset L^1(A^0(E))$.*

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Corollary 1.2 *Let X be a compact manifold and $p, k \in N$ such that $1 \leq p < \infty$. Then the elements of $L^p_k(A^0(E))$ can be considered as almost everywhere defined sections.*

Definition 1.3 The elements of $L^p_k(A^0(E))$ will be called **weak sections.**

We shall also use the following multiplication theorems: (see [6])

Theorem 1.4 *Let X be a compact manifold of dimension n and $p, k \in N$ such that $k > \frac{n}{p}$ and $1 \leq p < \infty$. Then $L^p_k(A^0(X \times C))$ is a Banach algebra under pointwise multiplication and, for $0 \leq j \leq k$, $L^p_j(A^0(X \times C))$ is a $L^p_k(A^0(X \times C))$-module.*

Theorem 1.5 *Let X be a compact manifold of dimension n and $k \in N$ such that $k > \frac{n}{2}$. Then $L^2_k(A^0(X \times C))$ is a Hilbert algebra under pointwise multiplication and, for $0 \leq j \leq k$, $L^2_j(A^0(X \times C))$ is a $L^2_k(A^0(X \times C))$-module.*

These theorems generalise to the extensions to convenient Sobolev completions of any bilinear map $A^0(E) \times A^0(F) \longrightarrow A^0(G)$, where E, F and G are arbitrary hermitian vector bundles.

2 The curvature of a weak connection

Remark 2.1. Let X be a compact manifold and E a hermitian vector bundle over X. Then $\mathcal{A}(E) = a_0 + A^1(ad(E))$, for any $a_0 \in \mathcal{A}(E)$.

Let $a_0 \in \mathcal{A}(E)$ be a fixed smooth connection and $p, k \in \mathbf{N}$ such that $1 \leq p < \infty$:

Definition 2.2. $\mathcal{A}^p_k(E) := a_0 + L^p_k(A^1(ad(E)))$.

Definition 2.3. The elements of $\mathcal{A}^p_k(E)$ will be called *weak connections.*

Conventions. From now on we shall denote by X a Riemannian compact manifold of dimension n and by E a hermitian vector bundle of rank r.

We shall denote by $p,\ q,\ k,\ m, \ldots$ natural numbers and we shall assume that $1 \leq p < \infty$.

Lemma 2.4 *If $a \in \mathcal{A}(E)$ then a defines a first order differential operator d_a, which extends to continuous operators*

$$d_a : L^p_k(A^0(E)) \longrightarrow L^p_{k-1}(A^1(E))$$

and, more generally, to continuous operators, that we shall denote also by d_a

$$d_a : L^p_k(A^q(E)) \longrightarrow L^p_{k-1}(A^{q+1}(E))$$

and

$$d_a : L^p_k(A^q(End(E))) \longrightarrow L^p_{k-1}(A^{q+1}(End(E))).$$

Proof. a induces, in a natural way, a connection in $End(E)$ and thus we have the associated differential operators of order 1

$$d_a : A^q(E) \longrightarrow A^{q+1}(E)$$

and

$$d_a : A^q(End(E)) \longrightarrow A^{q+1}(End(E)).$$

Since any differential operator of order m has bounded extensions from the L^p_k to the L^p_{k-m} respective Sobolev completions (see [7]), the lemma is proved. •

Lemma 2.5. *If $a \in \mathcal{A}^p_k(E)$ and $k > \frac{n}{p}$, then a extends to a continuous operators*

$$d_a : L^p_k(A^q(E)) \longrightarrow L^p_{k-1}(A^{q+1}(E)).$$

Proof. $\mathcal{A}^p_k(E) = a_0 + L^p_k(A^1(ad(E)))$. For $a = a_0 + \alpha \in a_0 + L^p_k(A^1(ad(E)))$ and $s \in L^p_k(A^0(E))$, we define $d_a(s) := d_{a_0}(s) + \alpha \wedge s$. Now we use the previous lemma and the multiplication theorems. The lemma is proven. •

Lemma 2.6. *If $k > \frac{n}{2}$ then the curvature operator $F : \mathcal{A}(E) \longrightarrow A^2(ad(E))$ extends to a continuous operator*

$$F : \mathcal{A}^p_k(E) \longrightarrow L^p_{k-1}(A^2(ad(E))).$$

Proof. For $a = a_0 + \alpha \in a_0 + L^p_k(A^1(ad(E)))$ we define $F(a) := F(a_0) + d_{a_0}(\alpha) + \alpha \wedge \alpha$. $F(a_0)$ is a constant and if $\alpha \in L^p_k(A^1(ad(E)))$ then $d_{a_0}(\alpha) \in L^p_{k-1}(A^2(ad(E)))$. And we use again the previous lemma and the multiplication theorems. •

Lemma 2.7 *The exterior derivative $d : A^q(X) \longrightarrow A^{q+1}(X)$ extends to bounded operators*

$$d : L^p_k(A^q(X)) \rightarrow L^p_{k-1}(A^{q+1}(X))$$

satisfying $d \circ d = 0$.

Proof. Since d is a differential operator of order 1, it extends continuously to any Sobolev completion. The space $A^q(X)$ is dense in $L^p_k(A^q(X))$ and on this space $d \circ d = 0$. Since $d \circ d$ is continuous and vanishes on a dense subspace the lemma is proven.

We give now the *Bianchi identity for weak connections*:

Theorem 2.8. *If $a \in \mathcal{A}^p_k(E)$ and $k > \dfrac{n}{p}$, then $d_a(F_a) = 0$.*

Proof. From Corollary 1.2 it follows that we can use local coordinate systems and local trivializations. The objects we work with are not smooth but a.e. - defined sections in different vector bundles. If a has local connection form θ and local curvature form Θ, then $\Theta = d\theta + \theta \wedge \theta$. Our generalized Bianchi identity can be locally written as $d\Theta + [\theta, \Theta] = 0$ and is a direct consequence of lemma 2.7 and of the multiplication and extension theorems. •

In the particular case of complex line bundles, i.e. $r = 1$, it is more natural and it corresponds to the classical theory, to define the curvature of a connection $a = a_0 + \alpha \in \mathcal{A}^p_k(E)$ directly by the formula $F_a = F_{a_0} + d\alpha$. With this definition the following holds:

Proposition 2.9. *If E is a complex line bundle and $a \in \mathcal{A}^p_k(E)$, then $dF_a = 0$.*

Proof. As before, we consider α such that $a = a_0 + \alpha$. Then $F_a = F_{a_0} + d\alpha$ and $dF_a = dF_{a_0} + d(d\alpha)$. Since a_0 is a smooth connection in E, $dF_{a_0} = 0$ (the classical Bianchi identity) and from lemma 2.7 it follows that $d(d\alpha) = 0$. •

Remark 2.10. The particular version of the generalised Bianchi identity in complex line bundles case holds without the condition $k > \dfrac{n}{p}$ because it allows to avoid the use of the multiplication theorems.

Such conditions cannot be avoided when we consider the curvature as operator defined on weak sections.

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