RECENT ADVANCES IN THE THEORY OF HECKE GROUPS AND λ-CONTINUED FRACTIONS

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Abstract

The Rosen or λ-continued fractions (λCF_s) are related to G_q, $q \in \mathbb{N}^*$, $q \geq 4$, the Hecke (triangle) group of index q. Using explicit planar natural extensions for the associated interval maps, we find the transition operator corresponding to the dynamical system underlying λCF. We prove that the associated RSCC (random system with complete connections) is with contraction and its transition operator is regular with respect to $L(I)$ = the Banach space of Lipschitz functions on $I = \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, where $\lambda = 2\cos\frac{\pi}{q}$.

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1 Introduction

The Rosen fractions form an infinite family which generalizes the nearest-integer continued fractions (see [5]). Although D.Rosen [7] introduced his infinite family of continued fractions in the mid-1950s, it is very recently that there has been any investigation of their metric properties [8], [6], [3]. It has been found in 1998 (see [1]) explicit planar natural extensions for each of the interval maps associated to the Rosen fractions.

The Rosen maps are naturally divided into two subfamilies-those of odd index and those of even. The fine behaviour of the transformations is starkly different in these settings.

We mention some related theory. Underlying the family of transformations which we discuss here is an example of what J.Wolfart [10] called a *discrete deformation of Fuchsian groups*. Wolfart showed that for any Riemann surface with a cusp, there is a family of Riemann surfaces with quotient singularities which has the given cusped surface as its limit (in the Chabauty topology).

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The groups which underlie our interval maps are Fuchsian groups of the first kind, acting upon the Poincaré upper half-plane by Möbius (fractional linear) transformations, with all of $\mathbb{R}$ as their limit sets. We avoid most of this theory, but the reader may wish to consult [2] for related discussion.

2 Preliminaries

Let $\lambda = \lambda_q$ equal $2\cos\frac{\pi}{q}$ for $q \in \{3, 4, \ldots\}$; $S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Then the group G_q generated by S and T is called the *Hecke (triangle) group of index* q. All relations in the group arise from $T^2 = I$ and $U^q = I$, where $U = ST$ and I represents the *projective* identity.

D.Rosen [7] defined a continued fraction related to G_q, $q \geq 4$. Fix some such q and let $I_q = \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$. Then the map

$$\tau_q : I_q \longrightarrow I_q$$

(1)
$$\tau_q(x) = |x^{-1}| - \left[|x^{-1}|\lambda^{-1} + \frac{1}{2}\right]\lambda,\ x \neq 0; \quad \tau_q(0) = 0,$$

gives a shift map on continued fraction expansions of the type

(2)
$$x = \frac{\varepsilon_1}{a_1\lambda + \tau_q(x)} = \cfrac{\varepsilon_1}{a_1\lambda + \cfrac{\varepsilon_2}{a_2\lambda + \ddots}} = [\varepsilon_1 a_1, \varepsilon_2 a_2, \ldots],$$

where $\varepsilon_i \in \{\pm 1\}$ and $a_i \in \mathbb{N}$. We call this the *Rosen* or λ*-expansion* (λCF) of x. Of course, the various ε_i and a_i depend on x. So, we have

(3)
$$\varepsilon_n(x) = sgn(\tau_q^{n-1}(x)),\ a_n(x) = \left[\left|(\tau_q^{n-1}(x))^{-1}\right|\lambda^{-1} + \frac{1}{2}\right],\ n \in \mathbb{N}^*,$$

in case $\tau_q^{n-1}(x) \neq 0$, and $\varepsilon_n(x) = 0$, $a_n(x) = \infty$ in case $\tau_q^{n-1}(x) = 0$.

In analogy to the classical setting, we call x a G_q*-irrational* if x has a Rosen expansion of infinite length. There are restrictions on the set of admissible sequences of ε_i and a_i. For G_q-irrationals, these restrictions are determined simply by the orbit of $\frac{\lambda}{2}$.

Setting

$$\frac{p_n}{q_n} = [\varepsilon_1 a_1, \varepsilon_2 a_2, \ldots, \varepsilon_n a_n],$$

we find that

$$p_n = a_n\lambda p_{n-1} + \varepsilon_n p_{n-2},$$
$$q_n = a_n\lambda q_{n-1} + \varepsilon_n q_{n-2}.$$

If we replace the tail of a continued fraction, we have

$$[\varepsilon_1 a_1, \varepsilon_2 a_2, \ldots, \varepsilon_n(a_n\lambda + y)] = \frac{p_n + p_{n-1}y}{q_n + q_{n-1}y}.$$

To obtain precise values of specific expansions, we utilize a sequence of polynomials in λ, already discussed by H. Weber [9].

$$\begin{cases} B_0 = 0; B_1 = 1 \\ B_n = \lambda B_{n-1} - B_{n-2}, \quad n \geq 2. \end{cases}$$

Note that

$$U^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}.$$

Now, [9], gives

$$B_n = \frac{\sin\frac{n\pi}{q}}{\sin\frac{\pi}{q}}.$$

Interpreting U as a Möbius transformation, one finds $U^{-1}(x^{-1}) = \dfrac{1}{U(x)}$ and thus, $U^{q-j}(x^{-1})\cdot U(x) = 1$. Since $U(x) = \lambda - x^{-1}$, we have $U^j(\lambda) = \lambda - U^{q-j-2}(\lambda)$. Now, if $q = 2p$ is even one solves to find $U^{p-1}(\lambda) = \dfrac{\lambda}{2}$; similarly, if $q = 2h+3$ is odd, one finds $U^h(\lambda) = 1$. Slightly more involved again with $q = 2h+3$, $U^h \cdot S^2T \cdot U^{h-1}(\lambda) = \dfrac{2}{\lambda}$. The above is in accordance with the formulas of [7].

The orbit of $-\dfrac{\lambda}{q}$ under τ_q is of extreme importance. We define ϕ_j to be $\tau_q^j\left(-\dfrac{\lambda}{2}\right)$, with $\phi_0 = -\dfrac{\lambda}{2}$. Thus for even q, $\phi_j = S^{-1}T(\phi_{j-1})$, for $j \in \{1, \ldots, p-1\}$ and $\phi_{p-1} = 0$. For $q = 2h+3$, we have $\phi_j = S^{-1}T(\phi_{j-1})$, for $j \in \{0, \ldots, h-1\} \cup \{h+1, \ldots, 2h\}$; $\phi_{h+1} = S^{-2}T(\phi_h)$ and $\phi_{2h+1} = 0$. That is, ϕ_j has for its expansion the j-th shift of that of ϕ_0.

3 Natural extensions for the Rosen maps

Consider (see [1]) the so-called *natural extension* $\mathcal{T}$ for any Rosen interval maps, which is defined as follows: for any $q \geq 4$, fixed, let $\lambda = \lambda_q$, $\tau(x)$ be $\tau_q(x)$ and

$$\mathcal{T}(x, y) = \left(\tau(x), \frac{1}{a\lambda + \varepsilon y}\right)$$

where we have suppressed the dependence of $a = a_1$ and $\varepsilon = \varepsilon_1$ on x.

Also notice that $\mathcal{T}$ is a transformation which on the first coordinate is simply the interval map while on the second coordinate it is directly related to the "past" of the first coordinate.

It has been shown in [1], that $\mathcal{T}$ is a bijective transformation of a domain Ω in $\mathbf{R}^2$ except for a set of Lebesgue measure zero. We consider two cases.

3.1 Even indices

Let $q = 2p$, with $p \geq 2$. Now let I be the interval I_q. Putting $\phi_j = \tau^j\left(-\frac{\lambda}{2}\right)$, with $\phi_0 = -\frac{\lambda}{2}$, we construct a partition of I by considering the intervals

$J_j = [\phi_{j-1}, \phi_j)$ for $j \in \{1, \ldots, p-1\}$,

$J_p = \left[0, \frac{\lambda}{2}\right)$.

Furthermore, let $K_j = [0, L_j]$, $j \in \{1, \ldots, p-1\}$ and $K_p = [0, R]$, where L_j, $1 \leq j \leq p-1$, and R satisfy the system

$$\begin{cases} R = \lambda - L_{p-1} \\ L_1 = \dfrac{1}{\lambda + R} \\ L_j = \dfrac{1}{\lambda - L_{j-1}} \quad \text{for } j \in \{2, \ldots, p-1\} \\ R = \dfrac{1}{\lambda - L_{p-1}}. \end{cases}$$

Let $\Omega = \bigcup_{k=1}^{p} J_k \times K_k$. If the above system admits a unique real solution, then Ω is the domain in $\mathbf{R}^2$ on which $\mathcal{T}$ is bijective except for a set of Lebesgue measure zero. Moreover, for this solution, $R = 1$.

If $n \geq 2$, then

$$\mathcal{T}(x, y) = \begin{cases} \left(\tau(x), \dfrac{1}{n\lambda + y}\right), & \text{if } (x, y) \in \left(\dfrac{2}{(2n+1)\lambda}, \dfrac{2}{(2n-1)\lambda}\right) \times [0, 1], \\ \left(\tau(x), \dfrac{1}{n\lambda - y}\right), & \text{if } (x, y) \in \left(\dfrac{-2}{(2n-1)\lambda}, \dfrac{-2}{(2n+1)\lambda}\right) \times [0, L_{p-1}]. \end{cases}$$

If $n = 1$, then

$$\mathcal{T}(x, y) = \begin{cases} \left(\tau(x), \dfrac{1}{\lambda + y}\right), & \text{if } (x, y) \in \left[\dfrac{2}{3\lambda}, \dfrac{\lambda}{2}\right) \times [0, 1], \\ \left(\tau(x), \dfrac{1}{\lambda - y}\right), & \text{if } (x, y) \in \left(\bigcup_{j=1}^{p-2} J_j \times K_j\right) \cup \left(\left[\phi_{p-2}, \dfrac{-2}{3\lambda}\right) \times K_{p-1}\right). \end{cases}$$

Also, the normalized invariant measure for the planar natural extension $\mathcal{T}$ is readily found (see [6]). So, $\mathcal{T}$ preserves the probability measure ν with density $\frac{C}{(1+xy)^2}$, where C is a normalizing constant. Actually, for q even, the constant C such that ν is a probability measure on Ω is given by

$$C = \frac{1}{\ln\left[\frac{\left(1+\cos\frac{\pi}{q}\right)}{\sin\frac{\pi}{q}}\right]}.$$

3.2 Odd indices

Now, fix an odd q and recycle notation as above. Let I be the interval I_q and $\phi_j = \tau^j\left(-\frac{\lambda}{2}\right)$, with $\phi_0 = -\frac{\lambda}{q}$. Putting $h = \frac{q-3}{2}$, we construct a partition of I by considering the intervals J_j, $j \in \{1, \ldots, 2h+2\}$, where

$J_{2k} = [\phi_{h+k}, \phi_k)$ for $k \in \{1, \ldots, h\}$,

$J_{2k+1} = [\phi_k, \phi_{h+k+1})$ for $k \in \{0, 1, \ldots, h\}$,

$J_{2h+2} = \left[0, \frac{\lambda}{2}\right)$.

Let $K_j = [0, L_j]$, $j \in \{1, \ldots, 2h+1\}$, and let $K_{2h+2} = [0, R]$, where L_j, $1 \le j \le 2h+1$, and R satisfy by system

$$\begin{cases} R = \lambda - L_{2h+1} \\ L_1 = \dfrac{1}{2\lambda - L_{2h}} \\ L_2 = \dfrac{1}{2\lambda - L_{2h+1}} \\ L_j = \dfrac{1}{\lambda - L_{j-2}} \quad \text{for } 2 < j < 2h+2 \\ R = \dfrac{1}{\lambda - L_{2h}}. \end{cases}$$

Let $q = 2h+3$, with $h \ge 1$ and $\Omega = \bigcup_{j=1}^{2h+2} J_j \times K_j$. If the above system admits a unique real solution, then Ω is the domain in $\mathbf{R}^2$ on which $\mathcal{T}$ is bijective except for a set of Lebesgue measure zero. Moreover, for this solution, R satisfies the equation

$$R^2 + (2-\lambda)R - 1 = 0.$$

In particular, $\frac{\lambda}{2} < R < 1$. Also the normalizing constant C such that ν is a probability measure on Ω, with density $\frac{C}{(1+xy)^2}$, is given by $C = \frac{1}{\ln(1+R)}$.

4 The Associated Transition Operator

Let $(I, \mathcal{B}_I, \rho, \tau)$ be the dynamical system underlying λCF, where $I = \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, $\mathcal{B}_I$ is the collection of Borel sets on I, ρ is the invariant measure of τ (projecting ν, we find the invariant measure ρ, of τ) and τ is defined as in (1).

Let $(\Omega, \mathcal{B}_\Omega, \nu, \mathcal{T})$ be the natural extension of $(I, \mathcal{B}_I, \rho, \tau)$, where Ω, ν and $\mathcal{T}$ are defined in Section 3.

Let μ be an arbitrary non-atomic probability on $\mathcal{B}_I$ and define

$$F_n(x) = F_n(x, \mu) = \mu\left(\tau^n \in \left[-\frac{\lambda}{2}, x\right)\right), \quad n \in \mathbf{N},\ x \in I.$$

Clearly, $F_0(x) = \mu\left(\left[-\frac{\lambda}{2}, x\right)\right)$ because τ^0 is the identity map. Since $-\frac{\lambda}{2} \leq \tau^{n+1} < x$ iff

$$(x + a_{n+1}(x)\lambda)^{-1} < \varepsilon_{n+1}(x)\tau^n(x) \leq \left(-\frac{\lambda}{2} + a_{n+1}(x)\lambda\right)^{-1},$$

we can write the Gauss-Kuzmin type equation as

$$F_{n+1}(x) = \sum_{(l,i)\in X} l\left[F_n\left(\frac{l}{-\frac{\lambda}{2} + i\lambda}\right) - F_n\left(\frac{l}{x + i\lambda}\right)\right], \ n \in \mathbf{N},\ x \in I,$$

$$X = \{-1, 1\} \times \mathbf{N}^*.$$

Assuming that for some $m \in \mathbf{N}$ the derivative F'_m exists everywhere in I and is bounded, it is easy to see by induction that F'_{m+n} exists and is bounded for all $n \in \mathbf{N}^*$, and we have

$$F'_{n+1}(x) = \sum_{(l,i)\in X} \frac{1}{(x + i\lambda)^2} F'_n\left(\frac{l}{x + i\lambda}\right), \quad n \geq m,\ x \in I.$$

Further, write

$$f_n(x) = \frac{1 + xH_x}{CH_x} \cdot F'_n(x), \quad x \in I,$$

where

$$H_x = \begin{cases} L_j, & \text{if } x \in J_j,\ j \in \{1, \ldots, p-1\} \\ 1, & \text{if } x \in J_p \end{cases}$$

in the even case (see Subsection 3.1), and

$$H_x = \begin{cases} L_j, & \text{if } x \in J_j,\ j \in \{1, \ldots, 2h+1\} \\ R, & \text{if } x \in J_{2h+2} \end{cases}.$$

in the odd case (see Subsection 3.2), to get

$$f_{n+1} = V f_n, \quad n \geq m,$$

with V being the linear operator defined as

$$Vf(x) = \sum_{(l,i)\in X} q_{li}(x) f(v_{li}(x)), \quad f \in B(I),\ x \in I,$$

where $B(I)$ is the Banach space of bounded measurable complex-valued functions f on I under the supremum norm $|f| = \sup_{x\in I} |f(x)|$,

$$q_{li}(x) = \begin{cases} 0, & \text{if } x \in \left[-\frac{\lambda}{2}, \frac{2}{\lambda} - \lambda\right],\ (l,i) \in \{(-1,1),(1,1)\} \\ \dfrac{H_{\frac{1}{x+i\lambda}}}{H_x} \cdot \dfrac{1 + xH_x}{(x+i\lambda)\left(x + i\lambda + lH_{\frac{1}{x+i\lambda}}\right)}, & \text{otherwise} \end{cases}$$

and

$$v_{li}(x) = \begin{cases} \dfrac{\lambda l}{2}, & \text{if } x \in \left[-\frac{\lambda}{2}, \frac{2}{\lambda} - \lambda\right],\ (l,i) \in \{(-1,1),(1,1)\} \\ \dfrac{l}{x+i\lambda}, & \text{otherwise.} \end{cases}$$

Note that here V is the *transition operator* of the homogeneous Markov chain $(y_n)_{n\geq 0}$ defined as

$$y_n = [\varepsilon_n a_n, \varepsilon_{n-1} a_{n-1}, \ldots, \varepsilon_1 a_1], \quad n \in \mathbf{N}^*,$$

and $y_0 = 0$. Clearly, $(y_n)_{n\geq 0}$ satisfies the recursion equation

$$y_n = \frac{\varepsilon_n}{a_n\lambda + y_{n-1}},\ n \in \mathbf{N}^*,\ y_0 = 0,$$

and $y_n \in I$, $\forall n \in \mathbf{N}$.

5 The Main Result

Here we restrict our attention to the transition operator V introduced in the previous section. In this section we deal with the properties of V on function spaces different from $B(I)$.

In connection with the operator V, we obtain the following results. First, if we define

$$V^{\infty} f = \int_I f(x)\rho(dx), \; f \in B(I),$$

then we have $V^{\infty} V^n f = V^{\infty} f$ for all $f \in B(I)$ and $n \in \mathbf{N}^*$. Second, let $L(I)$ be the Banach space of all bounded complex-valued Lipschitz functions on I under the usual norm $||f||_L = |f| + s(f)$, where

$$s(f) = \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|}.$$

Then V takes boundedly $L(I)$ into itself. Moreover, we have the following result.

Theorem. *Let be given* $I = \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ *and* $\mathcal{B}_I$ *the collection of Borel sets on* I. *Consider* $X = \{-1, 1\} \times N^*$, $\mathcal{X} = \mathcal{P}(X)$,

$$v : I \times X \to I$$

$$v(x, (l, i)) = v_{li}(x)$$

and

$$Q : I \times X \to [0, 1]$$

$$Q(x, (l, i)) = q_{li}(x).$$

Then the (RSCC) $\{(I, \mathcal{B}_I), (X, \mathcal{X}), v, Q\}$ *is with contraction and its transition operator* V *is regular with respect to* $L(I)$.

Proof. We have for all $(l, i) \in X$

$$\sup_{x \in I} \left| \frac{d}{dx} v(x, (l, i)) \right| \leq \sup_{x \in (\frac{2}{\lambda} - \lambda, \frac{\lambda}{2}]} \frac{1}{(x + \lambda)^2} = \frac{\lambda^2}{4} < 1,$$

$$\sup_{x \in I} \left| \frac{d}{dx} Q(x, (l, i)) \right| < \infty.$$

Hence the requirements of definition of an RSCC with contraction are met with $k = 1$ (see Definition 3.1.15 in [4]). By Theorem 3.1.16 in [4], it follows that the Markov chain $(y_n)_{n \geq 0}$ associated with the RSCC $\{(I, \mathcal{B}_I), (X, \mathcal{X}), v, Q\}$ is a Doeblin-Fortet chain. Hence by Definition 3.2.1 in [4], the Markov chain $(y_n)_{n \geq 0}$ is compact and its transition operator is a Doeblin-Fortet operator.

To prove the regularity of V with respect to $L(I)$ let us define recursively $x_{n+1} = (x_n + 2)^{-1}$, $n \in \mathbf{N}$, with $x_0 = x$. A criterion of regularity is expressed in Theorem 3.2.13 in [4], in terms of the supports $\sum_n(x)$ of the n-step transition probability functions $Q^n(x, \cdot)$, $n \in \mathbf{N}^*$, where with the usual notation

$$Q(x, B) = \sum_{\{(l,i) \in X \mid v(x,(l,i)) \in B\}} Q(x, (l, i)), \quad x \in I, \; B \in \mathcal{B}_I.$$

Clearly $x_{n+1} \in \sum_1(x_n)$ and therefore Lemma 3.2.14 in [4] and an induction argument lead to the conclusion that $x_n \in \sum_n(x)$, $n \in \mathbb{N}^*$. But $\lim\limits_{n\to\infty} x_n = \sqrt{2}-1$ for any $x \in I$. Hence

$$d\left(\textstyle\sum_n(x), \sqrt{2}-1\right) \leq |x_n - \sqrt{2}+1| \to 0 \quad \text{as} \quad n \to \infty,$$

where $d(x,y) = |x-y|$, $\forall x, y \in I$.

Now, the regularity of V with respect to $L(I)$ follows from Theorem 3.2.13 in [4]. The proof is complete.

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