SELF-SIMILAR SETS FROM RANDOM AFFINE TRANSFORMATIONS

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Abstract

In Physics and Biology as well as in other natural sciences we can find models which generate fractal structures. Such kind of structures identifies systems in which increasing detail is revealed by increasing magnification, and the newly revealed structure looks the same as that one can observe at lower magnification. This implies invariance under changes of the length-scale or self-similarity.

In this paper, we obtain self-similar sets as invariants under random affine transformations. Some topological properties which prove connections between self-similar sets are discussed. Sufficient conditions which imply that a set is a self-similar one are introduced and studied. Finally, we illustrate our ideas with two relevant computer experiments. By using computer simulation, we obtain our examples of self-similar sets in R^2.

1 Preliminaries

Let (X, d) be a metric space. Recall that a mapping $w: X \to X$ is called *contractive* if there exists a constant $0 \leq r < 1$ such that $d(w(x), w(y)) \leq rd(x, y)$ for every $x, y \in X$. Any such number r is called a *contractivity factor* for w.

Let $\mathcal{S} = \{w_1, w_2, \ldots, w_N\}$ be a finite set of contraction mappings on R^n. The following definition is in order

Definition 1.1 *We say that the compact set $K \subset R^n$ is invariant with respect to $\mathcal{S}$ if*

$$K = \bigcup_{i=1}^{N} w_i(K).$$

Hutchinson, [4], proves the following result.

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Theorem 1.1 *Given a finite set S of contraction mappings on R^n, there exists a unique compact set K invariant with respect to S. Furthermore, K is the limit of various approximating sequences of sets which can be constructed from S.*

In order to make a background for our results in this paper, we need the following definition, [9].

Definition 1.2 *Two metric spaces X and Y are called similar if there is a surjection mapping $w: X \to Y$ and a positive constant r satisfying the equality $d(w(x), w(y)) = rd(x, y)$ for all $x, y \in X$. Such a mapping is called a similarity.*

We shall denote by $\mu_r: R^n \to R^n$ the homothety $\mu_r(x) = rx$ $(r \geq 0)$ and by $\tau_b: R^n \to R^n$ the translation $\tau_b(x) = x - b$. We remark that the following statements are equivalent:

i) $w: R^n \to R^n$ is a similarity;

ii) there exist some homothety μ_r, translation τ_b, and orthonormal transformation $\mathcal{O}$ such that $w = \mu_r \tau_b \mathcal{O}$.

We will conclude this section with some topological properties suggested by the considerations given above. For this purpose, let us consider in the following (X, τ) be a general topological space. If x is a point of X, then by $\mathcal{N}(x)$ we shall denote the set of all neighbourhoods of x.

Definition 1.3 *X is called self-similar if for any $U \in \tau$, $U \subseteq X$, there is a set $V \subseteq U$ such that V is similar to X. Moreover, if $IntV \neq \emptyset$ then X is called strongly self-similar.*

Proposition 1.1 *Any strongly self-similar space is self-similar.*

Definition 1.4 *X is called pointwise self-similar if for any point $x \in X$ and for any $U \in \mathcal{N}(x)$, there is a set V such that $x \in V \subseteq U$ and V is similar to X. Moreover, if $V \in \mathcal{N}(x)$ then X is called strongly pointwise self-similar.*

From these considerations we get the two results in the following.

Proposition 1.2 *If X is pointwise self-similar space then it is self-similar.*

Proposition 1.3 *Suppose X is strongly pointwise self-similar space. Then the following statements hold:*

a) *X is pointwise self-similar;*

b) *X is strongly self-similar.*

How could we obtain these self-similar sets? The answer is given in the next section.

2 Constructing self-similar sets

In this section, we exhibit a method of constructing self-similar sets. The idea of this method comes from [1] and its details are given in the following. We recall some common notions and after that we state our results.

For the following results of this paper, unless mentioned otherwise, we shall consider $X = R^n$.

Let w be an *affine transformation* of X given by $w(x) = Tx + b$ where T is an $n \times n$ matrix and b is some fixed vector in X. We shall refer to T as the *derivative* of w. We remark that a general affine transformation in X consists of a linear transformation T, which deforms space relative to the origin, followed by a translation specified by the vector b.

Let $w_i: X \to X$, $w_i(x) = T_i x + b_i$, $i \in \{1, 2, \dots, m\}$ be affine transformations of X. The matrix of the composite function $w_1 w_2 \cdots w_m$ is $T_1 T_2 \cdots T_m$. The matrix of w^n is T^n where

$$w^n(x) = \underbrace{w \circ w \circ \cdots \circ w}_{n \text{ times}}(x)$$

is the n^{th} iteration of w. So the study of w^n is transformed to that of T^n. Moreover, after calculation, we get

Proposition 2.1 *Suppose $I - T$ is invertible, where I is the $n \times n$ identity matrix. Then the orbit of a point z_0 under w is the sequence $(z_k)_{k \geq 0}$ defined by the formula*

$$\begin{aligned} &z_0 \quad \text{given} \\ &z_k = T^k z_0 + (I - T)^{-1}(I - T^k)b, \quad k \geq 1. \end{aligned}$$

The following result can be proved by induction

Lemma 2.1 *For each finite sequence, $w_{i_1}, w_{i_2}, \dots, w_{i_p}$ from the set $\{w_1, w_2, \dots w_m\}$ and for all $x, y \in X$, we have*

$$\|w_{i_1} w_{i_2} \cdots w_{i_p}(x) - w_{i_1} w_{i_2} \cdots w_{i_p}(y)\| = \|T_{i_1} T_{i_2} \cdots T_{i_p} x - T_{i_1} T_{i_2} \cdots T_{i_p} y\|.$$

Using Lemma 2.1 we find

Theorem 2.1 *An affine mapping w is a contraction if and only if its derivative is a contraction.*

It follows that an affine mapping of X is a contraction if and only if the norm of its derivative is less than 1. So, any similarity w is affine transformation, [6]. Moreover, if $\|T\| < 1$ then w is a contraction mapping on X. If $\mathcal{S}$ is a set of affine contractions on X, then the unique compact invariant K for the w_i given by Theorem 1.1 can be a self-similar set. The existence and the uniqueness of this set follow from the contraction mapping principle, [7], as follows.

We consider X be a compact metric space. If $A \subset X$ is a nonempty subset of X and r a positive number then we put

$$N_r(A) = \{x \in X | \text{there is } a \in A \text{ such that } d(a, x) < r\}.$$

As we know, for two nonempty compact subsets A and B of X, the Hausdorff metric h is defined as $h(A,B) = \inf\{r | A \subseteq N_r(B) \text{ and } B \subseteq N_r(A)\}$. Denote by $\mathcal{C}$ the hyperspace of all nonempty compact subsets of X endowed with the Hausdorff metric. Let $\mathcal{S}: \mathcal{C} \to \mathcal{C}$ be defined by

$$\mathcal{S}(A) = \bigcup_{i=1}^{N} w_i(A).$$

Then $\mathcal{S}$ is a contraction mapping having contractivity factor $\max_{1 \leq i \leq N} r_i$. Therefore, according to the contraction mapping principle, there exists exactly one fixed set under $\mathcal{S}$, that is a set $A \in \mathcal{C}$ such that $\mathcal{S}(A) = A$. Such a set A can be obtained as the limit of $\mathcal{S}^n(B)$ for any $B \in \mathcal{C}$. Then we will write $A = \mathcal{S}(X; \{w_i | i = \overline{1,N}\})$.

Taking into account Definition 1.1 and Theorem 1.1 we deduce

Theorem 2.2 *Suppose X is a compact metric space and let $\mathcal{S}$ be a finite set of contracting imbeddings of X. Then the invariant set of X with respect to $\mathcal{S}$ is self-similar.*

The next section deals with a simulation method of self-similar sets. This simulation method is based on products of random matrices (see Furstenberg and Kesten, [3]).

3 Illustrations

In this section we will exhibit the results of some numerical experiments. All calculations were performed on a regular *PC* with coprocessor and an *HP* printer. The computer programs used, [8], were all written in Turbo Pascal version 6.1 and needs arround 500 *K*. It takes between 5 and 15 minutes to produce an image. The program is available upon request from the first author on a $5\frac{1}{4}$-inch floppy disk.

To state our computer algorithm, in the following we deal with the set of affine mappings

$$\mathcal{F} = \{x \in X \mapsto w(x) = Tx + b\},$$

By all means, the pair $\{X, w(x)\}$ composes a dynamical system.

Let us consider without loss of generality $n = 2$ and the set $\mathcal{T} = \{T_0, T_1, \ldots, T_r\}$ of $(r+1)$ non-singular 2×2 matrices which correspond to some affine transformations of $\mathcal{F}$. Let us also consider $W = R^n$, $X = \mathcal{T}$, $\mathcal{W}$ the borelians of W, and $\mathcal{X}$ the set of all parts of X. If $T_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathcal{T}$, $n \geq 1$ then we build the sequence $(z_n)_{n \geq 0}$ as follows:

$$z_0 \text{ given}, \quad z_1 = w_1(z_0), \quad z_2 = (w_2 \circ w_1)(z_0), \ldots.$$

If we use the method studied in [2] by Corbu and Postolache we obtain the representation of this sequence with matrices as

$$\begin{aligned} &z_0 \quad \text{given} \\ &z_{k+1} = T^{(k)} z_0 + T^{(k)} \left[(T_k \cdots T_0)^{-1} b_k + \cdots + (T_1 T_0)^{-1} b_1 + T_0^{-1} b_0 \right] \end{aligned}$$

where $T^{(k)} = T_k T_{k-1} \cdots T_0$.

Let be given a probability measure $\{\rho_x(s),\ s \in W\}_{x\in X}$ with $\rho_x(s) = \rho_x > 0$ and $\sum_{x\in X} \rho_x(s) = 1$. Moreover, for all $s \in W$ we consider the family of applications $w: W \times X \to W$ which are $(\mathcal{W}\otimes\mathcal{X}, \mathcal{W})$–measurable, $w \in \mathcal{F}$. By these considerations, we have the following

Definition 3.1 *The tuple $\{(W,\mathcal{W}),(X,\mathcal{X}),w,\rho\}$ is called random system with complete connections.*

Remark 3.1 *The sequence $(z_n)_{n\geq 0}$, obtained above, where for each n a random $w_n(x)$ is choosen defines a Markov chain.*

The transition probability of the Markov chain in Remark 3.1 is

$$P(z, A) = \sum_{x\in X} I_A(w_x(z))\rho_x \tag{4.1}$$

for all $z \in W$ and $A \in \mathcal{W}$. This is the Markov chain attached to the random system with complete connections, [5].

Let us suppose, without loss of generality, that the matrices T_k are positive (the general case is similar). For practical reasons we have to study the asymptotic behaviour of the following product of random matrices $T^{(k)} = T_k \cdots T_1 T_0$. For this purpose, we consider a positive matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its decomposition as a product of a diagonal matrix and a stochastic one as follows:

$$D = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix}, \quad P = \begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b} \\ \frac{c}{c+d} & \frac{d}{c+d} \end{pmatrix}.$$

Therefore we have $T = D \cdot P$. Using this decomposition for $T^{(k)}$ we have

$$\begin{aligned} T^{(k)} &= D_k P_k T_{k-1} \cdots T_0 \\ &= D_k \bar{T}_{k-1} T_{k-2} \cdots T_0 \\ &= D_k D_{k-1} P_{k-2} T_{k-3} \cdots T_0 \\ &\ \vdots \\ &= D_0 \cdots D_k P_k. \end{aligned}$$

Here $D_0, \ldots, D_k$ are diagonal matrices, P_k is a stochastic one and $\bar{T}_{k-1} = P_k T_{k-1}$. In general, if we denote by $\bar{T}$ the product between a general matrix and a stochastic one, that is, $\bar{T} = TP$, thus from the componentwise interpretation we get

$$\min_{i,j} t_{ij} < \min_{i,j} \bar{t}_{ij} < \max_{i,j} \bar{t}_{ij} < \max_{i,j} t_{ij}. \tag{4.2}$$

If we denote

$$T^{(k)} = \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{pmatrix}, \quad D_0 \cdots D_k = \begin{pmatrix} h^{(k)} & 0 \\ 0 & g^{(k)} \end{pmatrix}, \quad P_k = \begin{pmatrix} p_{1k} & 1-p_{1k} \\ 1-p_{2k} & p_{2k} \end{pmatrix},$$

then we have

$$T^{(k)} = \begin{pmatrix} h^{(k)}p_{1k} & h^{(k)}(1-p_{1k}) \\ g^{(k)}(1-p_{2k}) & g^{(k)}p_{2k} \end{pmatrix}.$$

Based on the inequalities (4.2) and taking into account the positivity of T_k, it followss that $\delta > 0$ and $k_0 > 1$ exists such that $p_{1k} > \delta$, $p_{2k} > \delta$ for all $k > k_0$. Then we obtain

Proposition 3.1 *In the conditions described above, there exists the limit*

$$\lim_{k\to\infty} \frac{1}{k} \log a^{(k)}.$$

By Proposition 3.1 it follows that if k is large enough then $T^{(k)}$ approaches to a matrix with proportional rows $T^{(k)} = \begin{pmatrix} e^{kh} & e^{kg} \\ e^{kh} & e^{kg} \end{pmatrix}$, where h and g are positive random variables.

Remark 3.2 *Similar results to that in Proposition 3.1 for $b^{(k)}$, $c^{(k)}$ and $d^{(k)}$ can be obtained.*

The theoretical approach given above, suggest the following procedure for finding self-similar sets, [1]:

Algorithm 3.1 *Let X be a general compact metric space and $w_i: X \to X$ be contraction mappings with*

$$d(w_i(x), w_i(y)) \le rd(x,y), \quad \textit{for all } x, y \in X,$$

for $i = 1, 2, \ldots, N$, where $0 \le r < 1$. Let $\{p_1, p_2, \ldots, p_N\}$ be probabilities with $p_i > 0$ and $\sum p_i = 1$. Choose $x_0 \in X$ and pick recursively

$$x_n \in \{w_1(x_{n-1}), w_2(x_{n-1}), \ldots, w_N(x_{n-1}),\}, \quad \textit{for } n = 1, 2, \ldots, M$$

where M is a large integer and $p_i = P(x_n = w_i(x_{n-1}))$.

We have used Algorithm 3.1 as basic procedure for obtaining the self-similar sets included in this paper. In this respect, the twig in Fig. 1 is obtained as invariant set of just two affine transformations $w_i: R^2 \to R^2$ $(i = 1, 2)$ as follows

$$w_1(x, y) = (0.81x + 0.011y; 0.03x + 0.41y) + (0.009; 0.1)$$
$$w_2(x, y) = (0.12x + 0.78y; 0.8x + 0.05y) + (0.1; 0.21).$$

We generated $v_0 = (x_0, y_0) \in R^2$ as random point, [7], and we defined a set of points $\{v_n \in R^2/n = 0, 1, \ldots, 10^5\}$ recursively according to

$$v_{n+1} = \begin{cases} w_1 v_n & \text{with} \quad p_1 = 0.76 \\ w_2 v_n & \text{with} \quad p_2 = 0.24 \end{cases}$$

If those points from the set which lie in the square $\{(x,y)/-1 \leq x,y \leq 1\}$ are plotted the result will be similar to Fig. 1.

The self-similar set in Fig. 2 is obtained as invariant set of the following two affine transformations: $w_i\colon R^2 \to R^2$ $(i = 1,2)$

$$w_1(x,y) = (0.85x - 0.19y; 0.19x + 0.85y) + (0.1; 0.1)$$
$$w_2(x,y) = (0.19x + 0.6y; 0.2x - 0.03y) + (0.1; 0.2).$$

If $v_0 = (x_0, y_0) \in R^2$ as a random point, and we define the set of points $\{v_n \in R^2/n = 0, 1, \ldots, 10^5\}$ recursively according to

$$v_{n+1} = \begin{cases} w_1 v_n & \text{with} \quad p_1 = 0.88 \\ w_2 v_n & \text{with} \quad p_2 = 0.12 \end{cases}$$

then the result is similar to Fig. 2.

The resulting pictures in Figs. 1 and 2 respectively appear to be the same no matter which initial point v_0 is choosen. Also, it is obvious that these sets share the self-similarity property.

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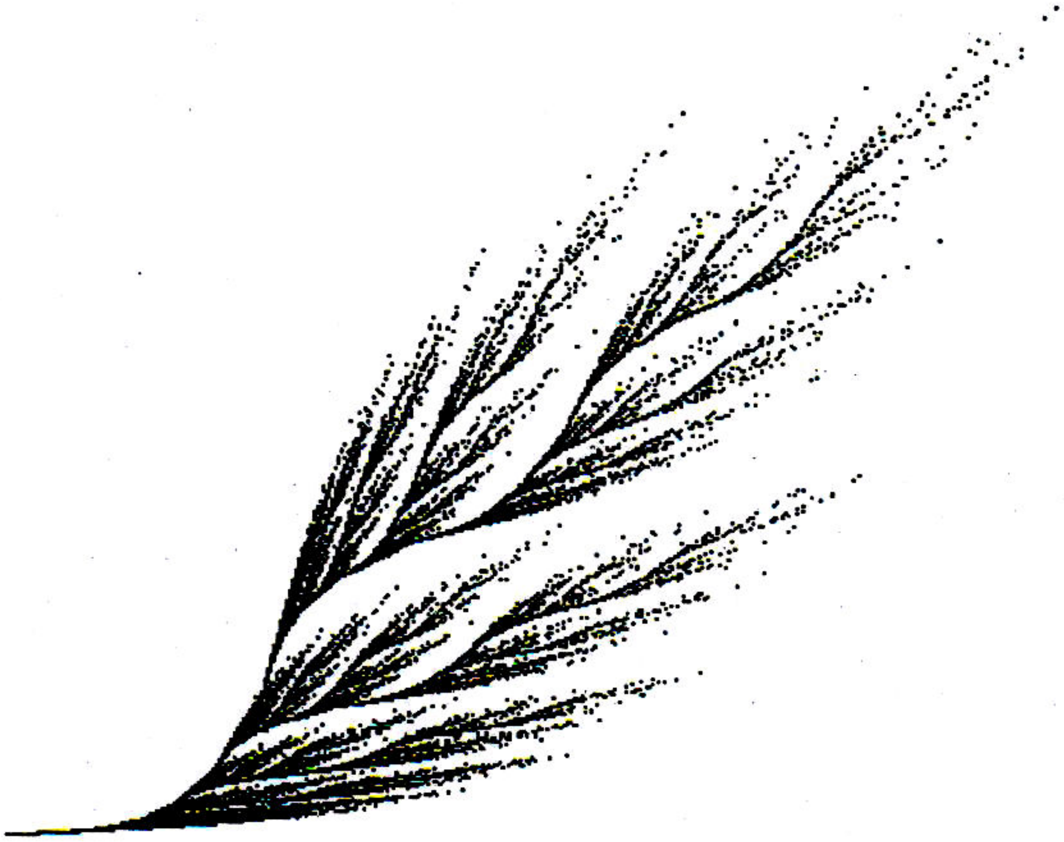

Fig. 1

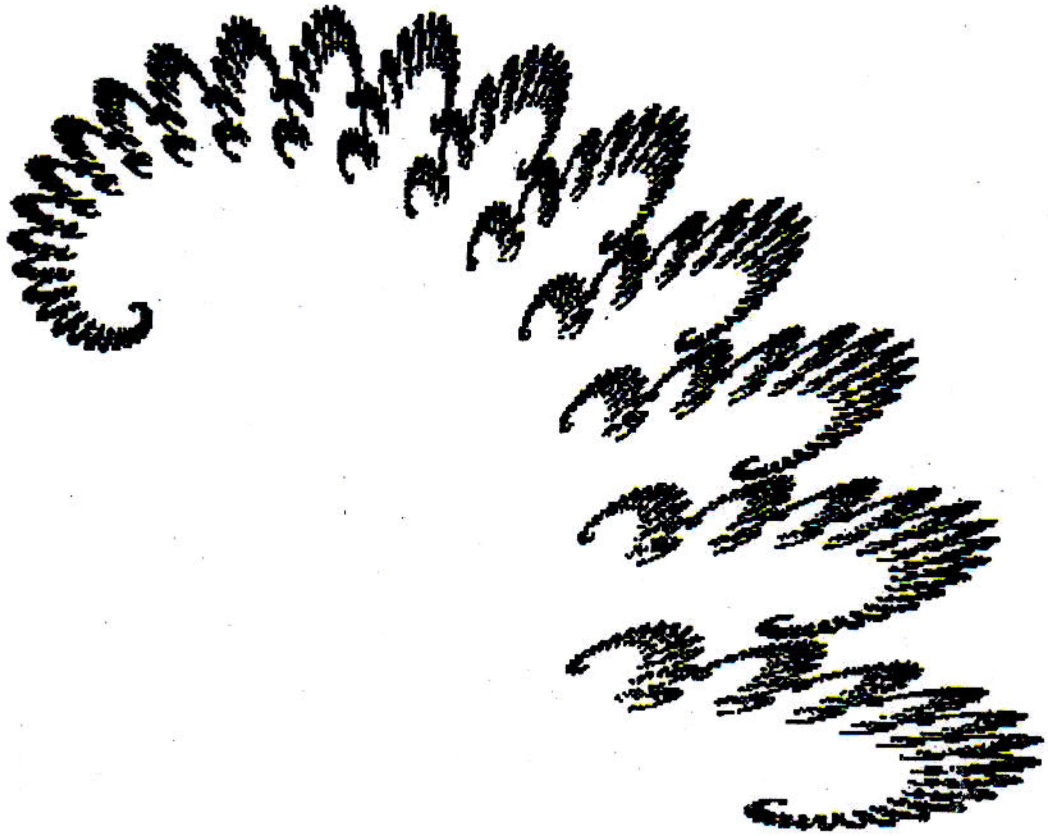

Fig. 2