EINSTEIN EQUATIONS FOR A GENERALIZED LAGRANGE SPACE OF ORDER 2 IN INVARIANT FRAMES

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Abstract

The study of higher order Lagrange spaces founded on the notion of bundle of velocities of order k has been given by Radu Miron and Gheorghe Atanasiu in [2].

The bundle of accelerations correspond in this study to k=2.

The notion of invariant geometry of order 2 was introduced by the author in [4].

In this paper we shall give the Maxwell equations of a generalized Lagrange space of order 2 in invariant frames.

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1. General Invariant Frames

Let us consider the bundle $E = \text{Osc}^2 M$, a nonlinear connection $N$ with the coefficients

$$
\begin{pmatrix}
N^i_j, & N^i_{j(1)} \\
(1) & (2)
\end{pmatrix}
$$

and the duals

$$
\begin{pmatrix}
M^i_j, & M^i_{j(1)} \\
(1) & (2)
\end{pmatrix}.
$$

The invariant frames adapted to the direct decomposition

$$
T_u(\text{Osc}^2 M) = N_0(u) \oplus N_1(u) \oplus V_2(u) \quad \forall u \in E
$$

will be $\mathcal{R} = (e^{(0)i}_\alpha, e^{(1)i}_\alpha, e^{(2)i}_\alpha)$ and the dual $\mathcal{R}^* = (f^{(0)i}_\alpha, f^{(1)i}_\alpha, f^{(2)i}_\alpha)$.

The duality conditions are:

$$
< e^{(A)i}_\alpha, f^{(B)i\alpha}_j > = \delta^A_j \delta^B_i, \quad (A, B = 0, 1, 2).
$$


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In this frame the adapted basis has the representation:

\[
\frac{\delta}{\delta x^i} = f^{(0)\alpha}_{\beta} \frac{\delta}{\delta s^{(0)\alpha}} \quad \frac{\delta}{\delta y^{(1)i}} = f^{(1)\alpha}_{\beta} \frac{\delta}{\delta s^{(1)\alpha}} \quad \frac{\delta}{\delta y^{(2)i}} = f^{(2)\alpha}_{\beta} \frac{\delta}{\delta s^{(2)\alpha}}
\]

and the cobasis

\[
\delta x^i = e^{(0)i\alpha} \delta s^{(0)\alpha} ; \delta y^{(1)i} = e^{(1)i\alpha} \delta s^{(1)\alpha} ; \delta y^{(2)i} = e^{(2)i\alpha} \delta s^{(2)\alpha}
\]

and we have the relations:

\[
\left( \frac{\delta}{\delta s^{(A)\alpha}}, \delta s^{(B)\beta} \right) = \delta^{\beta}_{\alpha} \delta_A^{B}, \quad (A, B = 0, 1, 2).
\]

This representation lead us to an invariant frames transformation group with the analytical expressions

\[
\sigma^{(A)i}_{\alpha} = A_{\alpha}^{\beta} (x, y^{(1)}, y^{(2)}), e^{(A)i}_{\alpha} ; \quad f^{(B)\alpha}_{\beta} = B_{\alpha}^{\beta} f^{(B)\beta}_{\alpha},
\]

isomorphic with the multiplicative nonsingular matrix group

\[
\begin{pmatrix}
0 & C_\beta^0 & 0 & 0 \\
0 & C_\beta^1 & 0 & 0 \\
0 & 0 & 2 & C_\beta^2
\end{pmatrix}.
\]

A N-linear connection D has in the frame \( \mathfrak{R} \) the coefficients:

\[
0^A_{\alpha \beta} = f^{(A)\gamma}_{\beta m} \left( \frac{\delta e^{(A)m}_{\beta}}{\delta s^{(0)\alpha}} + e^{(0)i\alpha}_{\beta} e^{(A)ij}_{\gamma} L^m_{ij} \right), \quad (A = 0, 1, 2),
\]

\[
B^A_{\alpha \beta} = f^{(A)\gamma}_{\beta m} \left( \frac{\delta e^{(A)m}_{\beta}}{\delta s^{(B)\alpha}} + e^{(B)i\alpha}_{\beta} e^{(A)ij}_{\gamma} C^m_{ij} (B) \right), \quad (A = 0, 1, 2 ; B = 1, 2).
\]

**Definition 1.** If the vector field \( X \in X(E) \) has the invariant components \( X^{(A)\alpha} \) \((A = 0, 1, 2)\) and we denote by \( ' \) and \( ' ' \) the h- and the \( v_B \), \( B = 1, 2 \), covariant invariant derivative operators then

\[
\begin{align*}
X^{(A)\alpha}_{(\beta')\beta} &= \frac{\delta X^{(A)\alpha}}{\delta s^{(0)\beta}} + 0^A_{\alpha \beta} X^{(A)} \phi, \\
X^{(A)\alpha}_{(B)\beta} &= \frac{\delta X^{(A)\alpha}}{\delta s^{(B)\beta}} + B^A_{\alpha \beta} X^{(A)} \phi.
\end{align*}
\]
The definition of the Lie bracket conduces us to the introduction of the non-holonomy coefficients of Vranceanu:

\[
\frac{\delta}{\delta s^{(A)\alpha}} : \frac{\delta}{\delta s^{(B)\beta}} = W^{\gamma}_{\alpha \beta} \frac{\delta}{\delta s^{(0)\gamma}} + W^{\gamma}_{\alpha \beta} \frac{\delta}{\delta s^{(1)\gamma}} + W^{\gamma}_{\alpha \beta} \frac{\delta}{\delta s^{(2)\gamma}},
\]

\((A, B = 0, 1, 2; A \leq B)\).

2. Einstein Equations

Let us consider a metric tensor \(G\) on \(Osc^2 M\), the invariant frames \(\mathcal{F}\) and \(\mathcal{F}^*\) so that the quadratic form associated to the metric has the canonical representation

\[
\omega^{(A)} = (\omega^{(A)1})^2 + .. + (\omega^{(A)iA})^2 - (\omega^{(A)iA+1})^2 + .. - (\omega^{(A)n})^2,
\]

where:

\[
\omega^{(A)} = g^{(A)ij} \delta y^{(A)i} \delta y^{(A)j}.
\]

We introduce the Vranceanu’s symbols:

\[
\varepsilon^{(A)}_{\alpha \beta} = \begin{cases} 
\delta_{\alpha \beta} & \alpha \leq i_A, \beta \leq i_A, \\
-\delta_{\alpha \beta} & \alpha > i_A, \beta > i_A, \\
0 & \text{in rest.}
\end{cases}
\]

**Theorem 2.1.** The Vranceanu’s symbols \(\varepsilon^{(A)}_{\alpha \beta}\) represents the invariant componentets of the tensors \(g^{(A)ij}\) in the frame \(\mathcal{F}\).

**Proposition 2.1.** The frame \(\mathcal{F}\) defined above is pseudoorthogonal.

**Proposition 2.2.** The invariant components \(\varepsilon^{(A)}_{\alpha \beta}\) of the metric tensor \(g\) of the total space \(E\) satisfy the relations:

\[
\varepsilon^{(A)}_{\alpha \beta \gamma} = 0, \quad \varepsilon^{(A)}_{\alpha \beta} \gamma^{(B)} = 0, \quad (A=0,1,2; B=1,2).
\]

**Proposition 2.3.** The invariant components of the canonical metrical \(N\)-linear con-nection \(\Gamma(N)\) satisfy the relations:

\[
0^A \Gamma_{\alpha \gamma}^{(A)} \varepsilon_{\alpha \beta}^{(A)} + 0^A \Gamma_{\beta \gamma}^{(A)} \varepsilon_{\alpha \varphi}^{(A)} = 0,
\]

\[
B^A \Gamma_{\alpha \gamma}^{(A)} \varepsilon_{\alpha \beta}^{(A)} + B^A \Gamma_{\beta \gamma}^{(A)} \varepsilon_{\alpha \varphi}^{(A)} = 0.
\]

The calculus of the Ricci’s tensor and the scalar curvature permit us to formulate the following result:
Theorem 2.2. The Einstein equations have the following invariant expressions in the frame $\mathbb{R}$:

\[
R_{\alpha\beta} - \frac{1}{2} \varepsilon_{\alpha\beta}^0 R = \kappa T_{\alpha\beta},
\]

\[
S_{\alpha\beta} - \frac{1}{2} \varepsilon_{\alpha\beta}^1 R = \kappa T_{\alpha\beta},
\]

\[
S_{\alpha\beta} - \frac{1}{2} \varepsilon_{\alpha\beta}^2 R = \kappa T_{\alpha\beta},
\]

with

\[
P_{\alpha\beta} = (-1)^{B+1} \kappa T_{\alpha\beta},
\]

where $T_{\alpha\beta}$ and $T_{\alpha\beta}^{(A)}$ are $d$-tensor fields and represent the invariant components of the energy-impulse tensor.

Theorem 2.3. Some of the equations which give the conservation law with respect to the canonical metrical $N$-linear connection $C\Gamma(N)$ are given by:

\[
\left( R_{\beta}^{\alpha} - \frac{1}{2} R^{\delta}_{\beta} \right)_{\alpha} + P_{\beta}^{\alpha}_{\alpha} + P_{\beta}^{\alpha} = 0,
\]

\[
\left( S_{\beta}^{\alpha} - \frac{1}{2} R^{\delta}_{\beta} \right)_{\alpha} - P_{\beta}^{\alpha} = 0,
\]

\[
\left( S_{\beta}^{\alpha} - \frac{1}{2} R^{\delta}_{\beta} \right)_{\alpha} - P_{\beta}^{\alpha} = 0,
\]

where $R^{\alpha}_{\beta} = \varepsilon^{(0)i}_{\alpha} R^{i}_{\beta}$.

3. An example of computation for Einstein equations

In this section we compute the Einstein equations in the particular case of the generalized Lagrange space $GL^{(2)}(M, g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})} \gamma_{ij}(x))$.

In some special cases it is not necessary to choose the frames where the quadratic forms $\omega$ have canonical shape but in these cases we must consider the restrictions $\varepsilon^{(0)i}_{\alpha} = \varepsilon^{(1)i}_{\alpha} = \varepsilon^{(2)i}_{\alpha} = e^{i}_{\alpha}$. For the considered generalized Lagrange space let us take the canonical nonlinear connection $N$ with the coefficients:

\[
N_{k}^{i} = \gamma_{kj}^{(1)k}.
\]
\[ N_{ij}^{(2)} = \frac{1}{2} \left( \frac{\partial \gamma_{kj}^i}{\partial x^r} - \gamma_{rh}^i \gamma_{kj}^b \right) y^{(1)k} y^{(1)r} + \gamma_{kj}^i y^{(2)k} \]

and the Berwald connection \( B\Gamma(N) = \left( \gamma_{jk}^i, \ 0, \ 0 \right) \).

**Theorem 3.1** The invariant components of the Berwald connection are obtained from \( B\Gamma(N) \) by a deviation induced by the invariant frames.

Direct calculus lead us to the following expressions of the invariant components of the Berwald connection:

\[ B\Gamma(N) = \left( \gamma_{\beta\alpha}^\eta + \frac{1}{2} W_{\beta\alpha}^{\eta} (00), \ \frac{1}{2} W_{\beta\alpha}^{\eta} (11), \ \frac{1}{2} W_{\beta\alpha}^{\eta} (22) \right). \]

**Theorem 3.2** In invariant frames the canonical metrical \( N \)-linear connection \( C\Gamma(N) \) (with respect to the metrical tensor proposed) have the coefficients

\[ C\Gamma(N) = \left( L_{\beta\alpha}^\gamma, C_{\beta\alpha}^\gamma (1), C_{\beta\alpha}^\gamma (2) \right), \]

with

\[ L_{\beta\alpha}^\gamma = L_{\beta\alpha}^\gamma + \Lambda_{\beta\alpha}^\gamma, \]

\[ C_{\beta\alpha}^\gamma (A) = C_{\beta\alpha}^\gamma (A) + \Lambda_{\beta\alpha} (A), \quad (A=1,2), \]

where \( L_{\beta\alpha}^\gamma \) and \( C_{\beta\alpha}^\gamma (A) \) are the coefficients of Berwald connection in invariant frames and \( \Lambda_{\beta\alpha} \) are the invariant components of the deviation tensor \( \Lambda_{jk}^i \) induced by the metric, which have the expressions:

\[ \Lambda_{jk}^i = \delta_{jk}^{(B)} \sigma_j + \delta_{jk}^{(B)} \sigma_k - \gamma_{jk}^i \sigma^i, \]

where

\[ \delta_{jk}^{(B)} \sigma_j = \frac{\delta \sigma}{\delta y^{(B)j}}, \quad \delta_{jk}^{(B)} \sigma_k = \gamma^{is} \sigma_s^{(B)}, \quad (B=1,2) \]

\( (y^{(0)i} = x^i) \).
Theorem 3.3 The deviation tensor $\Lambda_{\beta\gamma}^{\alpha}(B)$ with respect to $B\Gamma(N)$ is

$$
\Lambda_{\beta\gamma}^{\alpha}(B) = \frac{1}{2}g^{\alpha\eta}\left(\left(g_{\beta\eta})_{\gamma} + g_{\gamma\eta})_{\beta} - g_{\beta\gamma})_{\eta}\right) + \frac{1}{2}g_{\gamma\eta}\left(\left(g^{(B)}_{\beta\eta})_{\gamma} + g^{(B)}_{\gamma\eta})_{\beta} - g^{(B)}_{\beta\gamma})_{\eta}\right)\right)
$$

A direct calculus of curvature tensors, Ricci’s tensors and curvature scalars lead us to the following result:

Theorem 3.4 Consider the space $GL^{(2)n}$ endowed with the metric $g_{ij} = e^{2\sigma(x, y^{(1)}, y^{(2)})_{ij}}$ and the canonical metrical connection $C\Gamma(N)$. The Einstein equations in the invariant frames are:

$$
r_{\alpha\beta} + R^*_{\alpha\beta} + \tilde{R}_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}(r + R^* + \tilde{R}) = \kappa T_{\alpha\beta}^{0},
$$

$$
S^*_{\alpha\beta} + \tilde{S}_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}(r + R^* + \tilde{R}) = \kappa T_{\alpha\beta}^{A},
$$

$$
P^*_{\alpha\beta} + \tilde{P}^*_{\alpha\beta} = (-1)^A_{\alpha\beta} T_{\alpha\beta}^{0B},\quad (A,B=1,2).
$$

Theorem 3.5 The conservation law with respect to the canonical metrical connection $C\Gamma(N)$ in invariant frames are:

$$
\left( R^\alpha_{\beta} - \frac{1}{2}R^\delta_{\alpha\beta}\right)_{\beta} + \frac{1}{2}P^\alpha_{\beta\alpha\beta}^{(1)} + \frac{2}{3}P^\alpha_{\beta\alpha\beta}^{(2)} + \frac{1}{2}P^\alpha_{\beta\alpha\beta}^{(2)} = 0,
$$

$$
\left( S^\alpha_{\beta} - \frac{1}{2}R^\delta_{\alpha\beta}\right)_{\beta} = \frac{2}{3}P^\alpha_{\beta\alpha\beta}^{(1)} + \frac{2}{3}P^\alpha_{\beta\alpha\beta}^{(2)} = 0,
$$

$$
\left( S^\alpha_{\beta} - \frac{1}{2}R^\delta_{\alpha\beta}\right)_{\beta} = \frac{2}{3}P^\alpha_{\beta\alpha\beta}^{(1)} - \frac{2}{3}P^\alpha_{\beta\alpha\beta}^{(2)} = 0,
$$

where $'$ and $''$ ($B = 1, 2$) are the $h$- and the $v_B$-covariant invariant derivatives with respect to $C\Gamma(N)$. 
References


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