

# SOLUTIONS OF INVERSE PROBLEMS FOR VARIATIONAL CALCULUS

M. Neagu

## Abstract

In §1 the author presents a short history of the problem studied in this paper. §2 introduces the notion of harmonic map between a Riemannian space and a generalized Lagrange space, in a natural way. In §3 is proved that for certain systems of differential or partial differential equations, the solutions are harmonic maps, in the sense definite in §1. §4 describes the main properties of the generalized Lagrange spaces constructed in §3.

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**Key words:** generalized Lagrange spaces, harmonic maps, geodesics, differential equations, partial differential equations.

## 1 Introduction

The problem of finding a geometrical structure of Riemannian type on a manifold  $M$  such that the orbits of an arbitrary vector field  $X$  should be geodesics, was analysed by Sasaki [5]. Doing with this problem, Sasaki creates the well known almost contact structures on a manifold of odd dimension, although the initial problem rests open. After the introduction of generalized Lagrange spaces by Miron [2], the same problem is resumed by Udriște [6, 7, 8]. This succeeded to discover a Lagrange structure on  $M$ , dependent of the vector field  $X$  and a  $(1,1)$ -tensor field, such that the orbits of class  $C^2$  should be geodesics. Moreover, he formulates a more general problem [8], namely

1) There exist the structures of Lagrange type such that the solutions of certain PDEs of order one should be *harmonic maps*?

2) What is a *harmonic map* between two spaces endowed with structures of this type?

In this paper, the author tries to solve these problems. He will use the new notion of harmonic map on a direction, offering thus a partial answer to the Udriște's questions.

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## 2 Harmonic maps on a direction

Let  $(M^m, \varphi_{\alpha\beta}(a))$  be Riemannian manifold of dimension  $m$  and  $(N^n, h_{ij}(x, y))$  generalized Lagrange space of dimension  $n$ , where  $a = (a^\alpha)_{\alpha=1, \dots, m}$  are coordinates on  $M$  and  $(x, y) = (x^k, y^k)_{k=1, \dots, n}$  are coordinates on  $TN$ .

**Remark.** On  $M$ , the coordinates are indexed by  $\alpha, \beta, \gamma, \dots$  and, on  $N$ , respectively  $TN$ , the coordinates are indexed by  $i, j, k, \dots$ . Also, on  $M \times N$ , the first  $m$  coordinates will be indexed by  $\alpha, \beta, \gamma, \dots$  and the last  $n$  coordinates by  $i, j, k, \dots$ .

Let  $A \in \mathcal{X}(M)$  be an arbitrary vector field on  $M$ . If the manifold  $M$  is connected, compact and orientable, we can define the  $(\varphi, A, h)$ -energy functional or the energy functional on the direction  $A$  taking

$$E_{\varphi h}^A : C^\infty(M, N) \rightarrow R,$$

$$E_{\varphi h}^A(f) = \frac{1}{2} \int_M \varphi^{\alpha\beta}(a) h_{ij}(f(a), f_*(A)) f_\alpha^i f_\beta^j \sqrt{\varphi} da,$$

where  $f^i = x^i(f)$ ,  $f_\alpha^i = \frac{\partial f^i}{\partial a^\alpha}$ ,  $\varphi = \det(\varphi_{\alpha\beta})$  and  $f_* : TM \rightarrow TN$  is the differential of the map  $f$ .

**Definition.** The map  $f \in C^\infty(M, N)$  is  $(\varphi, A, h)$ -harmonic iff  $f$  is a critical point for the  $(\varphi, A, h)$ -energy functional  $E_{\varphi h}^A$ .

**Remarks.** i) If  $h_{ij}(x, y) = h_{ij}(x)$  is Riemannian metric, it recovers the classical definition of a harmonic map between two Riemannian manifolds [1].

ii) If we consider  $M = [a, b] \subset R$ ,  $\varphi_{11} = 1$  and  $A = d/dt$ , we obtain  $C^\infty(M, N) = \{x : [a, b] \rightarrow N \mid x - C^\infty \text{ differentiable}\} \stackrel{\text{not}}{=} \Omega_{a,b}(N)$  and the  $(1, d/dt, h)$ -energy functional should be

$$E_{1h}^{d/dt}(x) = \frac{1}{2} \int_a^b h_{ij}(x(t), \dot{x}(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} dt, \quad \forall x \in \Omega_{a,b}(N).$$

In conclusion, the  $(1, d/dt, h)$ -harmonic curves are exactly the geodesics of the generalized Lagrange space  $(N, h_{ij}(x, y))$  [2].

## 3 The geometrical interpretation of the solutions of certain PDEs of order one

Let  $f \in C^\infty(M, N)$  be a smooth map and let be the global section

$$\delta f \stackrel{\text{not}}{=} f_\alpha^i da^\alpha|_a \otimes \frac{\partial}{\partial y^i} \Big|_{f(a)} \in \Gamma(T^*M \otimes f^{-1}(TN)).$$

On  $M \times N$ , let  $T$  be one tensor of type  $(1, 1)$  with all components equal to zero excepting  $(T_\alpha^i)_{\substack{i=1, \dots, n \\ \alpha=1, \dots, m}}$ . Let be the system of partial differential equations

$$(E) \quad \delta f = T \text{ expressed locally by } \frac{\partial f^i}{\partial a^\alpha} = T_\alpha^i(a, f(a)).$$

If  $(M, \varphi_{\alpha\beta})$  and  $(N, \psi_{ij})$  are Riemannian manifolds we can build a scalar product on  $\Gamma(T^*M \otimes f^{-1}(TN))$  putting  $\langle T, S \rangle = \varphi^{\alpha\beta}(a) \psi_{ij}(f(a)) T_\alpha^i S_\beta^j$ , where  $T = T_\alpha^i da^\alpha \otimes \frac{\partial}{\partial y^i}$  and  $S = S_\beta^j da^\beta \otimes \frac{\partial}{\partial y^j}$ .

In these conditions we can prove the following

**Theorem.** *If  $(M, \varphi), (N, \psi)$  are Riemannian manifolds and  $f \in C^\infty(M, N)$  is solution of the system (E), then  $f$  is solution of the variational problem associated to the functional  $\mathcal{L}_T : C^\infty(M, N) \setminus \{f \mid \exists a \in M \text{ such that } \langle \delta f, T \rangle(a) = 0\} \rightarrow R_+$ ,*

$$\mathcal{L}_T(f) = \frac{1}{2} \int_M \frac{\|\delta f\|^2 \|T\|^2}{\langle \delta f, T \rangle^2} \sqrt{\varphi} da = \frac{1}{2} \int_M \frac{\|T\|^2}{\langle \delta f, T \rangle^2} \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i f_\beta^j \sqrt{\varphi} da.$$

**Proof.** In the space  $\Gamma(T^*M \times f^{-1}(TN))$ , the Cauchy inequality for the scalar product  $\langle, \rangle$  holds. It follows that we will have  $\langle T, S \rangle^2 \leq \|T\|^2 \|S\|^2$ ,  $\forall T, S \in \Gamma(T^*M \times f^{-1}(TN))$ , with equality iff there exists  $\mathcal{K} \in \mathcal{F}(M)$  such that  $T = \mathcal{K}S$ . Consequently, for every  $f \in C^\infty(M, N)$ , we will obtain

$$\mathcal{L}(f) = \frac{1}{2} \int_M \frac{\|\delta f\|^2 \|T\|^2}{\langle \delta f, T \rangle^2} \sqrt{\varphi} da \geq \frac{1}{2} \int_M \sqrt{\varphi} da = \frac{1}{2} Vol_\varphi(M).$$

Now, if  $f$  is solution of the system (E), we will conclude  $\mathcal{L}_T(f) = \frac{1}{2} Vol_\varphi(M)$ . This means that  $f$  is a global minimum point for  $\mathcal{L}_T$   $\square$ .

**Remarks.** i) In certain particular cases of the system (E), the functional  $\mathcal{L}_T$  becomes exactly a functional of type  $(\varphi, A, h)$ -energy.

ii) The global minimum points of the functional  $\mathcal{L}_T$  are solutions of the system  $\delta f = \mathcal{K}T$ , where  $\mathcal{K} \in \mathcal{F}(M)$  not necessarily with  $\mathcal{K} = 1$ .

**Examples.**

### 1. Orbits

For  $M = ([a, b], 1)$  and  $T = \xi \in \Gamma(x^{-1}(TN))$ , the system (E) becomes

$$(E_1) \quad \frac{dx^i}{dt} = \xi^i(x(t)), \quad x : [a, b] \rightarrow N,$$

that is the system of orbits for  $\xi$ , and the functional  $\mathcal{L}_\xi$  is

$$\mathcal{L}_\xi(x) = \frac{1}{2} \int_a^b \frac{\|\xi\|_\psi^2}{\langle \xi, x_*(d/dt) \rangle_\psi^2} \psi_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} dt.$$

Hence the functional  $\mathcal{L}_\xi$  is a  $(1, d/dt, h)$ -energy (see ii of first remarks of this paper). The fundamental metric tensor  $h_{ij} : TN \setminus \{y \mid \xi^b(y) = 0\} \rightarrow R$  is defined by

$$h_{ij}(x, y) = \frac{\|\xi\|_\psi^2}{\langle \xi, y \rangle_\psi^2} \psi_{ij}(x) = \psi_{ij}(x) \exp \left[ 2 \ln \frac{\|\xi\|_\psi}{|\langle \xi, y \rangle_\psi|} \right].$$

This case is studied, in other way, by Udriște in [6]-[8].

## 2. Pfaff systems

For  $N = (R, 1)$  and  $T = A \in \Lambda^1(T^*M)$ , the system (E) will become

$$(E_2) \quad df = A, \quad f \in \mathcal{F}(M),$$

that is a Pfaffian system, and the functional  $\mathcal{L}_T$  reduces to

$$\mathcal{L}_A(f) = \frac{1}{2} \int_M \frac{\|A\|_\varphi^2}{[f_*(A^!)]^2} \varphi^{\alpha\beta} f_\alpha f_\beta \sqrt{\varphi} da,$$

where  $A^! = \varphi^{\alpha\beta} A_\beta \frac{\partial}{\partial a^\alpha}$ . Hence the functional  $\mathcal{L}_A$  is a  $(g, A^!, h)$ -energy, where

$$g_{\alpha\beta}(a) = \frac{1}{\|A\|_\varphi^2} \varphi_{\alpha\beta}(a) \text{ and } h : TR \setminus \{0\} \rightarrow R, \quad h(x, y) = \frac{1}{y^2} = e^{-2 \ln |y|}.$$

## 3. Pseudolinear functions

We suppose that  $T_\beta^k(a, \dot{x}) = \xi^k(x) A_\beta(a)$ , where  $\xi^k$  is vector field on  $N$  and  $A_\beta$  is 1-form on  $M$ . In this case the system (E) will be

$$(E_3) \quad \frac{\partial f^k}{\partial a^\beta} = \xi^k(f) A_\beta(a)$$

and the functional  $\mathcal{L}_T$  is expressed by

$$\begin{aligned} \mathcal{L}_T(f) &= \frac{1}{2} \int_M \frac{\|\xi\|_\psi^2 \|A\|_\varphi^2}{\langle \xi, f_*(A^!) \rangle^2} \varphi^{\alpha\beta} \psi_{ij} f_\alpha^i f_\beta^j \sqrt{\varphi} da = \\ &= \frac{1}{2} \int_M g^{\alpha\beta}(a) h_{ij}(f(a), f_*(A^!)) f_\alpha^i f_\beta^j \sqrt{\varphi} da, \text{ where} \end{aligned}$$

$$g_{\alpha\beta}(a) = \frac{1}{\|A\|_\varphi^2} \varphi_{\alpha\beta}(a) \text{ and } h_{ij}(x, y) = \frac{\|\xi\|_\psi^2}{\langle \xi, y \rangle^2} \psi_{ij}(x) = \exp(-2 \ln |\langle \xi, y \rangle|) \psi_{ij}(x).$$

It follows that the functional  $\mathcal{L}_T$  becomes a  $(g, A^!, h)$ -energy.

**Remark.** Taking  $M$  an open subset in  $(R^n, \varphi = \delta)$  and  $N = (R, \psi = 1)$ , the system from the third example is

$$(PL) \quad \frac{\partial f}{\partial a^\alpha} = \xi(a) A_\alpha(f(a)), \quad \forall \alpha = \overline{1, m}.$$

Supposing that  $(grad f)(a) \neq 0, \forall a \in M$ , the solutions of this system are the well known *pseudolinear functions* [4]. These functions have the following property

–for every fixed point  $x_0 \in M$ , the hypersurface of constant level

$$M_{f(x_0)} = \{x \in M | f(x) = f(x_0)\}$$

is totally geodesic [4] (i. e. the second fundamental form vanishes identically).

In conclusion, these pseudolinear functions are examples of  $(g, A^\sharp, h)$ -harmonic functions on the Riemannian space  $\left(M, g_{\alpha\beta}(a) = \frac{1}{\|A\|^2} \delta_{\alpha\beta}\right)$  with values into the generalized Lagrange space  $\left(R, h(x, y) = \frac{\xi(x)}{y}\right)$ . For example, we have the following pseudolinear functions:

3. 1.  $f(a) = e^{\langle v, a \rangle + w}$ , where  $v \in M, w \in R$ , is solution for the system (PL) with  $\xi(a) = 1$  and  $A(f(a)) = f(a)v$ .

3. 2.  $f(a) = \frac{\langle v, a \rangle + w}{\langle v', a' \rangle + w'}$ , where  $v, v' \in M ; w, w' \in R$ , is solution for (PL) with  $\xi(a) = \frac{1}{\langle v', a \rangle + w'}$  and  $A(f(a)) = v - f(a)w$ .

**Remark.** A geometrical interpretation of the solutions of the system (E), in the general case, when the tensor  $T$  is expressed by  $T_\alpha^i(a, x) = \sum_{r=1}^t \xi_r^i(x) A_\alpha^r(a)$ , where  $\{\xi_r\}_{r=1, \overline{t}} \subset \mathcal{X}(N)$  is a family of vector fields on  $N$  and  $\{A^r\}_{r=1, \overline{t}} \subset \Lambda^1(T^*M)$  is a family of covector fields on  $M$ , will be treated, in other sense, in a subsequent paper.

### 4 The geometry asociated to PDEs of order one

For beginning, we remark that, in all above cases, the solutions of class  $C^2$  of the system  $\delta f = T$  becomes  $(\varphi, A, h)$ -harmonic maps, in the sense definite in this paper. Moreover, the generalized Lagrange structures constructed above are of type  $(M^n, e^{2\sigma(x,y)}\gamma_{ij}(x))$ , where  $\sigma : TM \setminus \{\text{Hyperplane}\} \rightarrow R$  is a smooth function. Now, we assume that a generalized Lagrange space  $(M^n, g_{ij}(x, y))$  satisfies the following axioms:

- a. 1. The fundamental tensor field  $g_{ij}(x, y)$  is of the form

$$g_{ij}(x, y) = e^{2\sigma(x,y)}\gamma_{ij}(x).$$

- a. 2. The space is endowed with the non-linear connection

$$N_j^i(x, y) = \Gamma_{jk}^i(x)y^k,$$

where  $\Gamma_{jk}^i(x)$  are the Christoffel symbols for the Riemannian metric  $\gamma_{ij}(x)$ .

Under these assumptions, our space verifies a constructive-axiomatic formulation of General Relativity due to Ehlers, Pirani and Schild [2]. This space represents a convenient relativistic model, since it has the same conformal and projective properties as the Riemannian space  $(M, \gamma_{ij})$ .

Developing the formalism presented in [2], [3] and denoting the curvature tensor field of the metric  $\gamma_{ij}(x)$  by  $r_{kij}^i$ , the following Maxwell's equations hold

$$\begin{cases} F_{ij|k} + F_{jk|i} + F_{ki|j} = - \sum_{(ijk)} g_{ip} r_{qjk}^h \frac{\partial \sigma}{\partial y^h} y^p y^q, \\ F_{ij|k} + F_{jk|i} + F_{ki|j} = -(f_{ij|k} + f_{jk|i} + f_{ki|j}), \\ f_{ij|k} + f_{jk|i} + f_{ki|j} = 0, \end{cases}$$

where the electromagnetic tensors  $F_{ij}$  and  $f_{ij}$  are

$$F_{ij} = \left( g_{ip} \frac{\delta \sigma}{\delta x^j} - g_{jp} \frac{\delta \sigma}{\delta x^i} \right) y^p, \quad f_{ij} = \left( g_{ip} \frac{\partial \sigma}{\partial y^j} - g_{jp} \frac{\partial \sigma}{\partial y^i} \right) y^p.$$

We will use the following notations:  $r_{ij} = r_{ijk}^k$ ,  $r = \gamma^{ij} r_{ij}$ ,  $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$ ,  
 $\sigma^H = \gamma^{kl} \frac{\delta \sigma}{\delta x^k} \frac{\delta \sigma}{\delta x^l}$ ,  $\sigma^V = \gamma^{ab} \frac{\partial \sigma}{\partial y^a} \frac{\partial \sigma}{\partial y^b}$ ,  $\bar{\sigma} = \gamma^{ij} \sigma_{ij}$ ,  $\dot{\sigma} = \gamma^{ab} \dot{\sigma}_{ab}$ ,

$$\text{where } \begin{cases} \sigma_{ij} = \frac{\delta \sigma}{\delta x^i} \Big|_j + \frac{\delta \sigma}{\delta x^i} \frac{\delta \sigma}{\delta x^j} - \frac{1}{2} \gamma_{ij} \sigma^H \\ \dot{\sigma}_{ab} = \frac{\partial \sigma}{\partial y^a} \Big|_b + \frac{\partial \sigma}{\partial y^a} \frac{\partial \sigma}{\partial y^b} - \frac{1}{2} \gamma_{ab} \sigma^V. \end{cases}$$

The Einstein's equations of the space  $(M, g_{ij}(x, y))$  take the form

$$\begin{cases} r_{ij} - \frac{1}{2} r \gamma_{ij} + t_{ij} = \mathcal{K} T_{ij}^H \\ (2-n)(\dot{\sigma}_{ab} - \dot{\sigma} \gamma_{ab}) = \mathcal{K} T_{ab}^V, \end{cases}$$

where  $T_{ij}^H$  and  $T_{ab}^V$  are the  $h$ - and the  $v$ -components of the energy momentum tensor field,  $\mathcal{K}$  is the gravific constant and

$$t_{ij} = (n-2)(\gamma_{ij} \bar{\sigma} - \sigma_{ij}) + \gamma_{ij} r_{st} y^s \gamma^{tp} \frac{\partial \sigma}{\partial y^p} + \frac{\partial \sigma}{\partial y^i} r_{tja}^a y^t - \gamma_{is} \gamma^{ap} \frac{\partial \sigma}{\partial y^p} r_{tja}^s y^t.$$

Consequently, in certain particular cases, it is possible to build a generalized Lagrange geometry (in Miron's sense) naturally attached to a system of partial differential equations. This idea was suggested by Udrishte in private discussions.

Open problem. Because the generalized Lagrange structure constructed in this paper is not unique, it arises a natural question:

-Is it possible to build a unique generalized Lagrange geometry naturally asociated to a given PDEs system?

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Author's address:

M. Neagu  
*Politehnica University of Bucharest*  
*Department of Mathematics I*  
*Splaiul Independenței 313,*  
*77206 Bucharest, Romania*  
*E-mail: mircea@mathem.pub.ro*