ON THE UPPER BOUND OF $varU^4f/varf$ IN GAUSS' PROBLEM

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Abstract

Taking up a problem raised in [3], we determine the upper bound of $varU^4 f/varf$ when f varies in the collection of non-constant monotone functions f on $I = [0,1]$. Then we show that the inequality $var\, U^4 f \leq 0,0625 varf$ found by Wirsing [9] improves significantly, as we'll find:

$$0,037134828 < \frac{var\, U^4 f}{var\, f} < 0,041204828.$$

The purpose here is to prove the conjecture $\delta^{(4)}(f) \leq \delta^{(4)}(f_4)(f(1) - f(0))$, which is true for some non-decreasing f.

1 Preliminaries

Let $I = [0,1]$, and $\Omega = I - \mathbf{Q}$. Any irrational number $x \in \Omega$, can be written in a unique way as a continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{\cfrac{1}{a_n + \ddots}}}} = [a_1, a_2, \ldots, a_n, \ldots] = \lim_{n\to\infty} [a_1(x), \ldots, a_n(x)] \quad (1)$$

and the corresponding sequence of convergents by $\left(\frac{p_n}{q_n}\right)_{n\geq -1}$. The sequences $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ satisfy the reccurence relations:

$$\begin{array}{l} p_{-1} = 1,\ p_0 = 0,\ p_n = a_n p_{n-1} + p_{n-2} \\ q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2} \end{array} \quad (2)$$

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where $a_0 = 0$.

We consider $BV(I)$ the Banach space of complex valued functions f of bounded variation on I under the norm

$$||f||_V = var\, f + |f| \tag{3}$$

Let U be the bounded linear operator defined by

$$Uf(x) = \sum_{i=1}^{\infty} \frac{x+1}{(x+i)(x+i+1)} f\left(\frac{1}{x+i}\right), \quad f \in BV(I). \tag{4}$$

In particular, it has been shown in [3] that U takes monotone functions into monotone functions and, moreover

$$var\, Uf \leq 0,5 var\, f$$

for any bounded monotone function f, the constant 0,5 being optimal.

We'll put $p_i(x) = \dfrac{x+1}{(x+i)(x+i+1)}$ and $u_i(x) = \dfrac{1}{x+i}$. So, we see that

$$(u_{i_n} \circ \ldots \circ u_{i_1})(x) = u_{i_n \ldots i_1}(x)$$

and

$$p_{i_1 \ldots i_n}(x) = p_{i_1}(x) \cdot p_{i_2}(u_{i_1}(x)) \ldots p_{i_n}\left(u_{i_{n-1} \ldots i_1}(x)\right), \tag{5}$$

written as a continued fraction

$$u_{i_n \ldots i_1}(x) = \cfrac{1}{i_n + \cfrac{1}{i_{n-1} + \cfrac{\ddots}{+\cfrac{1}{i_1 + x}}}}$$

We consider the egodic system $(\Omega, \mathcal{B}, \mu, \tau)$ to which is applied Birkhoff's theorem, when $\mathcal{B}$ is the collection of all Borel subsets on Ω, and μ is a measure of probability on $\mathcal{B}$. We denote

$$\tau : \Omega \to \Omega, \quad \tau(x) = \left\{\frac{1}{x}\right\}, \quad x = [x] + \{x\},$$

where $[\cdot]$ is the integer part, and $\{\cdot\}$ the fractional part.

Let γ be Gauss measure on $\mathcal{B}$,

$$\gamma(A) = \frac{1}{\ln 2} \int_A \frac{dx}{1+x}, \quad A \in \mathcal{B}.$$

Let us divide the unit interval I into intervals $B(k) = \left(\dfrac{1}{k+1}, \dfrac{1}{k}\right]$. For $x \in B(k)$, $\dfrac{1}{k+1} < x < \dfrac{1}{k}$, that is $k < \dfrac{1}{x} < k+1$, so $\left[\dfrac{1}{x}\right] = k$ and $\left\{\dfrac{1}{x}\right\} = \dfrac{1}{x} - \left[\dfrac{1}{x}\right] =$

$\frac{1}{x} - k$. Then $a_{n+1}(x) = a_1(\tau^n(x))$, $n \in \mathbf{N}^*$ with $a_1(x) = \left[\frac{1}{x}\right]$. We use the rest $r_n(x) = a_n(x) + [a_{n+1}(x), a_{n+2}(x), \ldots]$, and the set $\tau^{-n}((0,x)) = \{y \in \Omega \setminus \tau^n(y) < x\}$. Then we have $\tau^{n+1}(y) < x$ iff $\frac{1}{n+k} < \tau^n(y) < \frac{1}{k}$. Puting $F_n(x) = \mu(\tau^{-n}((0,x)))$, $x \in I$ we obtain Gauss' equation

$$F_{n+1}(x) = \sum_{k=1}^{\infty} \left(F_n\left(\frac{1}{k}\right) - F_n\left(\frac{1}{x+k}\right)\right), \quad n \in \mathbf{N},\ x \in I.$$

2 Results

In the paper [4], M.Iosifescu showed that for any natural n, the equality

$$U^n f = \sum_{i_1,\ldots,i_n} p_{i_1\ldots i_n}(x) f(u_{i_n\ldots i_1}(x)) \tag{6}$$

For n even it.yields

$$var\, U^n f = U^n f(0) - U^n f(1) =$$

$$\sum_{i_1,i_2,\ldots,i_n \in \mathbf{N}^*} (p_{i_1\ldots i_n}(1) f(u_{i_n\ldots i_1}(1)) - p_{i_1\ldots i_n}(0) f(u_{i_n\ldots i_1}(0))). \tag{7}$$

Without any loss of generality, throughout this section we assume that f is non-decreasing. In order to simplify the writing, we put

$$p_{i_1\ldots i_n}(0) = \alpha_{1\ldots i_n}$$
$$u_{i_n\ldots i_1}(0) = \beta_{i_n\ldots i_1}, \quad (\forall) i_1, \ldots, i_n \in \mathbf{N}^*.$$

For $n = 4$ we have

$$\begin{aligned} \alpha_{ijkl} &= \cfrac{1}{i + \cfrac{1}{j + \cfrac{1}{k + \cfrac{1}{l+1}}}} - \cfrac{1}{i + \cfrac{1}{j + \cfrac{1}{k + \cfrac{1}{l}}}} = [i,j,k,l,1] - [i,j,k,l] \\ \beta_{lkji} &= \cfrac{1}{l + \cfrac{1}{k + \cfrac{1}{j + \cfrac{1}{i}}}} = [l,k,j,i] \end{aligned} \tag{8}$$

As in [4], we put

$$\delta_{i_3\ldots i_n} = (-1)^{n-1} \sum_{i_2=1}^{\infty} \left(\alpha_{1 i_2\ldots i_n} - \sum_{i_1=1}^{\infty} \alpha_{(i_1+1) i_2 \ldots i_n} \right) \tag{9}$$

and in case when n even, we denote

$$f_{2k+2}(x) = \begin{cases} 1, & \text{if } \frac{c_{2k+1}}{c_{2k+2}} < x \leq 1 \\ 0, & \text{if } 0 \leq x \leq \frac{c_{2k+1}}{c_{2k+2}}, \end{cases}$$

for all $k \in \mathbb{N}^*$, where $c_k, k \in \mathbb{N}^*$ are the Fibonacci numbers, defined by $c_0 = c_1 = 1$, $c_k = c_{k-1} + c_{k-2}$, $k \geq 2$.

Let us compute $\delta^{(n)}(f_n)$ for n even.

It is easy to see that

$$\beta_{i_{2k+2}\ldots i_3} \geq \frac{c_{2k+1}}{c_{2k+2}}$$

holds iff

$$i_{2k+2} = \ldots = i_{2k-2m+2} = 1, \quad i_{2k-2m+1} \geq 2, \quad 0 \leq m \leq k-1,\ k \geq 1.$$

We then have

$$\delta^{(4)}(f_4) = \sum_{k,l=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)j(k+1)1} - \alpha_{1j(k+1)1} \right). \tag{10}$$

We denote

$$\gamma_4 = \delta^{(4)}(f_4) + \sum_{i=1}^{\infty} \alpha_{(i+1)111} \tag{11}$$

We want to state that

$$var\ U^4 f \leq \gamma_4 var\ f \tag{12}$$

and γ_4 cannot be lowered. In our case

$$\delta_{kl} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)jkl} - \alpha_{1jkl} \right).$$

So (7) become

$$var\ U^4 f \leq \sum_{k,l=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)jkl} - \alpha_{1jkl} \right) f(\beta_{lk}) + \right.$$

$$\left. + \left(\sum_{i=1}^{\infty} \alpha_{(i+1)1kl} \right) (f(\beta_{lk1}) - f(\beta_{lk})) \right).$$

We have

$$\delta^{(4)}(f) = \sum_{k,l=1}^{\infty} \delta_{kl} f(\beta_{lk})$$

Now the problem is to find the best upper bound for $\delta^{(4)}(f)$.

For all non-descreasing functions f, we put

$$f_4(x) = \begin{cases} 1, & \text{if } \frac{3}{5} < x \le 1 \\ 0, & \text{if } 0 \le x \le \frac{3}{5}. \end{cases} \tag{13}$$

It is easy to see that

$$p_{1111}(0) = \frac{1}{c_4 c_5} = \frac{1}{5 \cdot 8} = \frac{1}{40} = 0,025 \tag{14}$$

and

$$p_{1jkl}(0) < p_{1111}(0) \tag{14'}$$

for all $j, k, l \in \mathbf{N}^*$ such that $(j,k,l) \neq (1,1,1)$.

If we'll justify that

$$\sum_{i_2,\dots,i_n=1}^{\infty} (f(u_{i_n \dots i_2}(0)) - f(1)) \left(p_{1 i_2 \dots i_n}(0) - \sum_{i_1=1}^{\infty} p_{(i_1+1) i_2 \dots i_n}(0) \right) \ge 0,$$

the upper bound γ_n is the best we can have ans we'll state (12).

Theorem 1. For all $f \in BV(I)$ non-decreasing and f_4 from (13), the following inequality

$$\delta^{(4)}(f) \le \delta^{(4)}(f_4)(f(1) - f(0)) \tag{15}$$

takes place.

Proof. We'll evaluate $\delta^{(4)}(f_4)(f(1)-f(0))$ taking $f(1) = 1$ and $f(0) = 0$, in order to estimate γ_4 for all non-decreasing functions. We find

$$\delta^{(4)}(f_4)(f(1) - f(0)) = \sum_{i_2,i_3=1}^{\infty} \left(\sum_{i_1=1}^{\infty} \alpha_{(i_1+1) i_2 (i_3+1) 1} - \alpha_{1 i_2 (i_3+1) 1} \right)$$

$$\delta^{(4)}(f) = \sum_{i_3,i_4=1}^{\infty} \delta_{i_3 i_4} f(\beta_{i_4 i_3}).$$

The inequality (15) becomes

$$\sum_{j,k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)j(k+1)1} - \alpha_{1j(k+1)1} \right) \ge$$

$$\ge \sum_{k,l=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)jkl} - \alpha_{1jkl} \right) \right) f(\beta_{lk}) = \sum_{k,l=1}^{\infty} \delta_{kl} f(\beta_{lk}). \tag{16}$$

We'll take

$$f(x) = \begin{cases} 1 & \text{if } 0,5 \le x \le 1 \\ 0 & \text{if } 0 \le x < 0,5. \end{cases}$$

The computations below are based on tables in [1], and ψ denotes the digamma function, which can be expressed by the convergent series

$$\psi(x) = -c + \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+x-1}\right) = -c + \sum_{i=1}^{\infty}\frac{x-1}{i(i+x-1)}$$

c is Euler-Mascheroni constant, $c = 0,57721\ldots$

$$\delta_{kl} = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\left(\frac{1}{i+1+\cfrac{1}{j+\cfrac{1}{k+\cfrac{1}{l+1}}}} - \frac{1}{i+1+\cfrac{1}{j+\cfrac{1}{k+\cfrac{1}{l}}}}\right) - \alpha_{1jkl}\right) =$$

$$= \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\left(\frac{1}{i+1+\frac{kl+k+1}{j(kl+k+1)+l+1}} - \frac{1}{i+1+\frac{kl+1}{j(kl+1)+l}}\right) - \alpha_{1jkl}\right).$$

As is well-known, ψ satisfy the equations

$$\psi(1+x) = \psi(x) + \frac{1}{x}$$

$$\psi(2+x) = \psi(1+x) + \frac{1}{1+x} = \psi(x) + \frac{1}{1+x} + \frac{1}{x} \text{ for all } x \neq 0,-1,-2,\ldots \text{ and}$$

$$\psi(z_1) - \psi(z_2) = \sum_{i=1}^{\infty}\left(\frac{1}{i+z_2} - \frac{1}{i+z_1}\right).$$

We have

$$\delta_{kl} = \sum_{j=1}^{\infty}\left[-\frac{1}{1+\frac{(l+1)k+1}{j(k(l+1)+1)+l+1}} + \frac{1}{1+\frac{kl+1}{j(kl+1)+l}} - \right.$$

$$\left. -\psi\left(1+\frac{k(l+1)+1}{j(k(l+1)+1)+l+1}\right) + \psi\left(1+\frac{kl+1}{j(kl+1)+l}\right) - \alpha_{1jkl}\right]$$

The two terms from the above sum are equal with the last, as somebody can see after computation, and

$$\delta_{kl} = \sum_{j=1}^{\infty}\left(-2\alpha_{1jkl} - \psi\left(1+\frac{k(l+1)+1}{j(k(l+1)+1)+l+1}\right) + \psi\left(1+\frac{kl+1}{j(kl+1)+l}\right)\right)$$

$$\delta_{(k+1)l} = \sum_{j,k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)j(k+1)l} - \alpha_{1j(k+1)l} \right).$$

The inequality (15) is written

$$\sum_{k=1}^{\infty} \delta_{(k+1)l} \geq \sum_{k=1}^{\infty} \delta_{kl} f(\beta_{1k})$$

and we can reduce it to

$$-\delta_{11} f(\beta_{11}) + \sum_{k=2}^{\infty} \delta_{k1}(1 - f(\beta_{1k})) \geq 0.$$

The tables in [1] are not usefull, because the error is great.

We have to state that $\delta_{11} < 0$, and $\delta_{k1} > 0$ because $f(\beta_{1k}) < 1$ for all non-decreasing functions on $I = [0,1]$.

We'll estimate that

$$-0,188557 < \delta_{11} < -0,11276 \tag{17}$$

using diagamma function too. First

$$\begin{aligned}
\delta_{11} &= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)j11} - \alpha_{1j11} \right) = \\
&= \sum_{j=1}^{\infty} \left(-\psi\left(2 + \frac{3}{3j+2}\right) + \psi\left(2 + \frac{2}{2j+1}\right) - \alpha_{1j11} \right) = \\
&= \sum_{j=1}^{\infty} \left(-\psi\left(1 + \frac{3}{3j+2}\right) + \psi\left(1 + \frac{2}{2j+1}\right) - 2\alpha_{1j11} \right)
\end{aligned}$$

$$\begin{aligned}
2\alpha_{1j11} = 2\sum_{j=1}^{\infty} \frac{1}{(2j+3)(3j+5)} &= 2\sum_{j=1}^{\infty} \left(\frac{1}{j + \frac{3}{2}} - \frac{1}{j + \frac{5}{3}} \right) = \\
&= +2\left(-\psi\left(\frac{3}{5}\right) + \psi\left(\frac{5}{3}\right) \right).
\end{aligned}$$

Using [1] we find

$$-0,29598 < -2\sum_{j=1}^{\infty} \alpha_{1j11} < -0,28784, \tag{18}$$

and then

$$s = \sum_{j=1}^{\infty} \psi\left(1 + \frac{2}{2j+1}\right) - \psi\left(1 + \frac{3}{3j+2}\right) = \psi(1,66) - \psi(1,6) + \psi(1,4) - \psi(1,375) + \ldots$$

we have

$$0,11041 < s < 0,17508. \qquad (19)$$

Adding (18) and (19) we find (16). Then $f(\beta_{11} = f\left(\frac{1}{2}\right) > 0$. The values δ_{k1}, $k \geq 2$ are computed with MATCAD are positive.

Proposition 2. If the conjecture (15) is true, by (14) and the inequality (14'), for all non-decreasing functions f we have

$$var\, U^4 f \leq \gamma_4\, var\, f.$$

Proof. Using (11) in (7), we find

$$var\, U^4 f = \sum_{i,j,k,l=1}^{\infty} (p_{ijkl}(1) f(u_{lkji}(1)) - p_{ijkl}(0) f(u_{lkji}(0))) \leq$$

$$\leq \left(\delta^{(4)}(f_4) + \sum_{i=1}^{\infty} \alpha_{(i+1)111}\right) var\, f = \gamma_4\, var\, f$$

$$\gamma_4 = \sum_{j,k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)j(k+1)1} - \alpha_{1j(k+1)1}\right) + \sum_{i=1}^{\infty} \alpha_{(i+1)111} = a + b$$

$$a = \sum_{j,k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_{(i+1)j(k+1)1} - \alpha_{1j(k+1)1}\right) = \sum_{k=1}^{\infty} \delta_{(k+1)1} < 0,007774828...$$

$$b = \sum_{i=1}^{\infty} \alpha_{(i+1)111} = -\psi\left(2 + \frac{3}{5}\right) + \psi\left(2 + \frac{2}{3}\right) =$$

$$= -\frac{5}{8} - \psi(1,6) + \frac{3}{5} + \psi(1,(6)) = -0,15104 + \psi(1,(6)).$$

As

$$-0,1804 < \psi(1,(6)) < 0,18447,$$

we find

$$0,037134828 < \gamma_4 < 0,041204828.$$

Remark. If

$$f(x) = \begin{cases} 1, & \text{if} \quad \alpha < x \leq 1 \\ 0, & \text{if} \quad 0 \leq x < \alpha, \end{cases}$$

where

$$\alpha = \lim_{n \to \infty} \frac{c_{2n}}{c_{2n+1}} = \frac{\sqrt{5} - 1}{2} = 0,6180339...$$

it is easy to check that $var\, U^n f = \gamma_n\, var\, f$ so $var\, U^4 f = \gamma_4\, var\, f$. This improves significantly the inequality $var\, U^4 f \leq 0,0625\, var\, f$ from [9].

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