FINITE HOMOGENEOUS MARKOV CHAIN INDUCED BY A BRANCHING PROCESS IN RANDOM ENVIRONMENT

Sergiu CORBU and Mihai POSTOLACHE

Abstract

In this work, we study a finite Markov chain, on a partition of $W = [0,1]$, as an approximation of the Markov process associated to a branching process defined by Smith and Wilkinson [18]. We determine the state limit vectors of this Markov chain.

Mathematics Subject Classification: 60J80.
Key words: Markov process, Markov chain, branching process, probability measure.

1 Introduction

Let $E = \{1, 2, \dots, r\}$ be a finite set, $(\varepsilon)_{n\geq 1}$ be a sequence of independent and identically distributed E-random variables, defined on a field of probability $(\Omega, \mathcal{K}, P)$, and γ be a discrete distribution on E:

$$q_i > 0, \quad P(\varepsilon_1 = i) = q_i, \quad i \in E, \quad \sum_{i\in E} q_i = 1.$$

For every ε, and for all $w \in W = [0,1]$, we associate a generating function of probability

$$\varphi_\varepsilon(w) = \sum_{i=0}^{\infty} p_i(\varepsilon) w^i, \quad p_i > 0, \quad \sum_{i=0}^{\infty} p_i(\varepsilon) = 1.$$

The sequence $\{\varphi_{\varepsilon_n}(w)\}_{n\geq 1}$ has the same distribution as $(\varepsilon_n)_{n\geq 1}$ and is called *random environment*. Consider the matrix $P = (P_{ij})_{i,j}$, where

$$P_{ij} \text{ is the coefficient of } w^j \text{ in } E[\varphi_{\varepsilon_n}(w)]^i, \quad i, j \in \mathbf{Z}.$$

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1998*, 29-36

Since ε_n are independent, it follows that P_{ij} is random variable independent of n. We have

$$\sum_{j=0}^{\infty} P_{ij} w^j = E[\varphi_{\varepsilon_n}(w)]^i, \quad \sum_{j=0}^{\infty} P_{ij} = 1, \quad \forall i \in \mathbf{Z} \text{ and } w = 1.$$

On the set of integers, we define the sequence of random variables $\{Z_n\}_{n\geq 0}$ as follows

$$Z_{n+1} = \sum_{\ell=1}^{Z_n} \varepsilon_\ell, \quad \text{if } Z_n > 0 \text{ and } Z_{n+1} = 0 \quad \text{if } Z_n = 0.$$

Conditioned by $\varepsilon = (\varepsilon_n)_{n\geq 1}$, the sequence $(Z_n)_{n\geq 0}$ defines a Markov chain whose initial probability is $P(Z_0 = i) = \delta_{ij}$, where δ_{ij} is the Kroneker symbol, and the transition matrix P is defined above, called *branching process in random environment* [18]. This stochastic process represents a natural process of dividing particles, whose life period is given. Independent of the initial particle, each particle gives rise to descendants according to a generating function of probability and so on, each new generation having a different generating function of probability from a family Φ. Z_n represents the sum of all individuals from the generation n.

2 Associated Markov process

Let $W = [0,1]$ and the family of generating functions of probability $\Phi = \{\varphi_k\}_{k\in E}$

$$\varphi_k : W \to W, \quad \varphi_k(w) = \sum_{i=0}^{\infty} p_i^k w^i, \quad \sum_{i=1}^{\infty} i p_i^k < \infty, \quad k \in E.$$

For given w_0, fixed but arbitrary in W, we define the W-sequence of random variables $(X_n)_{n\geq 0}$ as $X_{n+1} = \varphi_{\varepsilon_{n+1}}(X_n)$, where $X_0 = w_0$. The process X_n is a random product of generating functions of probability from the family Φ, where the random point X_n is defined as

$$X_n = (\varphi_{\varepsilon_n} \circ \cdots \circ \varphi_{\varepsilon_1})(w_0), \quad n \geq 1, \quad X_0 = w_0.$$

Now we are interested to study the convergence in distribution of X_n to a limit probability measure on $\mathcal{W}$, the σ-algebra of Borel subsets of W.

To give an answer to our problem, we define the Markov operator

$$\mathcal{U} f(w) = \sum_{i\in E} q_i f(\varphi_i(w)), \quad w \in W, \quad f \in \mathcal{C}(W).$$

where $\mathcal{C}(W)$ is the set of all bounded continuous complex-valued functions on W. The iterares of $\mathcal{U}$ are

$$\begin{aligned} \mathcal{U}^n f(w) &= \sum_{i_1,\ldots,i_n\in E} q_{i_1}\cdots q_{i_n} f[(\varphi_{i_n} \circ \cdots \circ \varphi_{i_1})(w)] \\ &= E[f(\varphi^{(n)}(w))], \end{aligned}$$

where we put $\varphi^{(n)}(w) = (\varphi_{i_n} \circ \cdots \circ \varphi_{i_1})(w)$.

According to Kaijser [8], and Barnsley and Elton [3], the following result holds.

Theorem 2.1 *Suppose* $0 < r < 1$ *and* $E\left(\log \frac{|X_1(w') - X_1(w'')|}{|w' - w''|}\right) \leq \log r$. *Then there exists a probability measure* μ *such that* $\lim_{n\to\infty} \mathcal{U}^n f(w) = \int_W f d\mu$.

Moreover, according to [7], for a random environment ε the sequence Z_n converges a. s. to a random variable defined on $0, 1, 2, \ldots$ and the generating function of X_n converges to $E(\varphi_{\varepsilon_n} \circ \cdots \circ \varphi_{\varepsilon_1}(w))$, $w \in W$. Therefore, if $n \to \infty$, the random variable $\varphi^{(n)}(w) = (\varphi_{\varepsilon_n} \circ \cdots \circ \varphi_{\varepsilon_1})(w)$ is convergent in probability to a random variable $\xi(w)$, for all $w \in [0, 1]$. The sequence $\{X_n(w)\}$ is a Markov process with values in $[0, 1]$, studied by Smith and Wilkinson [18], called *dual process* associated to the branching process in random environment. This process is a Markov chain with transition probability

$$P(w, A) = \sum_{i \in E} q_i I_A(\varphi_{\varepsilon_i}(w)), \quad \forall w \in W,$$

where $A \in \mathcal{W}$, $I_A(\cdot)$ is the indicator function of the set A and the transition probability after n steps is

$$P^n(w; A) = \sum_{i_1, \ldots, i_n \in I} q_{i_1} \cdots q_{i_n} I_A[(\varphi_{i_n} \circ \cdots \circ \varphi_{i_1})(w)].$$

3 Induced Markov chain

In this section, we shall introduce a finite homogeneous Markov chain to approximate the process $\{X_n\}$. As random environment of this process, we shall consider a family $\Phi = \{\varphi_1, \ldots, \varphi_r\}$ of generating functions of probability. Suppose these functions ordered as follows:

a) $\varphi_1, \ldots, \varphi_r$ are supercritical generating functions of probability, that is $\varphi_i'(1) = m_i > 1$, $i = 1, 2, \ldots, k < r$, hence the fixed points $\bar{w}_1 < \bar{w}_2 < \cdots < \bar{w}_k$ are not 0 and 1 ($\varphi_i(\bar{w}_i) = \bar{w}_i$);

b) $\varphi_{k+1}, \ldots, \varphi_r$ are subcritical and critical generating functions of probability, that is $\varphi_i'(1) = m_i \leq 1$, $i = k+1, \ldots, r$, hence they do not admit fixed points on $[0, 1]$.

Using these points, we consider a partition of W as follows

$$S_0 = [0, \bar{w}_1], \quad S_1 = (\bar{w}_1, \bar{w}_2], \quad \cdots, \quad S_{k-1} = (\bar{w}_{k-1}, \bar{w}_k], \quad S_k = (\bar{w}_k, 1].$$

These points, determine on the first bisectrice some squares as in Fig. 1. Consider $A_{ij}^\ell = \{\omega;\ \varphi_{\varepsilon(\omega)}^{-1} \in S_j \mid w \in S_i\}$, that is the set of all generating functions of probability who map the point in S_j provided that $w \in S_i$. Of course, the transition probability from S_i to S_j will be

$$q_{ij}(w) = \sum_{A_{ij}^\ell} q_\ell, \quad i, j \in \{0, 1, \ldots, k\}.$$

Since $q_{ij} \geq 0$ and

$$\sum_{j \in E} q_{ij} = \sum_{\ell} \sum_{j \in E} q_\ell I_{A_{ij}^\ell}(w) = \sum_{\ell} q_\ell I_W(w) = \sum_{\ell} q_\ell = 1,$$

it follows that the matrix $Q = (q_{ij})_{i,j}$ is a stochastic one, and $q_{ii} \geq q_i + q_{i+1}$ (see Fig. 1).

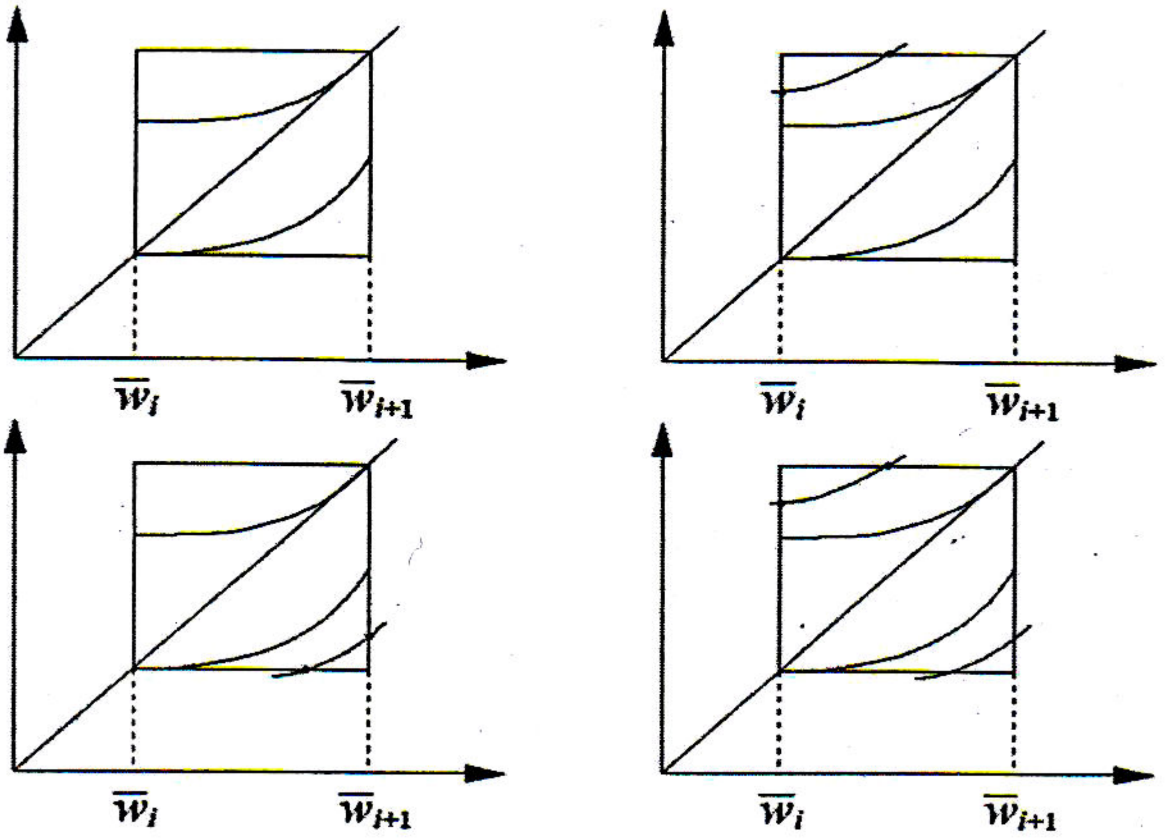

Fig. 1

We can easily prove that the transition matrix after n steps is

$$q_{ij}^{(n+1)}(w) = \sum_{\ell} \sum_{i_1,\ldots,i_n} q_{i_1} \cdots q_{i_n} I_{A_{ij}^{\ell}} (\varphi_{i_1} \circ \cdots \circ \varphi_{i_n})(w).$$

We shall consider the following remarkable cases:

1) If $\bar{w}_1 > 0$ ($\varphi_1(0) > 0$), then $q_{i0} = 0$ for every $i \geq 1$, and the transition matrix has the form

$$Q(w) = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0k} \\ 0 & q_{11} & \cdots & q_{1k} \\ \cdots & & & \\ 0 & q_{k1} & \cdots & q_{kk} \end{pmatrix}.$$

In this case, the state S_0 is transient and the states $S_1, \ldots, S_k$ are ergodic.

2) If $\bar{w}_1 = 0$ ($\varphi_1(0) = 0$), then there exists $i \geq 1$ such that $q_{i0} \neq 0$. In this case, we have

$$Q(w) = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0k} \\ q_{10} & q_{11} & \cdots & q_{1k} \\ \cdots & & & \\ q_{k0} & q_{k1} & \cdots & q_{kk} \end{pmatrix}.$$

It follows that all states composes a closed set.

3) If $k = r$, $\bar{w}_1 \neq 0$, $\bar{w}_k \neq 1$ (all generating functions of probability are supercritical), then

$$Q(w) = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0k-1} & 0 \\ 0 & q_{11} & \cdots & q_{1k-1} & 0 \\ \cdots & & & & \\ 0 & q_{k1} & \cdots & q_{kk-1} & q_{kk} \end{pmatrix}.$$

In this case, $q_{00} \cdot q_{kk} \neq 0$, therefore S_0 and S_k are asymptotic transient states, and $S_1, \ldots, S_{k-1}$ are asymptotic ergodic states.

4) If all generating functions of probability are subcritical ($m_i \leq 1$, $i \in E$), then the Markov chain has a unique asymptotic absorbant state $w = 1$. In the case 1), we remark that, S_0 is a "reflectant barrier" while in case 3) S_0 and S_k are reflectant barriers too. The transition matrix $Q = (q_{ij})$ and an initial vector of probabilities define a finite Markov chain on the set of states $\{S_0, S_1, \ldots, S_k\}$ denoted by $(Y_n)_{n \geq 0}$.

If $w_0 \in S_i$, then we may choose as initial vector $p_0 = (0, \ldots, 1, \ldots, 0)$ with 1 on the position i and 0 otherwise, and $p_n = p_0 Q_1 Q_2 \cdots Q_n = p_0 Q^{(n)}$ where $Q_n = Q(w_n)$.

To state some properties of the Markov chain $Y = (Y_n)_{n \geq 0}$ and its relation with the Markov chain $X = (X_n)_{n \geq 0}$, we shall suppose the following conditions:

Condition 1 *(boundedness).* If $P(w, A)$, $w \in W$, $A \in \mathcal{W}$, is the transition probability of the Markov chain X, suppose that there exists $c > 1$ such that

$$\frac{q_{ij}}{c} \leq P(w, S_j) \leq q_{ij} c, \quad w \in S_i, \quad i \neq j.$$

Condition 2 *(communication).* For all $w \in S_i$, suppose that there exists k such that $P^k(w, S_i) > 0$, where $P^k(\cdot, \cdot)$ is the transition probability in k steps of the chain X_0.

Using the definition of $P(\cdot, \cdot)$ and q_{ij}, we remark that

$$q_{ij} \leq P(w, S_j), \quad i, j \in \{0, 10, 2, \ldots, k\}.$$

For $i = j$, if denote by $\Gamma(\varphi_\ell)$ the graph of generating function of probability φ_ℓ, we have

$$r_i = P(w, S_i) = \sum{}^{(*)} q_\ell, \quad w \in S_i$$

the sum $(*)$ is for all ℓ from 0 to k such that the intersection between $\Gamma(\varphi_\ell)$ and $S_i \times S_i$ is not empty.

An important variable in the study of our Markov chain is the moment of the first entrance in a state. Using the transition probabilities q_{ij} of the Markov chain Y, we shall estimate the probability for the Markov chain X to enter in a state S_j if it starts from the point $w \in S_i$, $i \neq j$. We shall denote by $q(w, A)$ the probability that the Markon chain X arrive, for the first time, in $A \in \mathcal{W}$ if it starts from $w \notin A$. Suppose $w \in S_i$, $i \neq j$. Then, starting from w, to arrive in S_j for the first time, without passing through an intermediate state S_k, $k \neq i$, $k \neq j$, either it has to pass from w to S_j or to stay two steps in a row to S_i, and then to pass to S_j, or to stay three steps in a row to S_i, and then to pass to S_j and so on. This remark allows to write

$$q(w,S_j) = P(w,S_j) + \int_{S_i} P(w,du)P(u,S_j) + \int_{S_i} P(w,du_1)\int_{S_i} P(u_1,du_2)P(u_2,S_j) + \cdots, \quad w \in S_i, \quad i \neq j.$$

Theorem 3.1 *In the conditions given above, we have*

$$\frac{q_{ij}}{c(1-r_i)} \leq q(w,S_j) \leq \frac{cq_{ij}}{1-r_i}, \quad w \in S_i, \quad i \neq j.$$

Proof. By condition 1, we have

$$\begin{aligned} q(w,S_j) &\leq c\left[q_{ij} + q_{ij}\int_{S_i} P(w,du) + q_{ij}\int_{S_i} P(w,du_1)\int_{S_i} P(u_1;du_2) + \cdots\right] \\ &= cq_{ij}\left[1 + P(w,S_i) + P^2(w,S_i) + \cdots\right] \\ &= \frac{cq_{ij}}{1 - P(w,S_i)} \\ &= \frac{cp_{ij}}{1-r_i}. \end{aligned}$$

Similarly, we deduce

$$\begin{aligned} q(w,S_j) &\geq \frac{1}{c}\left[q_{ij} + q_{ij}\int_{S_i} P(w,du) + q_{ij}\int_{S_i} P(w,du_1)\int_{S_i} P(u_1,du_2) + \cdots\right] \\ &= cq_{ij}\left[1 + P(w,S_i) + P^2(w,S_i) + \cdots\right] \\ &= \frac{p_{ij}}{1-r_i}, \end{aligned}$$

where we have used the condition $0 < r_i < 1$.

The finite Markov chain obtained in this manner is nonhomogeneous and, in ou cases, the state limit vectors are convergent as follows:

1) $(0,\pi_1,\ldots,\pi_k)$, $\sum_{i=1}^{k} \pi_i = 1$;

2) $(\pi_0,\pi_1,\ldots,\pi_k)$, $\sum_{i=0}^{k} \pi_i = 1$;

3) $(0,\pi_1,\ldots,\pi_{k-1},0)$, $\sum_{i=1}^{k-1} \pi_i = 1$;

4) $(0,0,\ldots,1)$.

In cases 1), 2), 3) the invariant mesures are concentrated on the atractors of generatin functions of probability, that is the fixed points satisfying $|\varphi'(w)| < 1$. The fixe points w satisfying $|\varphi'(w)| > 1$ are called repellors. For a generating function of probability, we can see that all fixed points from $(0, 1)$ are attractors. A similar result is obtained by Gora [5] for a single function compounded at random with the perturbation of the identity mapping. The invariant measure of the Markov process is concentrated on the mapping's attractors.

References

[1] K.B. Athreya and P.E. Ney: *Branching Processes*, Springer-Verlag, Heidelberg and New York, 1972.

[2] M. Barnsley and S. Demko: Iterated functions system and the global construction of fractals, *Proc. R. Soc., London*, **A399** (1985), 243-275.

[3] M. Barnsley and J. Elton: A new class of Markov processes for image encoding, *Adv. Appl. Prob.*, **20** (1988), 14-32.

[4] S. Corbu and M. Postolache: Fractals generated by Möbius transformations and applications, in *New Frontiers in Algebra, Groups and Geometries*, (Gr. Tsagas (Ed.)), Hadronic Press, 1996, 407.

[5] P. Gora: Random comparing mappings, small stochastic perturbations and attractors, *Z. wahr. verw. Gabiete.*, **69** (1985), 137-160.

[6] M. Iosifescu and S. Grigorescu: *Dependence with Complete Connections and Its Applications*, Cambridge Univ. Press, 1991.

[7] P. Jagers: *Branching Processes with Biological Applications*, Wiley, New York, 1975.

[8] T. Kaijser: A limit theorem for Markov chains in compact metric spaces with applications to products of random matrices, *Duke Math. J.*, **45** (1978), 311-349.

[9] N. Kaplan: Some result about multidimensional branching processes with random environments, *Ann. Prob.*, **2** (1974), 411-455.

[10] Y. Kifer: *Ergodic Theory of Random Transformations*, Birkhäuser, 1986.

[11] T. Morita: Asymptotic behaviour of one-dimensional random dynamical systems, *J. Math. Soc. Japan*, **37** (1985), 651-663.

[12] T. Morita: *Hiroshima Math. J.*, **18** (1988), 15.

[13] T. Ohno: Asymptopic behaviour of dynamical systems with random parameters, *Publ. RIMS Kyoto Univ.*, **19** (1983), 83-98.

[14] B. Oksendal: *Stochastic Differential Equations*, Springer-Verlag, Heidelberg and New York, 1995

[15] V.I. Oseledec: A multiplicative ergodic theorem, *Trudii Moscov Mat. Obsc.*, 19 (1968), 179-210.

[16] M. Postolache: Random affine transformations and fractal structures, to appear.

[17] M. Postolache: *Numerical Methods (Second Revised Edition)*, Sirius Publishers, Bucharest, 1994 (in Romanian).

[18] W. Smith and W. Wilkinson: On branching processes in random environment, *Ann. Math. Statistics*, 40 (1969), 814-827.

Author's address:

S. Corbu and M. Postolache
Politehnica University of Bucharest
Department of Mathematics I
Splaiul Independenţei 313,
77206 Bucharest, Romania
E-mail: mihai@mathem.pub.ro