

INFINITESIMAL SYMMETRIES OF CAMASSA-HOLM EQUATIONS

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Abstract

§1 recalls known results about symmetry group of a third order PDE. One determines the symmetry group G (Theorem 2) of the Camassa-Holm equation (7), and one finds the family of third order PDEs which admits the Lie group G as symmetry group (Theorem 3). This class contains the Camassa-Holm equation and the Rosenau-Hyman equation.

Key-words: symmetry group, infinitesimal symmetries, criterion of infinitesimal invariance, Camassa-Holm equation.

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Camassa and Holm derived a new completely integrable dispersive shallow water equation that is bi-Hamiltonian and thus possesses an infinite number of conservation laws in involution. The Camassa-Holm (CH) PDE is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler's equations in the shallow water regime.

Holm, Marsden and Ratiu [10] have shown that Camassa-Holm equation [6] in n dimensions describes geodesic motion on the diffeomorphism group of R^n with respect to metric given by the H^1 norm of Eulerian fluid velocity. Misiolek [13] has shown that the CH equation represents a geodesic flow on the Bott-Virasoro group. Kouranbaeva [11] has shown that the CH equation (for the case $k = 0$) is a geodesic spray of the weak Riemannian metric on the diffeomorphism group of the line or the circle obtained by the right translation of the H^1 inner product over the entire group.

In the first part of this article one makes a short presentation of the theory of the infinitesimal symmetries associated with a third order PDE [5], [14], [16], and in the second part one applies this theory to the case of the Camassa-Holm PDE. It should be noticed that compared to the paper [9], the present work provides the whole family of PDEs of order three which are invariant under the same group which preserves the Camassa-Holm PDE.

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1 Symmetry group of a third order PDE

Let $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$; $\pi(x, u) = x$, $x = (x^1, \dots, x^n)$, be the projection map. Let $U \subset \mathbb{R}^{n+1}$ be an open set and $U_0 = \pi(U)$.

Definition 1. A smooth map $s : U_0 \rightarrow U$, $s(x) = (x, u(x))$ is called *local section of π (on U_0)*.

For the function u , we note

$$u_{i_1 \dots i_p}(x) = \frac{\partial^p u}{\partial x^{i_1} \dots \partial x^{i_p}}(x), \quad x \in U_0, \quad 1 \leq i_1 \leq \dots \leq i_p \leq n, \quad p \geq 1.$$

Let consider

$$J^k(U) = \{(x^i, u, u_{i_1}, \dots, u_{i_1 \dots i_k}) | (x^i, u) \in U\},$$

$J^k(U) \subset \mathbb{R}^{n+1} \times \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_k}$, where $N_k = \frac{(n+k-1)!}{k!(n-1)!}$, $k > 0$, and the projection

$$\pi^k : J^k(U) \rightarrow U_0, \quad \pi^k(x, u, u_{i_1}, \dots, u_{i_1 \dots i_k}) = x.$$

Convention: for $k = 0$, $J^0(U) = U$ and $\pi^0 = \pi$.

Definition 2. Let $s : U_0 \rightarrow U$ be a local section of π (over U_0). The section $j^k(s)$ of π^k over U_0 , called *the k -jet of s* , is defined as follows:

$$j^k(s)(x) = (x, u, u_{i_1}, \dots, u_{i_1 \dots i_k}), \quad x \in U_0,$$

$$u_{i_1 \dots i_r} = \frac{\partial^r u}{\partial x^{i_1} \dots \partial x^{i_r}}(p), \quad 0 \leq r \leq k.$$

The space $J^k(U)$ is called *the k th order jet space*.

Let $\Omega_k^q(U)$ be the vector space of q -forms on $J^k(U)$ with exterior differential d . In particular $\Omega_0^q(U) = \Omega^q(U)$ is the exterior algebra of q -forms on U and $\Omega_k^0 = C^\infty(J^k(U))$ is the algebra of real-functions

$$f = f(x^i, u, u_{i_1}, \dots, u_{i_1 \dots i_k})$$

on $J^k(U)$. A basis for $\Omega_k^1(U)$ (as a module $C^\infty(J^k(U))$) consists of the 1-forms $dx^i, du, du_{i_1}, \dots, du_{i_1 \dots i_k}$.

For $f \in C^\infty(J^k(U))$ we have

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial u_{i_1}} du_{i_1} + \dots + \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \frac{\partial f}{\partial u_{i_1 \dots i_k}} du_{i_1 \dots i_k}.$$

Definition 3. A form $\omega \in \Omega_k^q(U)$ is called *basic* if

$$\omega = A_{j_1 \dots j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

where $A_{j_1 \dots j_q}$ are differentiable functions on $J^k(U)$.

We note by $\mathcal{B}_k^q(U)$, the space of the basic q -forms.

Definition 4. The operator $D : \mathcal{B}_k^q(U) \rightarrow \mathcal{B}_{k+1}^{q+1}(U)$,

$$Df = \left(\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial u} u_j + \frac{\partial f}{\partial u_{i_1}} u_{i_1 j} + \dots + \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \frac{\partial f}{\partial u_{i_1 \dots i_k}} u_{i_1 \dots i_k j} \right) dx^j,$$

for $f \in C^\infty(J^k(U))$, and

$$D\omega = DA_{j_1 \dots j_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

for

$$\omega = A_{j_1 \dots j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

is called the total exterior derivative on $\mathcal{B}_k^q(U)$.

Convention: $Df = D_i f dx^i$, where $D_i f$ is the total derivative of f with respect to x^i , and

$$D_{ij} = D_i D_j, \quad D_{ijk} = D_i D_{jk}.$$

Let $\Omega_3^{n+1}(U)$ be the space of $(n+1)$ -forms, and $\theta \in \Omega_3^{n+1}(U)$,

$$\theta = F(x, u, u_i, u_{ik}, u_{ijk}) du \wedge dx^1 \wedge \dots \wedge dx^n.$$

Definition 5. A solution of the equation

$$\theta = 0,$$

on $U_0 \subset \mathbb{R}^n$, is a section $s : U_0 \rightarrow U$ such that $\theta \cdot j^3(s) = 0$.

In other words, θ determines the PDE

$$(1) \quad F(x, u, u_i, u_{ik}, u_{ijk}) = 0,$$

a solution of which is a function $u = u(x)$ such that

$$F \left(x, u(x), \frac{\partial u}{\partial x^i}(x), \frac{\partial^2 u}{\partial x^i \partial x^k}(x), \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}(x) \right) = 0, \quad \forall x \in U_0.$$

We note $u^{(3)} = (u, u_i, u_{ik}, u_{ijk})$.

Definition 6. The PDE (1) is called of maximal rang if the Jacobi matrix

$$J_F(x, u^{(3)}) = (F_x; F_u; F_{u_i}; F_{u_{ik}}; F_{u_{ijk}})$$

has rank 1 whenever $F(x, u^{(3)}) = 0$.

Then the subset

$$S = \{(x, u^{(3)}) \in J^{(3)}(U) | F(x, u^{(3)}) = 0\}$$

is a hypersurface.

Definition 7. A symmetry group of PDE (1) is a local transformations Lie group G acting on an open set U of the associated space of independent and dependent variables, with the properties:

(a) if $u = f(x, y)$ is a solution of the equation and if $g \cdot f$ has sense for $g \in G$, then $v = g \cdot f(x, y)$ is also a solution.

(b) any solution of the equation can be obtained by a DE associated to PDE (hence any solution is G -invariant $g \cdot f = f, \forall g \in G$).

For the determination of the symmetry group of PDE (1) is used the following criterion of infinitesimal invariance ([14]).

Theorem 1. Let (1) be a PDE of maximal rank defined on an open set U_0 . If G is a local group of transformations acting on U and

$$(2) \quad \text{pr}^{(3)}X[F(x, u^{(3)})] = 0 \quad \text{whenever} \quad F(x, u^{(3)}) = 0,$$

for every infinitesimal generator X of G , then G is a symmetry group of the given equation.

Let

$$X = \zeta^i(x, u) \frac{\partial}{\partial x^i} + \phi(x, u) \frac{\partial}{\partial u}$$

be vector field on U . The first, second and respectively third order prolongations of the vector field X are

$$(3) \quad \begin{aligned} \text{pr}^{(1)}X &= X + \Phi_i \frac{\partial}{\partial u_i}, \\ \text{pr}^{(2)}X &= \text{pr}^{(1)}X + \Phi_{ij} \frac{\partial}{\partial u_{ij}}, \\ \text{pr}^{(3)}X &= \text{pr}^{(2)}X + \Phi_{ijk} \frac{\partial}{\partial u_{ijk}}, \end{aligned}$$

where

$$\begin{aligned} \Phi_i &= D_i(\phi - \zeta^k u_k) + \zeta^k u_{ik} = D_i\phi - u_k D_i(\zeta^k), \\ \Phi_{ij} &= D_{ij}(\phi - \zeta^k u_k) + \zeta^k u_{ijk} = D_{ij}(\phi) - u_k D_{ij}(\zeta^k) - u_{ki} D_j(\zeta^k) - u_{kj} D_i(\zeta^k), \\ \Phi_{ijk} &= D_{ijk}(\phi - \zeta^l u_l) + \zeta^l u_{ijkl} = D_{ijk}(\phi) - u_l D_{ijk}(\zeta^l) - u_{li} D_{ij}(\zeta^l) - \\ &\quad - u_{li} D_{jk}(\zeta^l) - u_{lj} D_{ik}(\zeta^l) - u_{lkj} D_i(\zeta^l) - u_{lki} D_j(\zeta^l) - u_{lji} D_k(\zeta^l). \end{aligned}$$

Proposition. Let be a PDE of the maximal rank defined on U_0 . The set of all infinitesimal symmetries of the equation forms a Lie algebra on U . Moreover, if this Lie algebra is finite-dimensional, the symmetry group of the equation is a local Lie group of transformations acting on U .

Algorithm for finding the symmetry group of PDE (1)

-one considers a vector field X on U and one writes the infinitesimal invariance condition (2);

-one eliminates any dependence between partial derivatives of the function u , determined by the PDE (1):

-one writes the condition (2) like a polynomial in the partial derivatives of u ;
 -one equates with zero the coefficients of partial derivatives of u in (2), written as a polynomial in the derivatives of the function u ; it follows a PDEs system with respect to the unknown functions ζ^i , ϕ , and this system defines the Lie symmetry group G of the given PDE.

Every s -parametric subgroup H of the group G determines a family of group-invariant solutions. The problem of classification of group-invariant solutions reverts to the problem of classification of Lie subalgebras of Lie algebra \mathfrak{g} of the group G ([14], 186). For 1-dimensional algebras one considers a general element X , and we simplify this as much as possible using the adjoint transformations.

Remark. We will compute the adjoint representation $Ad G$ of the underlying Lie group G , by using the Lie series

$$(4) \quad Ad(\exp(\varepsilon X)Y) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (adX)^n(Y) = Y - \varepsilon[X, Y] + \frac{\varepsilon^2}{2}[X, [X, Y]] - \dots$$

In the case $n = 2$, we note

$$\begin{aligned} x^1 &= x, \quad x^2 = y, \quad \zeta^1 = \zeta, \quad \zeta^2 = \eta, \quad \Phi_1 = \Phi^x, \quad \Phi_2 = \Phi^y, \\ \Phi_{11} &= \Phi^{xx}, \quad \Phi_{12} = \Phi^{xy}, \quad \Phi_{22} = \Phi^{yy}, \\ \Phi_{111} &= \Phi^{xxx}, \quad \Phi_{112} = \Phi^{xxy}, \quad \Phi_{122} = \Phi^{xyy}, \quad \Phi_{222} = \Phi^{yyy} \end{aligned}$$

and

$$u^{(3)} = (u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}).$$

If

$$X = \zeta(x, y, u) \frac{\partial}{\partial x} + \eta(x, y, u) \frac{\partial}{\partial y} + \phi(x, y, u) \frac{\partial}{\partial u},$$

is the infinitesimal generator of the symmetry group of the PDE

$$(5) \quad F(x, y, u^{(3)}) = 0,$$

then the infinitesimal condition (2) becomes

$$\begin{aligned} (6) \quad & \zeta \frac{\partial F}{\partial x} + \eta \frac{\partial F}{\partial y} + \phi \frac{\partial F}{\partial u} + \Phi^x \frac{\partial F}{\partial u_x} + \Phi^y \frac{\partial F}{\partial u_y} + \\ & + \Phi^{xx} \frac{\partial F}{\partial u_{xx}} + \Phi^{xy} \frac{\partial F}{\partial u_{xy}} + \Phi^{yy} \frac{\partial F}{\partial u_{yy}} + \\ & + \Phi^{xxx} \frac{\partial F}{\partial u_{xxx}} + \Phi^{xxy} \frac{\partial F}{\partial u_{xxy}} + \Phi^{xyy} \frac{\partial F}{\partial u_{xyy}} + \Phi^{yyy} \frac{\partial F}{\partial u_{yyy}} = 0. \end{aligned}$$

2 Symmetry Lie group of Camassa-Holm equation

Let us consider the Camassa-Holm PDE ([6])

$$(7) \quad uu_{xxx} + u_{xy} + 2u_x u_{xx} - 3uu_x - u_y = 0.$$

The Jacobi matrix of the function

$$F(x, y, u^{(3)}) = uu_{xxx} + u_{xy} + 2u_x u_{xx} - 3uu_x - u_y$$

is

$$J_F = (0, 0; u_{xxx} - 3u_x; 2u_{xx} - 3u, -1; 2u_x, 0, 0; u, 1, 0, 0).$$

Because $\text{rank} J_F = 1$, it results that the PDE (7) is of maximal rank.

Let

$$X = \zeta(x, y, u) \frac{\partial}{\partial x} + \eta(x, y, u) \frac{\partial}{\partial y} + \phi(x, y, u) \frac{\partial}{\partial u}$$

be the infinitesimal generator of the symmetry group of the PDE (7). In this case, the condition (6) turns in

$$\phi(u_{xxx} - 3u_x) + \Phi^x(2u_{xx} - 3u) - \Phi^y + 2u_x \Phi^{xx} + u \Phi^{xxx} + \Phi^{xy} = 0.$$

On the other hand, the relations (3) implies

$$\Phi^x = \phi_x + (\phi_u - \zeta_x)u_x - \eta_x u_y - \zeta_u u_x^2 - \eta_u u_x u_y,$$

$$\Phi^y = \phi_y - \zeta_y u_x + (\phi_u - \eta_y)u_y - \zeta_u u_x u_y - \eta_u u_y^2,$$

$$\begin{aligned} \Phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \zeta_{xx})u_x - \eta_{xx} u_y + (\phi_{uu} - 2\zeta_{xu})u_x^2 - \\ &- 2\eta_{xu} u_x u_y - \zeta_{uu} u_x^3 - \eta_{uu} u_x^2 u_y + (\phi_u - 2\zeta_x)u_{xx} - 2\eta_x u_{xy} - \\ &- 3\zeta_u u_x u_{xx} - \eta_u u_y u_{xx} - 2\eta_u u_x u_{xy}, \end{aligned}$$

$$\begin{aligned} \Phi^{xy} &= \phi_{xy} + (\phi_{uy} - \zeta_{xy})u_x + (\phi_{ux} - \eta_{xy})u_y - \zeta_{uy} u_x^2 + (\phi_{uu} - \zeta_{ux} - \\ &- \eta_{uy})u_x u_y - \eta_{ux} u_y^2 - \zeta_y u_{xx} + (\phi_u - \zeta_x - \eta_y)u_{xy} - \eta_x u_{yy} - \\ &- \zeta_u u_y u_{xx} - 2\eta_u u_y u_{xy} - 2\zeta_u u_x u_{xy} - \eta_u u_x u_{yy} - \zeta_{uu} u_x^2 u_y - \eta_{uu} u_x u_y^2, \end{aligned}$$

$$\begin{aligned} \Phi^{yy} &= \phi_{yy} + (2\phi_{uy} - \eta_{yy})u_y - \zeta_{yy} u_x + (\phi_{uu} - 2\eta_{uy})u_y^2 - 2\zeta_{uy} u_x u_y - \\ &- \eta_{uu} u_y^3 - \zeta_{uu} u_x u_y^2 + (\phi_u - 2\eta_y)u_{yy} - 2\zeta_y u_{xy} - 3\eta_u u_y u_{yy} - \\ &- \zeta_u u_x u_{yy} - 2\zeta_u u_y u_{xy}, \end{aligned}$$

and respectively

$$\begin{aligned} \Phi^{xxx} &= \phi_{xxx} + u_x(3\phi_{xuu} - \zeta_{xxx}) - u_y \eta_{xxx} + 3u_x^2(\phi_{xuu} - \zeta_{xuu}) - 3u_x u_y \eta_{xuu} + \\ &+ u_x^3(\phi_{uuu} - 3\zeta_{xuu}) - 3u_x^2 u_y \eta_{xuu} - u_x^4 \zeta_{uuu} - u_x^3 u_y \eta_{uuu} + 3u_{xx}(\phi_{xu} - \\ &- \zeta_{xx}) - 3u_{xy} \eta_{xx} + 3u_x u_{xx}(\phi_{uu} - 3\zeta_{xu}) - 3u_y u_{xx} \eta_{xu} - 6u_x u_{xy} \eta_{xu} - \\ &- 6u_x^2 u_{xx} \zeta_{uu} - 3u_x u_y u_{xx} \eta_{uu} - 3u_x^2 u_{xy} \eta_{uu} - 3u_{xx}^2 \zeta_u - 3u_{xx} u_{xy} \eta_u + \\ &+ u_{xxx}(\phi_u - 3\zeta_x) - 3u_{xy} \eta_x - 4u_x u_{xxx} \zeta_u - u_y u_{xxx} \eta_u - 3u_x u_{xy} \eta_u, \end{aligned}$$

$$\begin{aligned} \Phi^{xzy} = & \phi_{xzy} + u_x(2\phi_{zyu} - \zeta_{xzy}) + u_y(\phi_{xzu} - \eta_{xzy}) + u_x^2(\phi_{yuu} - 2\zeta_{zyu}) + \\ & + u_x u_y(2\phi_{zuu} - \zeta_{xzu} - 2\eta_{zyu}) - u_y^2 \eta_{xzu} - u_x^3 \zeta_{yuu} + u_x^2 u_y(\phi_{uuu} - \eta_{yuu} - \\ & - 2\zeta_{xuu}) - 2u_x u_y^2 \eta_{xuu} - u_x^3 u_y \zeta_{uuu} - u_x^2 u_y^2 \eta_{uuu} + u_{xx}(\phi_{yu} - 2\zeta_{xy}) + \\ & + u_{xy}(2\phi_{zu} - \zeta_{xz} - 2\eta_{zy}) - u_{yy} \eta_{xz} - 3u_x u_{xz} \zeta_{yu} + u_y u_{xz}(\phi_{uu} - \eta_{yu} - \\ & - 2\zeta_{xu}) + 2u_x u_{xy}(\phi_{uu} - 2\zeta_{xu} - \eta_{yu}) - 4u_y u_{xy} \eta_{xu} - 2u_x u_{yy} \eta_{xu} - \\ & - 3u_x u_y u_{xz} \zeta_{uu} - 4u_x u_y u_{xy} \eta_{uu} - 3u_x^2 u_{xy} \zeta_{uu} - u_y^2 u_{xz} \eta_{uu} - u_x^2 u_{yy} \eta_{uu} - \\ & - 3u_{xz} u_{xy} \zeta_u - 2u_{xy}^2 \eta_u - u_{xz} u_{yy} \eta_u - u_{xxx} \zeta_y + u_{xzy}(\phi_u - \eta_y - 2\zeta_x) - \\ & - 2u_{xyy} \eta_x - u_y u_{xxx} \zeta_u - 2u_y u_{xxy} \eta_u - 3u_x u_{xzy} \zeta_u - 2u_x u_{xyy} \eta_u, \end{aligned}$$

$$\begin{aligned} \Phi^{zyy} = & \phi_{zyy} + u_x(\phi_{yyu} - \zeta_{xyy}) + u_y(2\phi_{zyu} - \eta_{zyy}) - u_x^2 \zeta_{yyu} + u_x u_y(2\phi_{yuu} - \\ & - 2\zeta_{xyu} - \eta_{yyu}) + u_y^2(\phi_{xuu} - 2\eta_{zyu}) + u_x u_y^2(\phi_{uuu} - \zeta_{xuu} - 2\eta_{yuu}) - \\ & - 2u_x^2 u_y \zeta_{yuu} - u_y^3 \eta_{xuu} - u_x u_y^3 \eta_{uuu} - u_x^2 u_y^2 \zeta_{uuu} - u_{xz} \zeta_{yy} + u_{xy}(2\phi_{yu} - \\ & - \eta_{yy} - 2\zeta_{xy}) + u_{yy}(\phi_{zu} - 2\eta_{zy}) - 2u_y u_{xz} \zeta_{yu} - 4u_x u_{xy} \zeta_{yu} + \\ & + 2u_x u_{xy}(\phi_{uu} - 2\eta_{yu} - \zeta_{xu}) + u_x u_{yy}(\phi_{uu} - \zeta_{xu} - 2\eta_{yu}) - 3u_y u_{yy} \eta_{xu} - \\ & - u_x^2 u_{yy} \zeta_{uu} - 4u_x u_y u_{xy} \zeta_{uu} - 3u_x u_y u_{yy} \eta_{uu} - u_y^2 u_{xz} \zeta_{uu} - 3u_y^2 u_{xy} \eta_{uu} - \\ & - u_{xz} u_{yy} \zeta_u - 2u_{xy}^2 \zeta_u - 3u_{xy} u_{yy} \eta_u - 2u_{xzy} \zeta_y + u_{zyy}(\phi_u - \zeta_x - 2\eta_y) - \\ & - u_{yyy} \eta_x - 2u_x u_{zyy} \zeta_u - u_x u_{yyy} \eta_u - 2u_y u_{xzy} \zeta_u - 3u_y u_{xyy} \eta_u, \end{aligned}$$

$$\begin{aligned} \Phi^{yyy} = & \phi_{yyy} - u_x \zeta_{yyy} + u_y(3\phi_{yuu} - \eta_{yyy}) - 3u_x u_y \zeta_{yyu} + 3u_y^2(\phi_{yuu} - \eta_{yyu}) - \\ & - 3u_x u_y^2 \zeta_{yuu} + u_y^3(\phi_{uuu} - 3\eta_{yuu}) - u_x u_y^3 \zeta_{uuu} - u_y^4 \eta_{uuu} - 3u_{xy} \zeta_{yy} + \\ & + 3u_{yy}(\phi_{yu} - \eta_{yy}) - 3u_x u_{yy} \zeta_{yu} + 3u_y u_{yy}(\phi_{uu} - 3\eta_{yu}) - 6u_y u_{xy} \zeta_{yu} - \\ & - 3u_x u_y u_{yy} \zeta_{uu} - 3u_y^2 u_{xy} \zeta_{uu} - 6u_y^2 u_{yy} \eta_{uu} - 3u_{xy} u_{yy} \zeta_u - 3u_{yy}^2 \eta_u - \\ & - 3u_{xyy} \zeta_y + u_{yyy}(\phi_u - 3\eta_y) - 3u_y u_{xyy} \zeta_u - u_x u_{yyy} \zeta_u - 4u_y u_{yyy} \eta_u. \end{aligned}$$

One substitutes the functions Φ^x , Φ^y , Φ^{xx} , Φ^{xxx} , Φ^{xzy} , and one eliminates any dependence among the derivatives of the function u , by substituting

$$u_{xzy} = -u u_{xxx} - 2u_x u_{xz} + 3u u_x + u_y$$

in the above relation. Thus, one finds

$$\begin{aligned}
& -3u\phi_x - \phi_y + u\phi_{xxx} + \phi_{xxy} + u_x(-3\phi - 3u\zeta_x - 3u\eta_y + \zeta_y - 9u^2\eta_x - u\zeta_{xxx} - \\
& -\zeta_{xxy} + 2\phi_{xx} + 3u\phi_{xxu} + 2\phi_{xyu}) + u_y(-2\zeta_x - u\eta_{xxx} - \eta_{xxy} + \phi_{xxu}) + u_x^2(-6u\zeta_u - \\
& -9u^2\eta_u - 2\zeta_{xx} - 3u\zeta_{xxu} - 2\zeta_{xyu} + 3u\phi_{xuu} + \phi_{yuu} + 4\phi_{xu}) - u_x u_y(2\zeta_u + 6u\eta_u + \\
& + 2\eta_{xx} + 3u\eta_{xxu} + \zeta_{xzu} + 2\eta_{xyu} - 2\phi_{xuu}) - u_y^2(\eta_u + \eta_{xzu}) - u_x^3(4\zeta_{xu} + 3u\zeta_{xuu} + \\
& + \zeta_{yuu} - 2\phi_{uu} - u\phi_{uuu}) - u_x^2 u_y(4\eta_{xu} + 3u\eta_{xuu} + \eta_{yuu} + 2\zeta_{xuu} - \phi_{uuu}) - 2u_x u_y^2 \eta_{xuu} - \\
& - u_x^4(2\zeta_{uu} + u\zeta_{uuu}) - u_x^3 u_y(2\eta_{uu} + u\eta_{uuu} + \zeta_{uuu}) - u_x^2 u_y^2 \eta_{uuu} + u_{xx}(2\phi_x + 3u\phi_{xu} - \\
& - 3u\zeta_{xx} + \phi_{yu} - 2\zeta_{xy}) - u_{xy}(\zeta_{xx} + 2\eta_{xy} + 3u\eta_{xx} - 2\phi_{xu}) - u_{yy}\eta_{xx} + u_x u_{xx}(-2\zeta_x + 2\eta_y + \\
& + 2\phi_u + 6u\eta_x - 9u\zeta_{xu} - 3\zeta_{yu} + 3u\phi_{uu}) - u_y u_{xx}(2\eta_x + 3u\eta_{xu} + 2\zeta_{xu} + \eta_{yu} - \phi_{uu}) - \\
& - 2u_x u_{xy}(2\eta_x + 3u\eta_{xu} - \phi_{uu} + 2\zeta_{xu} + \eta_{yu}) - 4u_y u_{xy}\eta_{xu} - 2u_x u_{yy}\eta_{xu} - 2u_x^2 u_{xx}(\zeta_u - \\
& - 3u\eta_u + 3u\zeta_{uu}) - 3u_x u_y u_{xx}(\zeta_{uu} + u\eta_{uu}) - u_y^2 u_{xx}\eta_{uu} - u_x^2 u_{xy}(4\eta_u + 3u\eta_{uu} + 3\zeta_{uu}) - \\
& - 4u_x u_y u_{xy}\eta_{uu} - u_x^2 u_{yy}\eta_{uu} - 3u_{xx}^2 u\zeta_u - 3u_{xx} u_{xy}(u\eta_u + \zeta_u) - u_{xx} u_{yy}\eta_u - \\
& - 2u_{xy}^2 \eta_u + u_{xxx}(\phi - u\zeta_x - \zeta_y + u\eta_y + 3u^2\eta_x) - 2u_{xyy}\eta_x + \\
& + u_x u_{xxx}(-u\zeta_u + 3u^2\eta_u) + u_y u_{xxx}(-\zeta_u + u\eta_u) - 2u_x u_{xyy}\eta_u = 0.
\end{aligned}$$

Looking at this condition as a polynomial in the partial derivatives of the function u , and identifying with the polynom zero, we obtain the following PDEs system

$$\begin{cases} \zeta_x = 0 \\ \zeta_y = 0 \\ \zeta_u = 0 \\ \eta_x = 0 \\ \eta_u = 0 \end{cases} \quad \begin{cases} \eta_{yy} = 0 \\ \phi_x = 0 \\ \phi_y = 0 \\ \phi_{uu} = 0 \\ \phi + u\eta_y = 0, \end{cases}$$

for which

$$\begin{cases} \zeta = C_1 \\ \eta = C_3 y + C_2 \\ \phi = -C_3 u, \end{cases}$$

is the general solution, where $C_1, C_2, C_3 \in \mathbb{R}$.

We get

$$X = C_1 \frac{\partial}{\partial x} + C_2 \frac{\partial}{\partial y} + C_3 \left(y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} \right).$$

Theorem 2. *The Lie algebra \mathfrak{g} of the infinitesimal transformations associated with the Camassa-Holm PDE is described by the vector fields*

$$(8) \quad \begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u}. \end{aligned}$$

Remarks.

1. The structure equations of the Lie algebra \mathfrak{g} of the symmetry group G are

$$[X_1, X_2] = 0, \quad [X_2, X_3] = X_2, \quad [X_3, X_1] = 0.$$

2. If $u = f(x, y)$ is a solution of the PDE (7), then each function

$$\begin{aligned} u^{(1)} &= f(x - \epsilon, y), \\ u^{(2)} &= f(x, y - \epsilon), \\ u^{(3)} &= e^{-\epsilon} f(x, e^{-\epsilon} y), \quad \epsilon \in R, \end{aligned}$$

is another solution for the equation.

3. By using the Lie series (4), the adjoint representation AdG of the Lie group G , is described by the table

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$X_3 - \epsilon X_2$
X_3	X_1	$e^\epsilon X_2$	X_3

Now we shall study the converse of the Theorem 2: given the Lie group G of transformations, determine the third order PDE which admits G like symmetry group.

The algorithm of determination of PDE ([5], 303), implies the using of a maximal chain of Lie subalgebras of the algebra \mathfrak{g} of the Lie group G , in the case in which \mathfrak{g} is solvable.

Because in the case of the Camassa-Holm equation, the Lie algebra \mathfrak{g} is solvable, we have the next chain of Lie subalgebras of it

$$\{X_1\} \subset \{X_1, X_2\} \subset \{X_1, X_2, X_3\}.$$

Theorem 3. *The third order PDE*

$$H(x, y, u^{(3)}) = 0$$

whose Lie algebra of the infinitesimal symmetries is described by the vectors fields (8), has the form

$$(9) \quad h \left(\frac{u_x}{u}, \frac{u_y}{u^2}, \frac{u_{xx}}{u}, \frac{u_{xy}}{u^2}, \frac{u_{yy}}{u^3}, \frac{u_{xxx}}{u}, \frac{u_{xxy}}{u^2}, \frac{u_{xyy}}{u^3}, \frac{u_{yyy}}{u^3} \right) = 0.$$

Proof. We consider the condition (6) for each above subalgebras.

1. $\{X_1\}$: $X_1 = \frac{\partial}{\partial x}$ și $pr^{(3)}X_1 = \frac{\partial}{\partial x}$. The condition (6) implies $pr^{(3)}X_1(H) = 0$, i.e.,

$$\frac{\partial H}{\partial x} = 0.$$

It results

$$H = f(y, u^{(3)}).$$

2. $\{X_1, X_2\}$: $X_2 = \frac{\partial}{\partial y}$ and $pr^{(3)}X_2 = \frac{\partial}{\partial y}$. Analogously, we get

$$H = g(u^{(3)}).$$

3. $\{X_1, X_2, X_3\}$: $X_3 = y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u}$, and

$$\begin{aligned} pr^{(3)}X_3 &= y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} - u_x \frac{\partial}{\partial u_x} - 2u_y \frac{\partial}{\partial u_y} - u_{xx} \frac{\partial}{\partial u_{xx}} - 2u_{xy} \frac{\partial}{\partial u_{xy}} - \\ &- 3u_{yy} \frac{\partial}{\partial u_{yy}} - u_{xxx} \frac{\partial}{\partial u_{xxx}} - 2u_{xxy} \frac{\partial}{\partial u_{xxy}} - 3u_{xyy} \frac{\partial}{\partial u_{xyy}} - 3u_{yyy} \frac{\partial}{\partial u_{yyy}}, \end{aligned}$$

which implies

$$H = h \left(\frac{u_x}{u}, \frac{u_y}{u^2}, \frac{u_{xx}}{u}, \frac{u_{xy}}{u^2}, \frac{u_{yy}}{u^3}, \frac{u_{xxx}}{u}, \frac{u_{xxy}}{u^2}, \frac{u_{xyy}}{u^3}, \frac{u_{yyy}}{u^3} \right).$$

Particular cases.

In the paper [9] P.A.Clarkson, E.L.Mansfield and T.J.Priestley studied the symmetries of a special class of third order PDE

$$(10) \quad u_y - \varepsilon u_{xxy} + 2ku_x = uu_{xxx} + \alpha uu_x + \beta u_x u_{xx},$$

where ε , k , α and β are arbitrary constants.

If the PDE (9) is written in the form

$$u_y = u^2 \varphi \left(\frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xy}}{u^2}, \frac{u_{yy}}{u^3}, \frac{u_{xxx}}{u}, \frac{u_{xxy}}{u^2}, \frac{u_{xyy}}{u^3}, \frac{u_{yyy}}{u^3} \right),$$

then, for

$$\varphi(C_1, \dots, C_8) = C_6 + C_5 - 3C_1 + 2C_1 C_2,$$

we obtain the *Camassa-Holm PDE* (the case $\varepsilon = 1$, $\alpha = -3$, $\beta = 2$ and $k = 0$ in (10)).

For

$$\varphi(C_1, \dots, C_8) = \varepsilon C_6 + C_5 + \alpha C_1 + \beta C_1 C_2,$$

we get a subclass of the class of PDEs (10), for the parameter $k = 0$. In this case, for the parameters $\varepsilon = 0$, $\alpha = 1$, $\beta = 3$ and $k = 0$, it results the *Rosenau-Hyman PDE*, which is considered in [9] also.

Remark. There exists PDEs of order one or two which admit (8) like infinitesimal symmetries. For example: the elementary nonlinear wave PDE [14, 300]

$$u_y = uu_x,$$

and the Liouville-Țițeica PDE [3]

$$uu_{xy} - u_x u_y = u^3.$$

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