EXTREMA OF p-ENERGY FUNCTIONAL ON A FINSLER MANIFOLD

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Abstract

In §1 the authors present the basic properties of the scalar product along a curve on a Finsler manifold. In §2 they investigate the variational formulae for the p-energy functional ($p \in R - \{0\}$). This concept generalises the notions of length ($p = 1$) and energy ($p = 2$) of a curve. §3 analyses the extrema of p-energy when the Finsler space has constant curvature.

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§1. Preliminaries

Let (M, F) be a connected n-dimensional Finsler manifold whose fundamental function $F : TM \to R$ verifies the following axioms:

(F1) $F(x, y) > 0$; $\forall x \in M$, $\forall y \neq 0$.

(F2) $F(x, \lambda y) = |\lambda| F(x, y)$; $\forall \lambda \in R, \forall (x, y) \in TM$.

(F3) the fundamental tensor $g_{ij}(x, y) = \dfrac{1}{2} \dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite.

(F4) F is C^∞ at every point $(x, y) \in TM$ with $y \neq 0$ and continuous at every $(x, 0) \in TM$. Then, the absolute Finsler energy is $F^2(x, y) = g_{ij}(x, y) y^i y^j$.

Let $c : [a, b] \to M$ be a C^∞ regular curve on M. For any two vector fields $X(t) = X^i(t) \dfrac{\partial}{\partial x^i}\Big|_{c(t)}$, $Y(t) = Y^i(t) \dfrac{\partial}{\partial x^i}\Big|_{c(t)}$ along the curve c, we introduce [1], [6] the *scalar product* $g(X, Y)(c(t)) = g_{ij}(c(t), \dot{c}(t)) X^i(t) Y^j(t)$ *along the curve* c.

Remarks. i) If $X = Y$, then we obtain $\|X\| = \sqrt{g(X, X)}$.
ii) The vector fields X and Y *are orthogonal along the curve* c and we write $X \perp Y$ iff $g(X, Y) = 0$.

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Let $C\Gamma(N) = (L^i_{jk}, N^i_j, C^i_{jk})$ be the *Cartan canonical N-linear connection* determined by the fundamental tensor $g_{ij}(x,y)$. The coefficients of this connection are expressed by

$$L^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\delta g_{mk}}{\delta x^j} + \frac{\delta g_{jm}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^m}\right),\ C^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial y^j} + \frac{\partial g_{jm}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^m}\right),$$

$$N^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\left(\Gamma^i_{kl}y^k y^l\right) = \frac{1}{2}\frac{\partial \Gamma^i_{00}}{\partial y^j},\ \Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m}\right),$$

where $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} + N^j_i \dfrac{\partial}{\partial y^j}$.

Let X be a vector field along the curve c expressed locally by $X(t) = X^i(t)\left.\dfrac{\partial}{\partial x^i}\right|_{c(t)}$. Using the Cartan N-linear connection we define *the covariant derivative along the curve* c, by

$$\frac{\nabla X}{dt} = \left\{\frac{dX^i}{dt} + X^m\left[L^i_{mk}(c(t),\dot c(t))\frac{dc^k}{dt} + C^i_{mk}(c(t),\dot c(t))\frac{\delta}{dt}\left(\frac{dc^k}{dt}\right)\right]\right\}\left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

Since $\dfrac{\delta}{dt}\left(\dfrac{dc^k}{dt}\right) = \dfrac{d^2c^k}{dt^2} + N^k_l(c(t),\dot c(t))\dfrac{dc^l}{dt}$ we obtain

$$\frac{\nabla X}{dt} = \left\{\frac{dX^i}{dt} + X^m\left[\Gamma^i_{mk}(c(t),\dot c(t))\frac{dc^k}{dt} + C^i_{mk}(c(t),\dot c(t))\frac{d^2c^k}{dt^2}\right]\right\}\left.\frac{\partial}{\partial x^i}\right|_{c(t)},$$

where $\Gamma^i_{mk}(c(t),\dot c(t)) = L^i_{mk}(c(t),\dot c(t)) + C^i_{ml}(c(t),\dot c(t))N^l_k(c(t),\dot c(t))$.

Remarks. i) c is a *geodesic* iff $\dfrac{\nabla \dot c}{dt} = 0$.

ii) Since $C\Gamma(N)$ is a metrical connection we have

$$\frac{d}{dt}\left[g(X,Y)\right] = g\left(\frac{\nabla X}{dt}, Y\right) + g\left(X, \frac{\nabla Y}{dt}\right).$$

§2. Variations and extrema of p-energy functional

Let $x_0, x_1 \in M$ be two points not necessarily distinct. We denote $\Omega \overset{\text{not}}{=} \{c : [0,1] \to M \mid$ c is piecewise C^∞ regular curve, $c(0) = x_0$, $c(1) = x_1\}$.

For every $p \in R - \{0\}$ we define the $p-energy\ functional$ $E_p : \Omega \to R_+$,

$$E_p(c) = \int_0^1 \left[g_{ij}(c(t),\dot c(t))\frac{dc^i}{dt}\frac{dc^j}{dt}\right]^{p/2} dt = \int_0^1 [g(\dot c,\dot c)]^{p/2}dt = \int_0^1 \|\dot c\|^p dt.$$

Remarks. i) This general functional was studied for the first time by Udriste [7]-[10] on Riemannian manifolds. In their papers [2], [3], de Cecco and Palmieri study the same functional for $p \in (1,\infty)$, but from a topological point of view, ignoring the geometrical structure.

ii) For $p = 1$ we obtain the *length functional* $L(c) = \int_0^1 \|\dot{c}\| dt$, and for $p = 2$ we obtain the *energy functional* $E(c) = \int_0^1 \|\dot{c}\|^2 dt$.
iii) For any naturally parametrized curve (i.e. , $\|\dot{c}\| =$ constant) we have $E_p(c) = = (L(c))^p = (E(c))^{p/2}$.
iv) The p-energy of a curve is dependent of parametrization if $p \neq 1$.

For every $c \in \Omega$ we denote $T_c\Omega = \{X : [0,1] \to TM | X$ is continuos, piecewise $C^\infty, X(t) \in T_{c(t)}M, \ \forall t \in [0,1], \ X(0) = X(1) = 0\}$. Let $(c_s)_{s \in (-\epsilon,\epsilon)} \subset \Omega$ be one parameter variation of the curve $c \in \Omega$. We denote $X(t) = \frac{dc_s}{ds}(0,t) \in T_c\Omega$. Using the equality $g\left(\frac{\nabla \dot{c}_s}{\partial s}, \dot{c}_s\right) = g\left(\frac{\nabla}{\partial t}\left(\frac{\partial c_s}{\partial s}\right), \dot{c}_s\right)$ we can prove the following

Theorem. *The first variation of the p-energy is*

$$\frac{1}{p}\frac{dE_p(c_s)}{ds}(0) = -\sum_t g(X, \Delta_t(\|\dot{c}\|^{p-2}\dot{c})) -$$

$$-\int_0^1 \|\dot{c}\|^{p-4} g\left(X, \|\dot{c}\|^2 \frac{\nabla \dot{c}}{dt} + (p-2) g\left(\frac{\nabla \dot{c}}{dt}, \dot{c}\right)\dot{c}\right) dt,$$

where $\Delta_t(\|\dot{c}\|^{p-2}\dot{c}) = (\|\dot{c}\|^{p-2}\dot{c})_{t+} - (\|\dot{c}\|^{p-2}\dot{c})_{t-}$ *represents the jump of* $\|\dot{c}\|^{p-2}\dot{c}$ *at the discontinuity point* $t \in (0,1)$.

Corollary. *The curve c is a critical point of* E_p *iff c is a geodesic.*

Remark. For $p = 1$ the curve c is a reparametrized geodesic.

Now, let $c \in \Omega$ be a critical point for E_p (i.e. the curve c is a geodesic). Let $(c_{s_1 s_2})_{s_1, s_2 \in (-\epsilon,\epsilon)} \subset \Omega$ be a two parameter variation of c. Using the notations: $X(t) = \frac{\partial c_{s_1 s_2}}{\partial s_1}(0,0,t) \in T_c\Omega$, $Y(t) = \frac{\partial c_{s_1 s_2}}{\partial s_2}(0,0,t) \in T_c\Omega$, $\|\dot{c}\| = v =$constant and $I_p(X,Y) = \frac{\partial^2 E_p(c_{s_1 s_2})}{\partial s_1 \partial s_2}(0,0)$, we obtain the following

Theorem. *The second variation of the p-energy is*

$$\frac{1}{pv^{p-4}} I_p(X,Y) = -\sum_t g\left(Y, v^2 \Delta_t\left(\frac{\nabla X}{dt}\right) + (p-2) g\left(\Delta_t\left(\frac{\nabla X}{dt}\right), \dot{c}\right)\dot{c}\right) -$$

$$-\int_0^1 g\left(Y, v^2\left[\frac{\nabla}{dt}\frac{\nabla X}{dt} + R^2(X,\dot{c})\dot{c}\right] + (p-2) g\left(\left[\frac{\nabla}{dt}\frac{\nabla X}{dt} + R^2(X,\dot{c})\dot{c}\right], \dot{c}\right)\dot{c}\right) dt,$$

where $\Delta_t\left(\frac{\nabla X}{dt}\right) = \left(\frac{\nabla X}{dt}\right)_{t+} - \left(\frac{\nabla X}{dt}\right)_{t-}$ *represents the jump of* $\frac{\nabla X}{dt}$ *at the discontinuity point* $t \in (0,1)$ *and, if* $R^l_{ijk}(c(t), \dot{c}(t))$ *represents the components of Finsler h-curvature, then*

$$R^2(X,\dot{c})\dot{c} = R^l_{ijk}(c(t),\dot{c}(t)) \frac{dc^i}{dt}\frac{dc^j}{dt} X^k \frac{\partial}{\partial x^l} = R^l_{jk}(c(t),\dot{c}(t)) \frac{dc^j}{dt} X^k \frac{\partial}{\partial x^l}.$$

Remark. It is well known that we have

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j},\ R^i_{hjk} = \frac{\delta L^i_{hj}}{\delta x^k} - \frac{\delta L^i_{hk}}{\delta x^j} + L^s_{hj}L^i_{sk} - L^s_{hk}L^i_{sj} + C^i_{hs}R^s_{jk}.$$

Moreover, using the Ricci identities for the deflection tensors, we also have

$$R^i_{jk} = R^i_{mjk}y^m = R^i_{0jk}.$$

Corollary. *$I_p(X,Y) = 0; \forall\, Y \in T_c\Omega \Leftrightarrow X$ is a Jacobi field in the sense of Matsumoto (i.e. $\frac{\nabla}{dt}\frac{\nabla X}{dt} + R^2(X,\dot c)\dot c = 0$. See [6], pp. 289).*

In these conditions we have the following

Definition. A point $c(b)$, $0 \le a < b < 1$, of a geodesic $c \in \Omega$ is called a *conjugate point* of a point $c(a)$ along the curve c, if there exists a non-zero Jacobi field which vanishes at $t \in \{a,\ b\}$.

Now, integrating by parts and using the property of metrical connection we find

$$\frac{1}{pv^{p-4}}I_p(X,Y) = \int_0^1 \left\{ v^2\left[g\left(\frac{\nabla X}{dt}, \frac{\nabla Y}{dt}\right) - R^2(X,\dot c,Y,\dot c)\right] + \right.$$

$$\left. +(p-2)g\left(\dot c, \frac{\nabla X}{dt}\right) g\left(\dot c, \frac{\nabla Y}{dt}\right)\right\} dt,$$

where $R^2(X,\dot c,Y,\dot c) = g(R^2(Y,\dot c)\dot c, X) = R_{0i0j}(c(t),\dot c(t))X^iY^j$.

Remark. Let $R_{ijk} = g_{jm}R^m_{ik}$. In any Finsler space it is satisfied the identity,

$$R_{ijk} + R_{jki} + R_{kij} = 0,$$

obtained by the Bianchi identities. Because $R_{0i0j} = R_{i0j} = R_{j0i} = R_{0j0i}$ we obtain $R^2(X,\dot c,Y,\dot c) = R^2(Y,\dot c,X,\dot c)$.

The quadratic form associated to the Hessian of the p-energy is

$$I_p(X) \overset{\text{not}}{=} I_p(X,X) = \int_0^1 \left\{ v^2\left[\left\|\frac{\nabla X}{dt}\right\|^2 - R^2(X,\dot c,X,\dot c)\right] + \right.$$

$$\left. +(p-2)\left[g\left(\dot c, \frac{\nabla X}{dt}\right)\right]^2\right\} dt.$$

Lemma 1. *Let $T_c^{\perp}\Omega = \{X \in T_c\Omega | g(X,\dot c) = 0\}$ and $T_c'\Omega = \{X \in T_c\Omega | X = f\dot c$, where $f : [0,1] \to R$ is continuous, piecewise C^∞, $f(0) = f(1) = 0\}$. Then*
i) $T_c\Omega = T_c^{\perp}\Omega \oplus T_c'\Omega$; ii) $I_p(T_c^{\perp}\Omega, T_c'\Omega) = 0$.

Proof. ii) Let $X \in T_c^{\perp}\Omega$ and $Y = f\dot{c} \in T_c^{'}\Omega$. Since $X \perp \dot{c}$ and c is a geodesic, we have $g\left(\dot{c}, \frac{\nabla X}{dt}\right) = 0$. In these conditions it follows

$$\frac{1}{pv^{p-4}} I_p(X, f\dot{c}) = \int_0^1 \left\{ v^2 \left[g\left(\frac{\nabla X}{dt}, f'\dot{c}\right) - R^2(X, \dot{c}, f\dot{c}, \dot{c}) \right] + \right.$$
$$\left. +(p-2) g\left(\dot{c}, \frac{\nabla X}{dt}\right) g\left(\dot{c}, f'\dot{c}\right) \right\} dt.$$

Hence $I_p(X, f\dot{c}) = 0$. □

Remark. According to the preceding lemma, the spaces $T_c^{\perp}\Omega$ and $T_c^{'}\Omega$ are orthogonal with respect to the bilinear form I_p and consequently, the study of the signature of the quadratic form I_p is reduced to the study of signatures of its restrictions to $T_c^{'}\Omega$ and $T_c^{\perp}\Omega$.

Proposition 1. *Let c be a geodesic and $p \in R - \{0, 1\}$. Then*
i) $I_p(T_c^{'}\Omega) \geq 0$ for $p \in (-\infty, 0) \cup (1, \infty)$,
ii) $I_p(T_c^{'}\Omega) \leq 0$ for $p \in (0, 1)$.
Moreover, in both cases: $I_p(X) = 0 \Leftrightarrow X = 0$.

Proof. Let $X = f\dot{c} \in T_c^{'}\Omega$. Then we have

$$\frac{1}{v^{p-4}} I_p(X) = p \int_0^1 \left\{ v^2 \left[g(f'\dot{c}, f'\dot{c}) - R^2(f\dot{c}, \dot{c}, f\dot{c}, \dot{c}) \right] + (p-2) \left[g(\dot{c}, f'\dot{c}) \right]^2 \right\} dt =$$
$$= p \int_0^1 \left[v^4 (f')^2 + (p-2) v^4 (f')^2 \right] dt = \int_0^1 p(p-1) v^4 (f')^2 dt.$$

Moreover, if $I_p(X) = 0 \Leftrightarrow f' = 0 \Leftrightarrow f$ is constant. The conditions $f(0) = f(1) = 0$ imply $f = 0$. □

Because $I_p(T_c^{'}\Omega)$ is positive definite for $p \in (-\infty, 0) \cup (1, \infty)$ and negative definite for $p \in (0, 1)$, it is sufficient to study the behaviour of I_p restricted to $T_c^{\perp}\Omega$. Since $X \perp \dot{c}$ and the curve c is a geodesic it follows $g\left(\dot{c}, \frac{\nabla X}{dt}\right) = 0$. Hence, for all $X \in T_c^{\perp}\Omega$, we have

$$\frac{1}{pv^{p-2}} I_p(X) = \int_0^1 \left[\left\| \frac{\nabla X}{dt} \right\|^2 - R^2(X, \dot{c}, X, \dot{c}) \right] dt \overset{\text{not}}{=} I(X).$$

Lemma 2. *The following statements are equivalent:*
i) the curve c has no conjugate points to $x_0 = c(0)$,
ii) $I|_{T_c^{\perp}\Omega}$ is positive definite.

Proof. The proof of the lemma follows closely the proof of Kobayashi for the case of a Riemannian manifold (See [4], vol 2, pp 72-76).

i)⇒ ii). Let $J_{c(0)}^{\perp} = \{X : [0,1] \to TM \mid \ X(t) \in T_{c(t)}M, \ \ \forall t \in [0,1], X$ is Jacobi field, $X(0) = 0, X \perp \dot{c}\}$. Then $dim_R J_{c(0)}^{\perp} = n - 1$, where $n = dim\ M$. Let $\{Y_1, Y_2, ..., Y_{n-1}\}$ be a basis in $J_{c(0)}^{\perp}$. Since the geodesic c has no conjugate point to $x_0 = c(0)$ it follows that $\{Y_1, Y_2, ..., Y_{n-1}\} \subset T_c^{\perp}\Omega$ is a basis for $T_c^{\perp}\Omega$.

Let $X \in T_c^\perp\Omega$. There exist the functions $f_1(t), f_2(t), ..., f_{n-1}(t)$ such that $X = \sum_{i=1}^{n-1} f_i Y_i$. We have

$$g\left(\frac{\nabla X}{dt}, \frac{\nabla X}{dt}\right) - R^2(X, \dot{c}, X, \dot{c}) =$$

$$g\left(\frac{\nabla X}{dt}, \frac{\nabla X}{dt}\right) - \sum_i f_i g(R^2(Y_i, \dot{c})\dot{c}, X) = g\left(\frac{\nabla X}{dt}, \frac{\nabla X}{dt}\right) + \sum_i f_i g\left(\frac{\nabla}{dt}\frac{\nabla Y_i}{dt}, X\right) =$$

$$= g\left(\sum_i f_i' Y_i, \sum_j f_j' Y_j\right) + 2g\left(\sum_i f_i' Y_i, \sum_j f_j \frac{\nabla Y_j}{dt}\right) +$$

$$+g\left(\sum_i f_i \frac{\nabla Y_i}{dt}, \sum_j f_j \frac{\nabla Y_j}{dt}\right) + g\left(\sum_i f_i \frac{\nabla}{dt}\frac{\nabla Y_i}{dt}, \sum_j f_j Y_j\right).$$

On the other hand, we find

$$\frac{d}{dt}\left[g\left(\sum_i f_i Y_i, \sum_j f_j \frac{\nabla Y_j}{dt}\right)\right] = g\left(\sum_i f_i' Y_i, \sum_j f_j \frac{\nabla Y_j}{dt}\right) +$$

$$+g\left(\sum_i f_i \frac{\nabla Y_i}{dt}, \sum_j f_j \frac{\nabla Y_j}{dt}\right) + g\left(\sum_i f_i Y_i, \sum_j f_j' \frac{\nabla Y_j}{dt}\right) +$$

$$+g\left(\sum_i f_i Y_i, \sum_j f_j \frac{\nabla}{dt}\frac{\nabla Y_j}{dt}\right).$$

Combining these equalities we obtain

$$g\left(\frac{\nabla X}{dt}, \frac{\nabla X}{dt}\right) - g(R^2(X, \dot{c})\dot{c}, X) = g\left(\sum_i f_i' Y_i, \sum_j f_j' Y_j\right) +$$

$$+\frac{d}{dt}\left[g\left(\sum_i f_i Y_i, \sum_j f_j \frac{\nabla Y_j}{dt}\right)\right] + g\left(\sum_i f_i' Y_i, \sum_j f_j \frac{\nabla Y_j}{dt}\right) -$$

$$-g\left(\sum_i f_i Y_i, \sum_j f_j' \frac{\nabla Y_j}{dt}\right).$$

Because we have $R^2(X, \dot{c}, Y, \dot{c}) = R^2(Y, \dot{c}, X, \dot{c})$, any two Jacobi fields X and Y such that $X(0) = Y(0) = 0$ satisfy $g\left(X, \frac{\nabla Y}{dt}\right) = g\left(\frac{\nabla X}{dt}, Y\right)$. Particularly, $g\left(Y_i, \frac{\nabla Y_j}{dt}\right) = g\left(\frac{\nabla Y_i}{dt}, Y_j\right)$. In these conditions we have

$$g\left(\sum_i f_i' Y_i, \sum_j f_j \frac{\nabla Y_j}{dt}\right) - g\left(\sum_j f_j Y_j, \sum_i f_i' \frac{\nabla Y_i}{dt}\right) = 0,$$

and we obtain

$$I(X) = \int_0^1 g\left(\sum_i f_i' Y_i, \sum_j f_j' Y_j\right) dt + g\left(\sum_i f_i Y_i, \sum_j f_j \frac{\nabla Y_j}{dt}\right)_{t=1} =$$

$$= \int_0^1 g\left(\sum_i f_i' Y_i, \sum_j f_j' Y_j\right) \geq 0.$$

We have $I(X) = 0$ iff $f_i' = 0$, $\forall i = \overline{1, n-1}$ iff X is a Jacobi field. Since the geodesic c has no conjugate points it follows $X = 0$.

ii) $\Rightarrow$ i). We assume that $\exists\, x_{t_0} = c(t_0)$, a point which is conjugate to $x_0 = c(0)$ and $t_0 \in (0,1)$. Then $\exists\, Y$ a nonzero Jacobi field such that $Y(0) = Y(t_0) = 0$. Let U be a sufficiently small convex neighborhood and let $\delta > 0$ such that $c(t_0 - \delta), c(t_0 + \delta) \in U$. Then there exists a unique Jacobi field W determined by the boundary values $W(t_0 - \delta) = Y(t_0 + \delta)$ and $W(t_0 + \delta) = 0$. The vector field X is defined along c by

$$X = \begin{cases} Y & \text{from } c(0) \text{ to } c(t_0 - \delta) \\ W & \text{from } c(t_0 - \delta) \text{ to } c(t_0 + \delta) \\ 0 & \text{from } c(t_0 + \delta) \text{ to } c(1). \end{cases}$$

Denoting $I_a^b(X) = \displaystyle\int_a^b \left(\left\|\frac{\nabla X}{dt}\right\|^2 - R^2(X, \dot{c}, X, \dot{c})\right) dt$ and using results from the proof i) $\Rightarrow$ ii) we obtain $0 = I_0^{t_0}(Y) = I_0^{t_0-\delta}(Y) + I_{t_0-\delta}^{t_0}(Y)$, since Y is a Jacobi field. In conclusion: $I(X) = I(X) - I_0^{t_0}(Y) = I_0^{t_0-\delta}(Y) + I_{t_0-\delta}^{t_0+\delta}(W) - I_0^{t_0-\delta}(Y) - I_{t_0-\delta}^{t_0}(Y) =$ $= I_{t_0-\delta}^{t_0+\delta}(W) - I_{t_0-\delta}^{t_0}(Y)$. Let

$$\overline{Y} = \begin{cases} Y \text{ from } c(t_0 - \delta) \text{ to } c(t_0) \\ 0 \text{ from } c(t_0) \text{ to } c(t_0 + \delta) \end{cases}$$

be a piecewise Jacobi field. Then $I_{t_0-\delta}^{t_0+\delta}(W) < I_{t_0-\delta}^{t_0+\delta}(\overline{Y}) = I_{t_0-\delta}^{t_0}(Y)$ implies $I(X) < 0$, contradiction. □

With Lemma 2 and the relation between I_p and I we have

Proposition 2. *Let $c \in \Omega$ be a geodesic and $p \in R - \{0, 1\}$.*

i) If c has no conjugate points to $x_0 = c(0)$, then
$I_p(T_c^\perp \Omega) \geq 0$ for $p \in (0,1) \cup (1, \infty)$ and $I_p(T_c^\perp \Omega) \leq 0$ for $p \in (-\infty, 0)$.
Moreover, in both cases $I_p(X) = 0 \Leftrightarrow X = 0$.

ii) If c has conjugate points to $x_0 = c(0)$, then
$\exists X \in T_c^\perp \Omega$ such that $I_p(X) < 0$ for $p \in (0,1) \cup (1, \infty)$ and
$\exists X \in T_c^\perp \Omega$ such that $I_p(X) > 0$ for $p \in (-\infty, 0)$.

Combining the propositions 1 and 2 we obtain

Corollary (extrema of the p-energy).

Let $p \in R - \{0,1\}$ and $c \in \Omega$ be a geodesic such that

A) has no conjugate points to $x_0 = c(0)$. Then

i) c did not even minimize, did not even maximize E_p for $p \in (-\infty, 0) \cup (0,1)$;

ii) c not maximizes E_p for $p \in (1, \infty)$;

B) has conjugate points to $x_0 = c(0)$. Then

i) c not maximizes E_p for $p \in (-\infty, 0)$;

ii) c not minimizes E_p for $p \in (0,1)$;

iii) c did not even minimize, did not even maximize E_p for $p \in (1, \infty)$.

Remarks. 1) According to the property (F2) imposed to the Finsler metric, the preceding consequence is valid replacing x_0 with x_1 by symmetry.

2) For the case $p \in (1, \infty)$, supposing that exists a *minimal geodesic* $\gamma \in \Omega$ (i.e. it minimizes the length functional), then γ is a global minimum point for the p-energy E_p since $E_p(\gamma) = (L(\gamma))^p \leq (L(c))^p \leq E_p(c), \forall c \in \Omega$, where the last inequality is the Hőlder inequality (For details, see [10]). On the other hand, we have the Hőlder inequality for the case $p \in (0,1)$

$$\int_0^1 |fg| dt \geq \left(\int_0^1 |f|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |g|^q dt \right)^{\frac{1}{q}},$$

where $q = \dfrac{p}{p-1} \in (-\infty, 0)$. In these conditions it follows

i) $E_p(c) \leq (L(c))^p$ for $p \in (0,1)$

ii) $E_p(c) \geq (L(c))^p$ for $p \in (-\infty, 0)$, for any curve $c \in \Omega$.

In conclusion we have

i) $E_p(\gamma) = (L(\gamma))^p \leq (L(c))^p \geq E_p(c)$ for $p \in (0,1)$

ii) $E_p(\gamma) = (L(\gamma))^p \geq (L(c))^p \leq E_p(c)$ for $p \in (-\infty, 0)$.

It follows that, in the cases $p \in (0,1) \cup (-\infty, 0)$, the Hőlder formula did not decide upon the role of minimal geodesics as extremum points of E_p. Actually, the statement A) of the preceding consequence solves this problem.

§3. Extrema of p-energy on constant curvature Finsler spaces

We assume the Finsler space (M, F) is complete, of dimension $n \geq 3$ and of constant curvature $K \in R$. Hence, we have $H_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$, where H_{ijkl} are the components of the h-curvature tensor H of the Berwald connection $B\Gamma$. It follows that

$$R_{ijk} = KF\left(g_{ik}\frac{y_j}{F} - g_{ij}\frac{y_k}{F} \right),$$

where $y_j = g_{jk}y^k$. We also have $R_{i0k} = R_{ijk}y^j = K(g_{ik}F^2 - y_i y_k)$. Hence, along the geodesic $c \in \Omega$, we obtain $R^2(X, \dot{c})\dot{c} = K\{\|\dot{c}\|^2 X - g(X, \dot{c})\dot{c}\}$.

Remark. This equality is also true in the case of constant h-curvature for the Cartan canonical connection.

As in Matsumoto (see [6], pp 292) we have *i) If $K \leq 0$, then the geodesic c has no conjugate points to $x_0 = c(0)$.*
ii) If $K \geq 0$ and the geodesic c has conjugate points to $x_0 = c(0)$, then the number of conjugate points is finite (Morse index theorem for a Finsler manifold).

Moreover, in the case ii), choosing an orthonormal frame of vector fields $\{E_i\}_{i=\overline{1,n-1}} \in T_c^{\perp}\Omega$ parallelly propagated along the geodesic c, we can build a basis $\{U_i, V_i\}_{i=\overline{1,n-1}}$ in the set of Jacobi fields orthogonal to $\dot{c}$, defining

$$U_i(t) = \sin(\sqrt{K}vt)E_i \text{ and } V_i(t) = \cos(\sqrt{K}vt)E_i,$$

where $v = \|\dot{c}\| =$ constant. In conclusion, the distance between two consecutive conjugate points is $\frac{\pi}{\sqrt{K}}$.In these conditions we can prove the following

Theorem. *Let (M, F) be a Finsler space, as above, and let $c = c_p \in \Omega$ be a global extremum point for the p-energy functional E_p, where p is a number in $R - \{0, 1\}$.In these conditions we have*
i)If $p \in (-\infty, 0)$, then c has conjugate points, $K > 0$ and

$$\left[\frac{(m(c)+1)\pi}{\sqrt{K}}\right]^p \leq E_p(c) \leq \left[\frac{m(c)\pi}{\sqrt{K}}\right]^p,$$

where $m(c)$ is the maximal number of conjugate points to $x_0 = c(0)$ along the geodesic c.
ii) If $p \in (0, 1)$, then c has conjugate points, $K > 0$ and

$$\left[\frac{m(c)\pi}{\sqrt{K}}\right]^p \leq E_p(c) \leq \left[\frac{(m(c)+1)\pi}{\sqrt{K}}\right]^p.$$

iii) If $p \in (1, \infty)$, then c is a minimal geodesic (i.e. it minimizes the length functional).

Proof. i) If $p \in (-\infty, 0)$ and c is an extremum point for the p-energy E_p, then c is a minimum point and the curve c must have conjugate points to x_0, respectively to x_1, and hence $K > 0$. Let $x_0^1, x_0^2, ..., x_0^{m(c)}$ be the consecutive conjugate points to x_0. Since the distance between two consecutive conjugate points is $\frac{\pi}{\sqrt{K}}$ it follows $\frac{m(c)\pi}{\sqrt{K}} \leq L(c) \leq \frac{(m(c)+1)\pi}{\sqrt{K}}$. On the other hand $E_p(c) = (L(c))^p$, and hence, the above inequality is true.
ii) By analogy to i).
iii) By the above Remark 2), if $\gamma \in \Omega$ is a minimal geodesic, then $E_p(\gamma) \leq E_p(c)$. But c is a minimum point for E_p, and hence $E_p(c) \leq E_p(\gamma)$. In conclusion, we have $E_p(\gamma) = (L(\gamma))^p = (L(c))^p = E_p(c)$ and consequently $L(\gamma) = L(c)$. Hence c is a minimal geodesic. □

If we denote $m = \sup\{m(c) \mid c \in \Omega,\ c - \text{geodesic}\} \in N$, we obtain the following

Corollary.*If there is $c \in \Omega$ a global extremum point for the p-energy functional E_p, where $p \in (-\infty, 0) \cup (0, 1)$, we must have $m < \infty$ and $m(c) = m$.*

Remarks. i) If x_1 is not a conjugate point to x_0, then it follows $||\dot{c}|| = \frac{(m+1)\pi}{\sqrt{K}}$ and $E_p(c) = \left[\frac{(m+1)\pi}{\sqrt{K}}\right]^p$, because the p-energy is dependent of parametrization.
ii) If x_1 is a conjugate point to x_0, then we obtain $E_p(c) = \left(\frac{m\pi}{\sqrt{K}}\right)^p$ and $v = \frac{m\pi}{\sqrt{K}}$.

One example. In the case of Riemannian unit sphere $S^n \subset R^{n+1}$, $n \geq 2$, it is well known that the geodesics are precisely the great circles, that is the intersections of S^n with the hyperplanes trough the center of S^n. Moreover, two arbitrary points on S^n are conjugate along a geodesic γ if they are antipodal points. In these conditions, for any two points x_0 and x_1 on the sphere S^n, there is no geodesic trough these points which has a finite maximal number of conjugate points, because we can surround the sphere infinite times. Hence, for the unit sphere S^n, we have $m = \infty$. In conclusion, in the case $p \in (-\infty, 0) \cup (0, 1)$, the p-energy functional on the sphere has no global extremum points.

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