SYMmetric INVARIANT NON-DEGENERATE
BILINEAR FORMS ON NILPOTENT LIE
ALGEBRAS

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Abstract

Let $g$ be a Nilpotent Lie algebra of finite dimension. The aim of the present work is to study symmetric invariant non-degenerate bilinear forms on $g$.

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1 Introduction

Let $g$ be a nilpotent Lie algebra of finite dimension $n$ over the field $K = \mathbb{R}$ or $K = \mathbb{C}$. It is known that there is a complete classification of $g$ until dimension 7 ([2]) and, in some cases, of dimension 8 and 9 ([3, 4, 5, 6]). The existence of a symmetric invariant non-degenerate bilinear form on a nilpotent Lie algebra is an open problem.

The aim of the present paper is to find out which of these known nilpotent Lie algebras admit a symmetric invariant non-degenerate bilinear form.

The whole paper contains three sections. The first section contains the introduction. Some known results are given in the second section. From the complete classification of nilpotent Lie algebras up to dimension 7 ([2]) we determine which of them admit a symmetric, invariant, non-degenerate bilinear form. This is the content of the last section.

2 Some Known results

Let $g$ be a nilpotent Lie algebra of finite dimension $n$ over a field $K$ of characteristic zero. A symmetric bilinear form

$$
\varphi : g \times g \to K
$$

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is called non-degenerate, when \( N\varphi = \{ 0 \} \), where
\[
N\varphi = \{ x \in g \mid \varphi(x, v) = 0, \forall v \in g \}.
\]
This \( \varphi \) is called invariant, if we have:
\[
\varphi ([x, y], z) = \varphi (x, [y, z]) \quad \forall x, y, z \in g.
\]
The pair \((g, \varphi)\) is called regular quadratic Lie algebra.

The classification of regular quadratic Lie algebras of dimension less or equal than 7 is given in ([2]) and there are the following four Lie algebras:

1. \( g_1 \) : \( \dim g_1 = 1 \), Abelian and \( \varphi_1 \) the standard inner product.
2. \( w_3 \) : \( \dim w_3 = 5 \), whose structure constants are:
   \[
c_{12}^1 = -c_{21}^1 = 1, c_{13}^1 = -c_{31}^1 = 2, c_{23}^2 = -c_{32}^2 = -2, \text{ all the others are zero.}
\]
3. \( w_4 \) : \( \dim w_4 = 6 \), whose structure constants are:
   \[
c_{45}^3 = -c_{54}^3 = 1, c_{13}^2 = -c_{31}^2 = -1, c_{53}^6 = -c_{35}^6 = 1, \text{ all the others are zero.}
\]
4. \( w_5 \) : \( \dim w_5 = 7 \), whose structure constants are:
   \[
c_{12}^1 = -c_{21}^1 = 1, c_{13}^3 = -c_{31}^3 = 1, c_{41}^4 = -c_{14}^4 = 2, c_{45}^6 = -c_{54}^6 = -1, c_{25}^7 = -c_{52}^7 = 1, c_{43}^7 = -c_{34}^7 = 2, \text{ all the others are zero.}
\]

3 Main results

All nilpotent Lie algebras up to dimension seven which are regular quadratic are given by the below theorem:

**Theorem 1** The nilpotent Lie algebra up to dimension seven, which are regular quadratic, are the following:

(I) All the Abelian with \( \varphi \) the common inner product.

(II) The indecomposable ones, which are:
   
   \( (II)_a \ n_3^3 \) isometric to \((w_3, \varphi_3)\), \( \dim n_3^3 = 5 \);

   \( (II)_b \ n_6^{21} \) isometric to \((w_4, \varphi_4)\), \( \dim n_6^{21} = 6 \);

   \( (II)_c \ n_7^{12} \) isometric to \((w_5, \varphi_5)\), \( \dim n_7^{12} = 7 \).

(III) The decomposable ones, which are:

   \( (III)_a \ n_6^{32} = n_3^3 \oplus g_1 \), with \( \varphi = \varphi_1 + \varphi_3 \), \( \dim n_6^{32} = 6 \);

   \( (III)_b \ n_7^{153} = n_6^{21} \oplus g_1 \), with \( \varphi = \varphi_1 + \varphi_4 \), \( \dim n_7^{153} = 7 \);

   \( (III)_c \ n_7^{143} = n_5^3 \oplus g_2 \), with \( \varphi = \varphi_2 + \varphi_3 \), \( \dim n_7^{143} = 7 \),
where \( g_k, k = \frac{1}{2}, 1 \) are the Abelian Lie algebras with the standard symmetric, invariant, non-degenerate bilinear forms on \( g_k \), \( k = \frac{1}{2}, 1 \), respectively, the Lie algebras \( n^3_3, n^2_{21}, n^3_{12}, n^{15}_{143} \), and \( n^4_{143} \) are given in the classification ([2]), \( \varphi_1, \varphi_2 \) are the standard Euclidean inner product and \( \varphi_3, \varphi_4, \varphi_5 \) are the symmetric, invariant, non-degenerate bilinear forms on \( n^3_3, n^2_{21} \) and \( n^4_{143} \), respectively.

**Proof.** We distinguish three cases: Abelian, indecomposable and decomposable.

**(I) ABELIAN CASES**

Let \( g_k \) be an Abelian Lie algebra of dimension \( k \) over the field \( \mathbb{R} \). Let \( \{ e_1, e_2, \ldots, e_k \} \) be a base of \( g_k \). If \( a \) and \( b \) are two elements of \( g_k \), which can be written:

\[
a = \sum_{i=1}^{k} a_i e_i \quad \text{and} \quad b = \sum_{j=1}^{k} b_j e_j,
\]

the symmetric, invariant, non-degenerate bilinear form \( \varphi_k \) on \( g_k \) is defined by

\[
\varphi_k : g_k \times g_k \to \mathbb{R}, \quad \varphi_k (a, b) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_i b_j.
\]

In order to study the non-Abelian nilpotent Lie algebras \( g_k \), which admit a symmetric, invariant, non-degenerate, bilinear form \( \varphi \), we will use the classification which is given in [2, 102-124]. Therefore we try to determine the regular quadratic Lie algebras \( (g, \varphi) \).

We study separately each dimension.

**(II) INDECOMPOSABLE CASES**

**(II)’** \( \dim g = 3 \). There is only the \( n^3_3 \). This is not regular quadratic.

**Indeed:** Let \( \{ e_1, e_2, e_3 \} \) be a base of \( n^3_3 \) such that \([e_1, e_2] = e_3\) and the other Lie brackets are zero. Let \( \varphi \) be the required quadratic form on \( n^3_3 \). Therefore we have:

\[
\varphi(e_3, e_1) = \varphi([e_1, e_2], e_1) = (\varphi \text{ invariant}) = \varphi(e_1, [e_2, e_1]) = \varphi(e_1, -e_3) \Rightarrow 2\varphi(e_3, e_1) = 0.
\]

It can be also proved that

\[
\varphi(e_3, e_2) = 0 \quad \text{and} \quad \varphi(e_3, e_3) = 0
\]

and finally

\[
\varphi(e_3, a_1 e_1 + a_2 e_2 + a_3 e_3) = 0.
\]

The above relation implies that \( e_3 \in N \varphi = \{ 0 \} \), which means that \( n^3_3 \) is not regular quadratic.

**(II)”** \( \dim g = 4 \). The only nilpotent Lie algebras not decomposable is \( n^4_4 \). This is not regular quadratic.

**Indeed:** We prove this in the same way as for \( n^3_3 \).

**(II)’’** \( \dim g = 5 \). From the indecomposable Lie algebras of dimension five, which are: \( n^3_{21}, n^3_{143}, n^4_{143}, n^3_{143}, n^3_{143}, n^3_{143}, n^3_{143}, n^3_{143} \), the \( n^3_3 \) is the unique regular quadratic.
**Indeed:** Using the same technique as in the cases (II) we conclude that the only indecomposable nilpotent Lie algebra which is regular quadratic is $n_3^5$. All the other are not regular quadratic. It is obvious that $n_3^5$ is isometric onto $w_3$ provided with the symmetric, invariant, non-degenerate bilinear form $\varphi_3$, which is given in [1].

(II) $\dim g = 6$. It is known that the indecomposable Lie algebras of dimension six are $n_0^i, i = 1, 2, \ldots, 24$. The only regular quadratic is $n_6^{21}$.

**Indeed:** We use the same technique as the previous ones and we conclude that $n_6^{21}$ is the only regular quadratic. All the others are not regular quadratic. This nilpotent Lie algebra $n_6^{21}$ is isometric onto $w_4$, provided with symmetric, invariant, non-degenerate bilinear form $\varphi_4$, which is given in [1].

(II) $\dim g = 7$. The indecomposable nilpotent Lie algebras of dimension seven are the following:

- $n_7^{1,a}$,
- $n_7^i, i = 2, \ldots, 11$,
- $n_7^{13,a}$,
- $n_7^{i}, i = 14, \ldots, 41, 47, \ldots, 52, 60, 61$,
- $n_7^{62,a}$,
- $n_7^i, i = 63, \ldots, 69$,
- $n_7^{70,a}$,
- $n_7^i, i = 71, \ldots, 77$,
- $n_7^{78,a}$,
- $n_7^i, i = 79, \ldots, 131$,
- $n_7^{132,a}$,
- $n_7^i, i = 133, 142, 147, \ldots, 152, 157$,

where $a \neq 0$ is a parameter defined in ([2]). The $n_7^{12}$ is the only regular quadratic.

**Indeed:** If we use the same methods as in the case $\dim g = 3$, then we obtain that $n_7^{12}$ is the only regular quadratic. All the others are not regular quadratic. This nilpotent Lie algebra $n_7^{12}$ is isometric onto $w_5$ provided with the symmetric, invariant, non-degenerate bilinear form $\varphi_5$ which is given in [1].

**(III) DECOMPOSABLE CASES**

Let $(g_1, f_1)$ and $(g_2, f_2)$ be two regular quadratic Lie algebras with Lie brackets $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$, respectively. We consider the direct sum $g_1 \oplus g_2$. 
which is a Lie algebra with Lie bracket
\[ [X, Y] = [X_1, Y_1]_1 + [X_2, Y_2]_2, \]
where \( X = X_1 + X_2 \) and \( Y = Y_1 + Y_2 \), \( X_1, Y_1 \in g_1 \) and \( X_2, Y_2 \in g_2 \).

On the Lie algebra \( g = g_1 \oplus g_2 \) we define a symmetric bilinear form \( f \) as follows:
\[ f : g \times g \mapsto K, \]
\[ f : (X = X_1 + X_2, Y = Y_1 + Y_2) \mapsto f(X, Y) = f_1(X_1, Y_1) + f_2(X_2, Y_2), \]
where \( X_1, Y_1 \in g_1 \) and \( X_2, Y_2 \in g_2 \).

It can be easily seen that \( f \) is invariant and non-degenerate. Therefore \((g = g_1 \oplus g_2, f = f_1 \oplus f_2)\) is regular quadratic.

Below we study the decomposable nilpotent Lie algebras of dimension less or equal than seven which are regular quadratic.

\((III)'\) \( \dim g = 4 \). There is only one such Lie algebra which is \( n_3^2 = n_3^1 \oplus g_1 \). This is not regular quadratic.

**Indeed:** If \( \{e_1, e_2, e_3\} \) is a base of \( n_3^1 \) and \( \{f_1\} \) is a base of \( g_1 \), then \( \{(e_1, 0), (e_2, 0), (e_3, 0), (0, f_1)\} \) is a base of \( n_3^2 = n_3^1 \oplus g_1 \).

The Lie brackets of \( n_3^2 \) are the following:
\[ [(e_i, 0), (0, f_1)] = (0, 0), i = 1, 2, 3 \]
and all the others are zero.

If \( \varphi \) is a symmetric, invariant, bilinear form on \( n_3^2 \), then we have:
\[ \varphi \left( [(x_1, y_1), (x_2, y_2)], (z, w) \right) = \varphi \left( (x_1, y_1), [(x_2, y_2), (z, w)] \right), \]
for all \( x_1, x_2, z \in n_3^1 \) and \( y_1, y_2, w \in g_1 \). It can be easily proved that
\[ \varphi ((e_3, 0), (0, f_1)) = \varphi ((e_1, 0), (e_2, 0)) = 0, \]
\[ \varphi ((e_3, 0), (e_1, 0)) = \varphi ((e_3, 0), (e_2, 0)) = 0. \]

Therefore we obtain that
\[ (e_3, 0) \in N \varphi = \{ (0, 0) \}, \]
which implies that \( n_3^2 \) is not regular quadratic.

\((III)''\) \( \dim g = 5 \). All the decomposable nilpotent Lie algebras of dimension five are \( n_3^2, n_5^2, n_5^5 \) and \( n_5^9 \), which are not regular quadratic.

**Indeed:** This can be proved with the same method as in \((III)'\).

\((III)_{a} \) \( \dim g = 6 \). The decomposable nilpotent Lie algebras of dimension 6 are \( n_6^i, i = 25, \ldots, 35 \). The \( n_6^{32} \) is the only regular quadratic.
Indeed: Using the same method as in (III)' we can prove that only $n_{6}^{32} = n_{3}^{5} \oplus g_{1}$ with the bilinear form $\varphi = \varphi_{1} + \varphi_{3}$ is regular quadratic and the others $n_{b}^{i}, i = 25, \ldots, 31, 33$ are not regular quadratic.

(III)$_{b} +$ (III)$_{r}$, dim $g = 7$. The following Lie algebras:

$n_{i}^{7}, i = 42, 46, 53, 59, 134, 141; i = 143, \ldots, 146; i = 153, \ldots, 156; i = 158; i = 159$ of dimension seven are decomposable. The $n_{7}^{153}$ and $n_{7}^{143}$ are regular quadratic.

Indeed: Using the techniques of (III)' we obtain that only $n_{7}^{153} = n_{6}^{21} \oplus g_{1}$, with bilinear form $\varphi_{1} + \varphi_{4}$ and $n_{7}^{143}$ with the bilinear form $\varphi_{2} + \varphi_{3}$ are regular quadratic. All the others are not regular quadratic. □

Acknowledgments

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