RELATION BETWEEN TOPOLOGY AND THE KILLING VECTOR FIELDS

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Abstract

Let $(M,\, g)$ be a compact Riemannian manifold of dimension n. The aim of the present paper is to prove that the dimension of the vector space $K^1(M,\, R)$ of the Killing vector fields is not a topological invariant.

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Key words: Killing vector field, harmonic 1-form, Ricci tensor field, one-parameter family of isometries.

1 Introduction

Let $(M,\, g)$ be a compact Riemannian manifold of dimension n. The study of the existence of Killing vector fields on the manifold is an important problem of Differential Geometry.

The aim of the present paper is to generalize the results [6, 7]. We also have proved that the old conjecture "The dimension of the vector space $K^1(M,\, R)$ of the Killing vector fields on a compact Riemannian manifold $(M,\, g)$ is a topological invariant" is not true.

The whole paper contains five sections. Each of them is analyzed as follows. The second section deals with the fibre bundles over a compact Riemannian manifold $(M,\, g)$ and differential operators on the cross sections of these fibre bundles. The Killing vector fields can be considered as special cross sections of the tangent bundle $T(M)$ over $(M,\, g)$. In the third section we study the $\dim\big(K^1(M,\, R)\big)$ with respect to the Riemannian metric g on M. The existence of one-parameter family of transformations on $(M,\, g)$ which is related to the Riemannian metric g is contained in the fourth section. The last section gives a negative answer to the above mentioned conjecture, that is, $\dim\big(K^1(M,\, R)\big)$ is not topological invariant.

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2.

Let $(M,\, g)$ be a compact orientable Riemannian manifold without boundary of dimension n. We denote by TM and T^*M the tangent and cotangent bundle respectively over the manifold M. These two bundles are isomorphic with respect to the Riemannian metric g on M. Therefore we only consider the cotangent bundle T^*M over M and the results on it can be transferred on the tangent bundle TM.

Let $C^\infty\,(T^*M)$ be the cross sections on T^*M. We must notice that each exterior 1-form is a cross section on T^*M. The Laplace operator $\Delta = d\delta + \delta d$ is a second order elliptic differential operator on $C^\infty(T^*M)$, that is:

$$\Delta = d\delta + \delta d \,:\, C^\infty(T^*M) \mapsto C^\infty(T^*M),$$

$$\Delta = d\delta + \delta d \,:\, \alpha \mapsto \Delta(\alpha) = d\delta(\alpha) + \delta d(\alpha),$$

where d and δ are the first order differential operators defined by

$$d \,:\, C^\infty(T^*M) \mapsto C^\infty(\Lambda^2 T^*M),$$

$$\delta \,:\, C^\infty(T^*M) \mapsto C^\infty(M),$$

where $C^\infty(\Lambda^2 T^*M)$ and $C^\infty(M)$ are the cross sections on $\Lambda^2 T^*M$ and the differential functions on M respectively.

These differential operators are related by

$$\langle \alpha, \delta\beta \rangle = \langle d\alpha, \beta \rangle\,,\ \forall \alpha \in C^\infty(T^*M)\,, \forall \beta \in C^\infty(\Lambda^2 T^*M),$$

where $\langle\ ,\ \rangle$ is the global inner product on $C^\infty(\Lambda^q T^*M)$, $q = 1, 2$ and defined by

$$\langle \gamma, \delta \rangle = \int_M \langle \gamma, \delta \rangle_x dM(x),\ \forall \gamma, \delta \in C^\infty(\Lambda^q T^*M),$$

where $\langle \gamma, \delta \rangle_x$ is the inner product on $\Lambda^q T^*_x M$ induced by the metric g on M and dM is the measure on M for each $x \in M$.

Let $(x_1, \ldots, x_n)$ be a local coordinate system on the chart $(U,\, \varphi)$ and let $\{e_1, \ldots, e_n\}$ be the associated local orthonormal frame on U. If α is 1-form on M, which is a cross section on T^*M, that is $\alpha \in C^\infty(T^*M)$, then α with respect to the local system can be characterized by

$$\alpha(e_i) = \alpha_i, \quad i = 1, 2, \ldots, n.$$

The following formulas are known:

$$(d\alpha)_{ij} = e^{kl}_{ij} \nabla_k \alpha_l,\ \delta\alpha = -\nabla_l \alpha^l, \tag{1}$$

$$(\Delta\alpha)_i = -\nabla^k \nabla_k \alpha_i + \varepsilon^k_i \left(\nabla_l \nabla_k \alpha^l - \nabla_k \nabla_l \alpha^l \right), \tag{2}$$

where

$$\varepsilon^{kl}_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is even permutation of } (k,l), \\ -1 & \text{if } (i,j) \text{ is odd permutation of } (k,l), \\ 0 & \text{if } (i,j) \text{ is not permutation of } (k,l), \end{cases}$$

and

$$\varepsilon_i^k = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{if } k \neq i. \end{cases}$$

If α is an 1-form, then we have:

$$\frac{1}{2}\Delta|\alpha|^2 = (\alpha, \Delta\alpha) - |\nabla\alpha|^2 + \frac{1}{2}Q_1(\alpha), \tag{3}$$

where

$$|\alpha| \quad \text{is the local norm of } \alpha, \tag{4}$$

$$|\nabla\alpha|^2 = \nabla^k\alpha^i\nabla_k\alpha_i \text{ and} \tag{5}$$

$$Q_1(\alpha) = -2R_{kl}\alpha^k\alpha^l. \tag{6}$$

Let α be an 1-form. To this 1-form we can associate a vector field, denoted by $v(\alpha)$, which in the local system $(x_1, \ldots, x_n\,,\, e_1, \ldots, e_n)$ can be expressed as follows:

$$v(\alpha) \ : \ v(\alpha)^i = g^{ik}\alpha_k, \tag{7}$$

where (g^{ik}) is the inverse matrix of (g_{ik}) obtained by the metric g on M. The relation (7) gives an isomorphism between the vector space

$$\Lambda^1(M, \mathbf{R}) \equiv D^1(M, \mathbf{R}) \text{ and } D_1(M, \mathbf{R})$$

of 1-forms $D_1(M, \mathbf{R})$ and vector fields $D^1(M, \mathbf{R})$ respectively.

Therefore it is equivalent to substitute the notion of 1-form by the notion of vector field and conversely.

3.

Let α be an 1-form. This is called Killing 1-form, if its covariant derivative $\nabla\alpha$ is a 2-form, that means:

$$\nabla\alpha \in C\left(\Lambda^2 T^* M\right). \tag{8}$$

In the local system $(x_1, \ldots, x_n,\, e_1, \ldots, e_n)$ $\nabla\alpha$ can be expressed as follows:

$$\nabla_i\alpha_j + \nabla_j\alpha_i = (\nabla\alpha)_{ij} + (\nabla\alpha)_{ji} = 0. \tag{9}$$

If we apply Ricci's formula for α, then we have:

$$\nabla_l\nabla_k\alpha^l - \nabla_k\nabla_l\alpha^l = -\alpha^\nu R^l_{\nu kl} = -\alpha^\nu R_{k\nu}\,. \tag{10}$$

From the relations (9) and (10) we obtain that the 1-form $\alpha = (\alpha_i)$ satisfies the equations:

$$g^{jk}\nabla_j\nabla_k\alpha_i + R_i^\nu\alpha_\nu = 0. \tag{11}$$

Hence, if we consider the second order elliptic differential operator D on the cross sections on $\Lambda^1(M,\mathbf{R})$, that means:

$$\begin{aligned} D\ :\ C^\infty\left(\Lambda^1(M,\mathbf{R})\right) &\mapsto C^\infty\left(\Lambda^1(M,\mathbf{R})\right), \\ D\ :\ \alpha &\mapsto D(\alpha), \end{aligned} \tag{12}$$

which in the local system $(x_1,\ldots,x_n,\ ,e_1,\ldots,e_n)$ can be expressed as follows:

$$D(\alpha)_i = g^{jk}\nabla_j\nabla_k\alpha_i + R_i^\nu\alpha_\nu = 0. \tag{13}$$

The $Kern(D)$, that is

$$Kern(D) = \left\{\alpha \in C^\infty\left(\Lambda^1(M,\mathbf{R})\right)|D(\alpha) = 0\right\}$$

consists of the Killing 1-forms. Since

$$\Lambda^1(M,\mathbf{R}) = D_1(M,\mathbf{R}) \equiv D^1(M,\mathbf{R}),$$

we conclude that the vector space $K^1(M,\mathbf{R})$, where $K^1(M,\mathbf{R})$ is the vector space of the Killing vector fields on M, is isomorphic onto $Kern(D)$, that is

$$K^1(M,\mathbf{R}) \equiv Kern(D).$$

For any point $x \in M$, we define:

$$\begin{aligned} p(x) &= Sup\left\{R(\alpha,\alpha) = R_{kl}\alpha^k\alpha^l|\alpha \text{ unit vector in } T_xM\right\} \text{ and} && (14) \\ r &= \max\{p(x)|x \in M\}. && (15) \end{aligned}$$

Theorem 3.1 *Let $(M,\, g)$ be a compact orientable Riemannian manifold of dimension n without boundary. If $p(x) \leq 0$ and if there exists a point $x_0 \in M$ such that $p(x_0) < 0$, then $Kern(D) = \{0\}$. If $r = 0$, then*

$$\dim\left(Kern(D)\right) = \dim\left(K^1(M,\mathbf{R})\right) \leq n\,.$$

Proof. It is known that

$$(\alpha,\Delta\alpha) = \alpha_i(\Delta\alpha)_i\ , \tag{16}$$

which by means of (2) and Ricci's formula

$$\nabla_l\nabla_k\alpha^l - \nabla_k\nabla_l\alpha^l = -\alpha^\nu R^l_{\nu kl} = -\alpha^\nu R_{k\nu} \tag{17}$$

takes the form:

$$(\alpha,\Delta\alpha) = -2Q_1(\alpha). \tag{18}$$

The formula (3) by means of (18) becomes:

$$\frac{1}{2}\Delta|\alpha|^2 = -|\nabla_\alpha|^2 - \frac{3}{2}Q_1(\alpha). \tag{19}$$

If we integrate (19) on M we obtain:

$$2\int_M |\nabla\alpha|^2 \, dM = -3\int_M Q_1(\alpha)\, dM, \tag{20}$$

which by virtue of (6) takes the form:

$$\int_M |\nabla\alpha|^2 \, dM = 3\int_M R_{kl}\alpha^k\alpha^l \, dM. \tag{21}$$

Since we have:

$$|\nabla\alpha|^2 \geq 0, \tag{22}$$

$$p(x) = \left(R_{kl}\alpha^k\alpha^l\right)_x \leq 0, \quad \forall x \in M \setminus \{x_0\} \text{ and } p(x_0) < 0, \tag{23}$$

we conclude that

$$|\nabla\alpha|^2 = 0 \ , \ \alpha_x \ , \tag{24}$$

which implies $\alpha = 0$ on M and therefore

$$\dim\left(Kern(D)\right) = \dim\left(K^1(M, \mathbf{R})\right) = 0.$$

If $r = 0$, the formula (21) becomes:

$$\int_M |\nabla\alpha|^2 \, dM = 3\int_M R_{kl}\alpha^k\alpha^l \, dM, \tag{25}$$

which implies $|\nabla\alpha|^2 = 0$ and therefore $\nabla\alpha = 0$, that means α is a parallel vector field. Hence every Killing vector field on M is parallel. We take the curve $c(t)$ in M whose tangent at each point $x \in c(t)$ is the vector α_x. Therefore we obtain:

$$\alpha(t) = c(t) = \sum_{i=1}^{n} f_i(t) e_i$$

and since $\alpha(t)$ is parallel we conclude that the functions $f_i(t)$ are constants and hence $\dim\left(K^1(M, \mathbf{R})\right) \leq n$. $\square$

4.

An one-parameter group of differential transformations of M is a mapping

$$F : R \times M \mapsto M \ , \ F \ : \ (t, P) \mapsto F(t, P) = \varphi_t(P)$$

which satisfies the following conditions:

(i) For each $t \in \mathbf{R}$, φ_t is a transformation of M,

(ii) For all $t, s \in \mathbf{R}$ and $P \in M$, $\varphi_{s+t}(P) = \varphi_t(\varphi_s(P))$.

It is known that each one-parameter group of transformations induces a vector field X as follows. Let P be a point of M. We denote by X_P the tangent vector of the curve $f(t) = \varphi_t(P)$, when $t \in \mathbf{R}$. This is called the orbit of P at $P = \varphi_t(P)$. The orbit $f(t) = \varphi_t(P)$ is an integral curve of X starting at P.

The inverse is also true, that means a vector field X on M determines an one-parameter group of differentiable transformations of M. We consider a local coordinate system $(x_l, \ldots, x_n)$ in a neighborhood U of P_0 such that

$$x_1(P_0) = \cdots = x_n(P_0) = 0.$$

The vector field X on U can be written:

$$X = \lambda^1(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \cdots + \lambda^n(x_1, \ldots, x_n)\frac{\partial}{\partial x_n}. \tag{26}$$

We construct the following system of ordinary linear differential equations:

$$\frac{df^i}{dt} = \lambda^i\left(f^1(t), \ldots, f^n(t)\right), \quad i = 1, \ldots, n. \tag{27}$$

By the fundamental theorem of systems of linear differential equations, there exists a unique set of functions

$$f^1 = f^1(t, x_1, \ldots, x_n), \ldots, f^n = f^n(t, x_1, \ldots, x_n), \tag{28}$$

defined for $x = (x_1, \ldots, x_n)$ with $|x_i| < \delta,\ i = 1, \ldots, n$ and for $|t| < \varepsilon$, which form a solution of this system for each fixed x and satisfy the initial conditions:

$$f^i(0, x_1, \ldots, x_n) = x_i\,, \quad i = 1, \ldots, n. \tag{29}$$

The set $\varphi_t = \left\{f^1 = f^1(t, x_1, \ldots, x_n), \ldots, f^n = f^n(t, x_1, \ldots, x_n)\right\}$for $|t| < \varepsilon$ and for all $(x_1, \ldots, x_n) \in U$, such that:

$$U = \{(x_1, \ldots, x_n) || x_i| < \delta, i = 1, \ldots, n\}\,,$$

defines a local one-parameter group of local transformations on $I_\varepsilon \times U$, which can be extended to an one-parameter group of transformations on the manifold M.

If the vector field X on the Riemannian manifold (M, g) is Killing, with respect to the metric g, then the one-parameter group of differentiable transformations are isometries with respect to the Riemannian metric g.

From the above we have proved the theorem:

Theorem 4.1 *Let (M, g) be a compact orientable Riemannian manifold of dimension n without boundary. If $p(x) \leq 0$ and if there is a point $x_0 \in M$ such that $p(x_0) < 0$, then there exists no one-parameter family of isometries on M.*

If $r = 0$, then every Killing vector field X is parallel, that means:

$$X = c_1 e_1 + \cdots + c_n e_n, \tag{30}$$

where

$$e_1 = \frac{\partial}{\partial x_1}, \cdots, e_n = \frac{\partial}{\partial x_n} \tag{31}$$

and $c_1, \ldots, c_n$ are real constants. The system of ordinary linear differential equations (27) takes the form:

$$\frac{df^1}{dt} = c_1 \, , \; \frac{df^2}{dt} = c_2 \, , \cdots, \; \frac{df^n}{dt} = c_n \, , \tag{32}$$

which by integration gives:

$$f^1(t, x_1, \ldots, x_n) = c_1 t + k_1 \, , \; \ldots, \; f^n(t, x_1, \ldots, x_n) = c_n t + k_n \, , \tag{33}$$

where $k_1, \ldots, k_n$ are constants of integrations.

If we take under the consideration the conditions (29), then we obtain:

$$f^1(0, x_1, \ldots, x_n) = k_1 = x_1, \ldots, f^n(0, x_1, \ldots, x_n) = k_n = x_n \, . \tag{34}$$

Therefore the 1-parameter group of local transformations is defined by:

$$\{c_1 t + x_1, \ldots, c_n t + x_n\} \, . \tag{35}$$

Each transformation has the property:

$$\varphi_t \, : \, U \mapsto U, \; \varphi_t \, : \, (x_1, \ldots, x_n) \mapsto (c_1 t + x_1, \ldots, c_n t + x_n), \tag{36}$$

which is an isometry.

From the above we have the theorem:

Theorem 4.2 *Let $(M, \, g)$ be a compact orientable Riemannian manifold of dimension n without boundary. If $r = 0$, then there are no one-parameter families of isometries determined by (36).*

5.

Let $(M, \, g)$ be a compact orientable Riemannian manifold of dimension n. Let $H^1(M, \mathbf{R})$ be the vector space of harmonic 1-forms on $(M, \, g)$. Then the $\dim H^1(M, \mathbf{R}) = b_1$, is the first Betti number, which is a topological property of M, that means , it is independent of the Riemannian metric g on M.

There was the following conjecture.

Conjecture 5.1 *Let $(M, \, g)$ be a compact Riemannian manifold of dimension n. Let $K^1(M, \mathbf{R})$ be the vector space of the Killing vector fields on $(M, \, g)$. Is the $\dim K^1(M, \mathbf{R})$ a topological invariant?*

This conjecture is not true. This is a consequence of the following theorem.

Theorem 5.1 *The dimension of $K^1(M, \mathbf{R})$ is not a topological invariant on the compact manifold M.*

Proof. There are constants $\beta(n) > \gamma(n)$ depending only on the dimension n of the compact manifold M ([3]) such that M admits a complete metric g with Ricci curvature $p(g)$ satisfying the inequalities:

$$-\beta(n) < p(g) < -\gamma(n). \tag{37}$$

The inequalities (37), by means of (8) imply $\dim\big(K^1(M, \mathbf{R})\big) = 0$. This is valid for every compact manifold M. Hence $\dim\big(K^1(M, \mathbf{R})\big) = 0$ for a metric g, satisfying the inequalities (37), is not a topological invariant. □

Theorem 5.2 *Let M be a compact manifold without boundary. There is a Riemannian metric g on M such that there exists no one-parameter family of isometries on M with respect to the metric g.*

Proof. There exists a metric g on M such that its Ricci curvature $p(g)$ satisfies the inequalities:

$$-\beta(n) < p(g) < -\gamma(n),$$

where $\beta(n) > \gamma(n)$ are constants depending only on the dimension n of M. Hence there is not a Killing vector field on $(M,\, g)$. From this we conclude that there exists no one-parameter family of isometries of $(M,\, g)$. □

Theorem 5.3 *Let M be a compact manifold of dimension $n \geq 3$. We consider a Riemannian metric g on M with the property $p(g) < 0$. Then the group of isometries $I(M)$ of $(M,\, g)$ is finite.*

Proof. There exists no one-parameter family of isometries of $(M,\, g)$. From this we conclude that $I(M)$ is finite. This geometric restriction is sharp ([3]). If M is a compact manifold of dimension $n \geq 3$ and G is a subgroup of $Diff(M)$, then $G = I(M)$, where $I(M)$ is the group of isometries on M with respect to a metric g with the property $p(g) \leq 0$ if G is finite.

Let $(M,\, g)$ be a differential manifold of dimension n. We denote by $H(M)$ the space of all Riemannian metrics on the manifold M. The set $H(M)$ can be become a metric space with metric d defined by:

$$d\,:\, H(M) \times H(M) \mapsto I\!R_+, \quad d\,:\,(g_1, g_2) \mapsto d(g_1, g_2),$$

where $d(g_1, g_2)$ is the minimal distance between $(M,\, g_1)$ and $(M,\, g_2)$ for all isometric embeddings in any metric space M. □

It has been proved the following theorem ([3]):

Theorem 5.4 *Let $(M,\, g)$ be a compact manifold. The subset $\Lambda(M)$ of $H(M)$ with the property $\Lambda(M) = \{g \in H(M) \,|\, g$ Riemannian metric with Ricci curvature $p(g) < 0\}$.*

Then $\Lambda(M)$ is dense in the set of all metrics $H(M)$ with respect to the metric d.

Now, we can prove the following theorems:

Theorem 5.5 *Let M be a compact manifold of dimension n. There is an infinite number of metrics, whose set denoted by $\Lambda(M)$ such that $\dim K^1(M, g) = 0$, $\forall g \in \Lambda(M)$. As a matter of fact $\Lambda(M)$ is dense in the set of all metrics $H(M)$ with respect to the metric d.*

Proof. Let g be a Riemannian metric with negative Ricci curvature. Then we have:

$$\dim K^1(M, g) = 0. \tag{38}$$

Since $g \in \Lambda(M)$ and the subset $\Lambda(M)$ is dense in $H(M)$ with respect to the above mentioned metric d we conclude that (38) is valid for all metrics of $\Lambda(M)$. □

Theorem 5.6 *Let M be a compact manifold of dimension n. There is a set, denoted by $\Lambda(M)$, of infinite number of Riemannian metric on M such that the group of isometries $I(M, g)$ for every $g \in \Lambda(M)$ is finite.*

Proof. It is known that, for every metric g on M with the property $\dim K^1(M, \mathbf{R}) = 0$, the group of isometries of (M, g) is finite. From this and Theorem 5.4 the theorem follows. □

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