LIE AUTOMORFISMS OF SPECIAL NILPOTENT LIE ALGEBRAS

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Abstract

The aim of the present paper is to study the group of Lie automorphisms of characteristically Nilpotent Lie algebras of dimension seven.

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1 Introduction

Let g be a Lie algebra over a field F of characteristic zero. The problem of the determination of the group $Aut(g)$ of Lie automorphisms of g is an open one. We assume that g is characteristically nilpotent of dimension seven. The number of such non-isomorphic Lie algebras is eight. The purpose of this paper is to determine the groups of Lie automorphisms of such Lie algebras. This is an extension of the paper [2].

The whole paper contains four sections.

The first section is the introduction. Basic elements of Lie algebras, special Lie algebras and the group of Lie automorphisms are given in the second section. The third section includes the study of the group of Lie automorphisms of the characteristically nilpotent Lie algebras of dimension seven. Some other properties of this group are studied in the last section.

2 Basic constructions

Let g be a Lie algebra over the field F of characteristic zero of dimension n. It is known that from this algebra we can form the following sequences of ideals of g:

$$C^0g = g,\ C^1g = [g, g], \ldots,\ C^qg = \left[g, C^{q-1}g\right], \ldots, \tag{1}$$

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which is called descending central sequence.

$$C_0g = \{0\},\ C_1g = centre(g),\ \ldots C_qg = centre\,(g/C_{q-1}g)\,,\ldots, \tag{2}$$

which is called increasing central sequence and

$$D^0g = g,\ D^1g = [g,g],\ldots,\ D^qg = \left[D^{q-1}g, D^{q-1}g\right],\ldots, \tag{3}$$

which is called derived sequence.

If there exists an integer q such that $C^q = \{0\}$, then the Lie algebra g is called nilpotent of nilpotency q.

A linear mapping f on g is called derivation, if it satisfies the relation:

$$f[x,y] = [fx,y] + [x,fy]\,, \forall x,y \in g.$$

The set of all derivations f on g is denoted by $\Delta(g)$, that is

$$\Delta(g) = \{f | f : g \mapsto g,\ \text{linear and } f[x,y] = [fx,y] + [x,fy]\}\,.$$

The following mapping:

$$ad_x : g \mapsto g,\ ad_x : y \mapsto ad_x\, y = [x,y]$$

is a derivation which is called inner derivation. The set of all inner derivations is denoted by Δ_i. The other derivations on g are called outer, which are denoted by Δ_0. It is known that:

$$\Delta = \Delta_i \oplus \Delta_0,$$

where Δ_i is an ideal of Δ. There are some Lie algebras whose $\Delta_i = \Delta$.

The set of derivations Δ on g is another Lie algebra.

The Lie algebra g is called characteristically nilpotent if the Lie algebra Δ is nilpotent.

Let us consider a Lie algebra g over a field F of characteristic zero. A Lie automorphism f of g is defined as follows:

$$f : g \mapsto g,\ f \text{ linear mapping and}$$

$$f : [x,y] \mapsto f\,([x,y]) = [f(x), f(y)]\,.$$

The set of all Lie automorphisms of a given Lie algebra is denoted by $Aut(g)$ and defined by:

$$Aut(g) = \{f | f : g \mapsto g, f \text{ is linear},\ f\,([x,y]) = [f(x), f(y)]\}\,.$$

This is a group with the usual composition of automorphisms as its inner law.

A Lie automorphism is called unipotent if there is a base such that its representation with respect to the base $\{e_1,\ldots,e_n\}$ has the form:

$$\begin{bmatrix} 1 & b_{21} & b_{31} & \ldots & b_{n-1\,1} & b_{n\,1} \\ 0 & 1 & b_{32} & \ldots & b_{n-1\,2} & b_{n\,2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & b_{n\,n-1} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}. \tag{4}$$

3 The group of Lie authomorphisms

The characteristically nilpotent Lie algebras of dimension seven over the field F of characteristic zero are the following:

$$\begin{array}{lllll} g_{7,1}: & [x_2,x_1]=x_3 & [x_3,x_1]=x_4 & [x_4,x_1]=x_5 & [x_5,x_1]=x_6 \quad [x_6,x_1]=x_7 \\ & [x_3,x_2]=x_6 & [x_4,x_2]=x_7 & [x_5,x_2]=x_7 & \\ & [x_3,x_4]=x_7; & & & \end{array} \tag{5}$$

$$\begin{array}{lllll} g_{7,2}: & [x_1,x_2]=x_3 & [x_1,x_3]=x_4 & [x_1,x_4]=x_5 & [x_1,x_5]=x_6 \quad [x_1,x_6]=x_7 \\ & [x_2,x_3]=x_5+x_7 & [x_2,x_4]=x_6 & [x_2,x_5]=x_7; & \end{array} \tag{6}$$

$$\begin{array}{lllll} g_{7,3}: & [x_1,x_2]=x_3 & [x_1,x_3]=x_4 & [x_1,x_4]=x_5 & [x_1,x_5]=x_6 \quad [x_1,x_6]=x_7 \\ & [x_2,x_3]=x_6+x_7 & [x_2,x_4]=x_7; & & \end{array} \tag{7}$$

$$\begin{array}{lllll} g_{7,4}: & [x_1,x_2]=x_3 & [x_1,x_3]=x_4 & [x_1,x_4]=x_6+\lambda x_7 & [x_1,x_5]=x_7 \quad [x_1,x_6]=x_7 \\ & [x_2,x_3]=x_5 & [x_2,x_4]=x_7 & [x_2,x_5]=x_6 & \\ & [x_3,x_5]=x_7; & & & \end{array} \tag{8}$$

$$\begin{array}{lllll} g_{7,5}: & [x_1,x_2]=x_3 & [x_1,x_3]=x_4 & [x_1,x_4]=x_6+x_7 & [x_1,x_6]=x_7 \\ & [x_2,x_3]=x_5 & [x_2,x_5]=x_6 & [x_3,x_5]=x_7; & \end{array} \tag{9}$$

$$\begin{array}{lllll} g_{7,6}: & [x_1,x_2]=x_3 & [x_1,x_3]=x_4 & [x_1,x_4]=x_7 & [x_1,x_5]=x_6 \quad [x_1,x_6]=x_7 \\ & [x_2,x_3]=x_5 & [x_2,x_4]=x_6 & [x_2,x_5]=x_7 & \\ & [x_3,x_4]=x_7; & & & \end{array} \tag{10}$$

$$\begin{array}{lllll} g_{7,7}: & [x_1,x_2]=x_3 & [x_1,x_3]=x_4 & [x_1,x_4]=x_7 & [x_1,x_5]=x_7 \quad [x_1,x_6]=x_7 \\ & [x_2,x_3]=x_5 & [x_2,x_4]=x_7 & [x_2,x_5]=x_6 & \\ & [x_3,x_5]=x_7; & & & \end{array} \tag{11}$$

$$\begin{array}{lllll} g_{7,8}: & [x_1,x_2]=x_4 & [x_1,x_3]=x_7 & [x_1,x_4]=x_5 & [x_1,x_5]=x_6 \\ & [x_2,x_3]=x_6 & [x_2,x_4]=x_6 & [x_2,x_6]=x_7 & [x_4,x_5]=-x_7. \end{array} \tag{12}$$

Let Θ be a Lie automorphism of the characteristically nilpotent Lie algebra $g_{7,8}$. Let $\{e_1,\ldots,e_7\}$ be a base of $g_{7,8}$.

It is known that Θ can be represented with respect to the base $\{e_1,\ldots,e_7\}$ by the matrix:

$$T = \begin{bmatrix} b_{11} & b_{21} & \ldots & b_{71} \\ b_{12} & b_{22} & \ldots & b_{72} \\ \ldots & \ldots & \ldots & \ldots \\ b_{17} & b_{27} & \ldots & b_{77} \end{bmatrix} \tag{13}$$

and therefore

$$\Theta E = TE, \tag{14}$$

where

$$E^T = [e_1, e_2, \ldots, e_7].$$

If we apply this Lie automorphism Θ on the Lie brackets of the Lie algebra $g_{7,8}$, including those which are zero, then we obtain:

$$\Theta([x_1, x_2]) = \Theta(x_4), \quad \Theta([x_1, x_3]) = \Theta(x_7), \quad \Theta([x_1, x_4]) = \Theta(x_5), \tag{15}$$
$$\Theta([x_1, x_5]) = \Theta(x_6), \quad \Theta([x_2, x_3]) = \Theta(x_6), \quad \Theta([x_2, x_4]) = \Theta(x_6), \tag{16}$$
$$\Theta([x_2, x_6]) = \Theta(x_7), \quad \Theta([x_4, x_5]) = \Theta(-x_7), \quad \Theta([x_k, x_\lambda]) = 0, \tag{17}$$

where $[x_k, x_\lambda] = 0$ and all other Lie brackets are zero.

The relations (15)-(17) by means of (14) and the fact that Θ is a Lie automorphism, after the necessary calculations, give:

$$\begin{aligned}
&b_{12} = 0,\\
&b_{13} = b_{23} = 0,\\
&b_{14} = b_{24} = b_{34} = 0,\\
&b_{15} = b_{25} = b_{35} = b_{45} = 0,\\
&b_{16} = b_{26} = b_{36} = b_{46} = b_{56} = 0,\\
&b_{17} = b_{27} = b_{37} = b_{47} = b_{57} = b_{67} = 0,\\
&b_{21} = b_{42} = b_{43} = b_{53} = b_{54} = b_{65} = 0 = b_{31},\\
&b_{63} = -b_{41},\\
&b_{64} = b_{52} - b_{41},\\
&b_{74} = b_{32} - b_{41}b_{52} - b_{61},\\
&b_{75} = b_{51},\\
&b_{76} = -b_{41},\\
&b_{11} = b_{22} = b_{33} = b_{44} = b_{55} = b_{66} = b_{77} = 1.
\end{aligned} \tag{18}$$

From the above we have proved that Θ is unipotent. With the same methods we can prove that the Lie automorphisms of the Lie algebras $g_{7,1}$, $g_{7,2}$, $g_{7,3}$, $g_{7,4}$, $g_{7,7,}$, $g_{7,8}$ have the same property.

Theorem 3.1 *The characteristically nilpotent Lie algebras* $g_{7,1}$, $g_{7,2}$, $g_{7,3}$, $g_{7,4}$, $g_{7,7,}$, $g_{7,8}$ *of dimension 7 have the property that the group* $Aut(g)$ *consists of unipotent Lie automorphisms.*

For the characteristically nilpotent Lie algebra $g_{7,6}$ the system using similar techniques yields:

$$\begin{aligned}
&b_{12} = 0,\\
&b_{13} = b_{23} = 0,\\
&b_{14} = b_{24} = b_{34} = 0,\\
&b_{15} = b_{25} = b_{35} = b_{45} = 0,\\
&b_{16} = b_{26} = b_{36} = b_{46} = b_{56} = 0,\\
&b_{17} = b_{27} = b_{37} = b_{47} = b_{57} = b_{67} = 0,\\
&b_{22} = b_{44} = b_{66} = 1 \text{ and}\\
&\{b_{11}, b_{33}, b_{55}, b_{77}\} = \begin{cases} 1 \\ -1 \end{cases},\\
&b_{21} = b_{54} = 0,\\
&b_{43} = b_{11}b_{32},\\
&b_{53} = -b_{31},\\
&b_{63} = b_{11}b_{52} - b_{41},\\
&b_{64} = -b_{11}b_{31},\\
&b_{65} = b_{11}b_{32},\\
&b_{73} = b_{11}b_{42} + b_{31}b_{42} + b_{11}b_{62} - b_{41}b_{32} - b_{51},\\
&b_{74} = b_{32} + b_{11}b_{31}b_{32} - 2b_{11}b_{41} + b_{52},\\
&b_{75} = -b_{31} + b_{32} - b_{11}b_{42},\\
&b_{76} = b_{32}.
\end{aligned} \tag{19}$$

Therefore this Lie algebra, except the unipotent Lie automorphisms has other Lie automorphisms which can be described by the matrices:

$$T_{\varepsilon_1 1 \varepsilon_3 1 \varepsilon_5 1 \varepsilon_7} = \begin{bmatrix} \varepsilon_1 & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} & b_{71} \\ 0 & 1 & b_{32} & b_{42} & b_{52} & b_{62} & b_{72} \\ 0 & 0 & \varepsilon_3 & b_{43} & b_{53} & b_{63} & b_{73} \\ 0 & 0 & 0 & 1 & b_{54} & b_{64} & b_{74} \\ 0 & 0 & 0 & 0 & \varepsilon_5 & b_{65} & b_{75} \\ 0 & 0 & 0 & 0 & 0 & 1 & b_{76} \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_7 \end{bmatrix}, \tag{20}$$

where $\varepsilon_i = \begin{cases} 1 \\ -1 \end{cases}, \ i = 1, 3, 5, 7.$

Now we can state the following theorem:

Theorem 3.2 *Let g be the characteristically nilpotent Lie algebra $g_{7,6}$ described in (20).*

The group of Lie automorphisms $Aut(g)$ has the form:

$$Aut(g) = \{T_{\varepsilon_1 1 \varepsilon_3 1 \varepsilon_5 1 \varepsilon_7} \mid \ \text{where } T_{\varepsilon_1 1 \varepsilon_3 1 \varepsilon_5 1 \varepsilon_7} \text{ is given by (20)}\}.$$

The unipotent Lie automorphisms constitute of a proper subgroup of $Aut(g)$.

Theorem 3.3 *Let g be the characteristically nilpotent Lie algebra $g_{7,5}$ described in (19).*

The group of Lie automorphisms $Aut(g)$ has the form:

$$Aut(g) = \{T_{1\varepsilon_2\varepsilon_3\varepsilon_4 1\varepsilon_6\varepsilon_7} \mid \textit{ where } T_{1\varepsilon_2\varepsilon_3\varepsilon_4 1\varepsilon_6\varepsilon_7} \textit{ is the next matrix (21) } \} :$$

$$T_{1\varepsilon_2\varepsilon_3\varepsilon_4 1\varepsilon_6\varepsilon_7} = \begin{bmatrix} 1 & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} & b_{71} \\ 0 & \varepsilon_2 & b_{32} & b_{42} & b_{52} & b_{62} & b_{72} \\ 0 & 0 & \varepsilon_3 & b_{43} & b_{53} & b_{63} & b_{73} \\ 0 & 0 & 0 & \varepsilon_4 & b_{54} & b_{64} & b_{74} \\ 0 & 0 & 0 & 0 & 1 & b_{65} & b_{75} \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_6 & b_{76} \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_7, \end{bmatrix}, \tag{21}$$

where $\varepsilon_i = \begin{cases} 1 \\ -1 \end{cases}, \ i = 2, 3, 4, 6, 7.$

The unipotent Lie automorphisms constitute of a proper subgroup of $Aut(g)$.

4 Some properties of the group of Lie authomorphisms

In this section we study more properties of $Aut(g)$, where g is given by (5) – (12).

Theorem 4.1 *Let $g_{7,1}, g_{7,2}, g_{7,3}, g_{7,4}, g_{7,5}, g_{7,6}, g_{7,7}$ be the characteristically nilpotent Lie algebras of dimension seven given by (15) – (21). The group of Lie automorphisms of these Lie algebras can be generated by $\Theta(e_1)$ and $\Theta(e_2)$, where $\{e_1, \ldots, e_7\}$ is the given base of these Lie algebras.*

Proof. It is known that Θ can be described by a matrix:

$$T = \begin{bmatrix} \varepsilon_1 & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} & b_{71} \\ 0 & \varepsilon_2 & b_{32} & b_{42} & b_{52} & b_{62} & b_{72} \\ 0 & 0 & \varepsilon_3 & b_{43} & b_{53} & b_{63} & b_{73} \\ 0 & 0 & 0 & \varepsilon_4 & b_{54} & b_{64} & b_{74} \\ 0 & 0 & 0 & 0 & \varepsilon_5 & b_{65} & b_{75} \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_6 & b_{76} \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_7 \end{bmatrix},$$

where $\varepsilon_i = \begin{cases} 1 \\ -1 \end{cases}, \ i = 1, \ldots, 7.$

From the above system (19) we have proved that the only arbitrary invariants are:

$$b_{21},\ b_{31},\ \ldots,\ b_{71},\ b_{32},\ b_{42},\ \ldots,\ b_{72}\ .$$

Therefore Θ is completely determined by $\Theta(e_1)$ and $\Theta(e_2)$. □

With the same method as in theorem 4.1 we can prove the below theorem.

Theorem 4.2 *The $Aut(g_{7,8})$ of the Lie algebra $g_{7,8}$ can be determined by $\Theta(e_1)$, $\Theta(e_2)$ and the element b_{73} by $\Theta(e_3)$.*

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