HYPERBOLIC KAC-MOODY ALGEBRAS

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Abstract

The classification of solvable Lie algebras is still a hopeless problem. In 1945, Malcev reduced the solvable case to the nilpotent one and in [7] to each nilpotent Lie algebra n of finite dimension, a Kac-Moody algebra $g(A)$ has been associated (n is of type A). The aim of this paper is to connect the classification of nilpotent Lie algebra of maximal rank and of Hyperbolic type H_2 under the methode of Kac-Moody algebras. We will follow very closely Kac's book [4].

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1 Kac-Moody algebras.

Definition 1.1 Let $A(a_{ij})$ $1 \leq i, j \leq l$ be a square matrix with entries in $\mathbf{Z}$ with the following properties:

(i) $a_{ii} = 2$,

(ii) $a_{ij} \leq 0$ if $i \neq j$,

(iii) if $a_{ij} = 0$, then $a_{ji} = 0$, $i \neq j$

for $i, j = 1, ..., l$.
Then the square matrix A is called *generalized Cartan matrix* (denoted by $g.C.m.$).

Definition 1.2 A generalized Cartan matrix A is called a matrix of *hyperbolic type* if it is indecomposable, symmetrizable of indefinite type and if every proper connected subdiagram of $S(A)$ is of finite or affine type.

Definition 1.3 Let L be a Lie algebra with $3l$ generators x_i, h_i, y_i $(i = 1, \ldots, l)$ and A be a g.C.m. associated to the Lie algebras L, satisfying the relations:

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(i) $[h_i, h_j] = 0$, $[x_i, y_j] = \delta_{ij} h_i$;

(ii) $[h_i, x_j] = a_{ij} x_j$, $[h_i, y_j] = -a_{ij} y_j$;

(iii) $(ad\, x_i)^{1-a_{ij}} x_j = 0$, $(ad\ y_i)^{1-a_{ij}} y_j = 0$,

for $i, j = l, ..., l$.
Then the Lie algebra L is called a *Kac-Moody algebra*.

Any Kac-Moody algebra can be written as follows:

$$L = L_+ \oplus h \oplus L_-,$$

where $L_+ = \underset{\alpha\in\Delta_+}{\oplus} L_\alpha$, $L_- = \underset{\alpha\in\Delta_-}{\oplus} L_\alpha$, $h = \overset{n}{\underset{i=1}{\oplus}} Kh_i$.

2 The positive part of an Hyperbolic Kac-Moody algebra.

Definition 2.1 Let W be the Weyl group of $g(A)$, that means that the set generated by all the reflections r_i, which are defined by $r_i\alpha_j = \alpha_j - a_{ij}\alpha_i$ (a_{ij} elements of A). A root $\alpha \in \Delta$ is called *real* if there exists $w \in W$ such that $w(\alpha)$ is a simple root. We denote by Δ_+^{re} the set of all positive real roots, where $\Delta_+^{re} = W(\alpha_i)$.

Definition 2.2 A root a which is not real is called an imaginary root. Then the set of all positive imaginary roots are $\Delta_+^{im} = W(-\mathbf{C}^v)$.
By definition

$$\Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}.$$

Let $t_i \in Dern_+$ be defined by $t_i e_j = \delta_{ij} e_j$, where e_j are the generators of the Hyperbolic Kac-Moody algebra of type A. Let T be the maximal torus on n_+, then the root space decomposition corresponding to T is:

$$n_+ = \underset{\alpha\in\Delta_+}{\oplus} n^\alpha,$$

where

$$n^\alpha = \{x \in n_+\ ;\ tx = \alpha(t)x,\ (\forall)t \in T\},$$
$$\Delta_+ = \{\alpha \in T^*;\ n^\alpha \neq (0)\},$$
$$mult(\alpha) = \dim n^\alpha \text{ of } \alpha \in -\mathbf{C}^v,\ ([4]).$$

Theorem 2.1 *[7] Let n be a nilpotent Lie algebra of maximal rank.*
Then the mapping

$$G_l(n) : I \to L_+(n)/ \underset{\alpha\in I}{\oplus} n^\alpha$$

is a bijection from the set of all G_l -orbits of all the ideals $I(A)$ onto $Nil(max, n)$ (the set of all the nilpotent Lie algebras of maximal rank).

3 Nilpotent Lie algebras of maximal rank and of type of the Hyperbolic Kac- Moody Algebra of type H_2.

Let A be the g.C.m. of type H_z which is:

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

The positive part of the hyperbolic Kac-Moody algebra of type $A = H_2$ is the Lie algebra n_+ defined by the generators $\{e_1, e_2\}$ and the relations:

$$[e_1 [e_1 [e_1 [e_1 [e_1, e_2]]]]] = 0 = [e_1 [e_1 [e_1 [e_1 [e_2, e_1]]]]] .$$

The maximal torus is $T = Ct_1 \oplus Ct_2$.
The root system with multiplicity till the height ≤ 6 $(ht(k_1\alpha_1 + k_2\alpha_2) = k_1 + k_2)$.
The reflections are:

$$r_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Delta_+^{re} = \{\alpha_1, \alpha_2, 3\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2\} ,$$

$$\begin{aligned} \Delta_+^{im} = \{&\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 2\alpha_2, 3\alpha_1 + 2\alpha_2, \\ &2\alpha_1 + 3\alpha_2, 4\alpha_1 + 2\alpha_2, 3\alpha_1 + 3\alpha_2, 2\alpha_1 + 4\alpha_2\} . \end{aligned}$$

hThe Basic chain is:

$$N = \{2\alpha_1 + 2\alpha_2, 3\alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2, 4\alpha_1 + 2\alpha_2, 3\alpha_1 + 3\alpha_2, 2\alpha_1 + 4\alpha_2\}.$$

The automorphism group of the Dynkin diagram is $< (1,2) >$.
Finally we found 9 algebras, 3 continuous families with 1 parameter and 7 continuous families with 2 parameters.

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