ON THE TOPOLOGY OF JULIA SETS

M. Postolache and E. Ciobanu

Abstract

Topological aspects concerning the Julia sets for certain one-parameter polynomial functions are investigated in this paper. Using our algorithm and the iteration, we emphasize the spiraling into these sets what looks like limit cycles. These cycles become smaller and finally become the vertices of the Julia polygons that enclose the components of the Fatou set.

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1 Introduction

This paper is written by applied analysts who use the computer to produce pictures in order to get ideas about the behaviour of some geometric sets. For this purpose, we use our own simple software packages wich operate on PC's. All the pictures in this paper can be reproduced by any unexperienced user since this package is available upon request from the first author.

The aim of this paper is to summarize our computer-exploratory work concerning the Julia sets for one-parameter polynomial functions of the form

$$f_{n,c}(z) = z^n + c, \quad n \geq 2,$$

where both z and c are complex numbers. We shall debate about some topological properties of these sets using computer experiments and the iteration, [12]. First, we shall consider the case when the Julia set is a Cantor set. After that we shall state some results regarding the Julia sets as one-dimensional continua (for details, see [14]).

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The final form of this work has been made when the first author visited Tsukuba University supported by the Japanese Society for Promotion Science.

On the whole, this work keeps on some ideas of Mandelbrot, [10], which was the first person to investigate some sets using computer graphics. Through his research via computers, this scientist marks a new stage in the investigation of these sets. Although this paper uses some ideas of Mandelbrot, through its proofs, computer algorithms, and examples this paper is new.

Beside Mandelbrot, others who developed the theory of Julia sets are Julia, [9], Devaney, [5]. For additional background material into this subject the reader may refer to [2], [3], [15].

2 Filled Julia sets

The *Julia set* for a nonlinear rational transformation on the Riemannian spehere is the closure of the sets of all repulsive cycles of the transformation. More accurately we have, [4]

Definition 2.1 *The filled Julia set of $f_{n,c}$ is the set of points*

$$\{z, \ /|f^k_{n,c}(z)| \not\to \infty \text{ as } k \to \infty\}$$

that is the set of points whose orbits under $f_{n,c}$ are bounded. The Julia set of $f_{n,c}$ is the frontier of the filled Julia set. The Fatou set is the complement of the Julia set.

Proposition 2.1 *If the number of components of the Fatou set is finite then there are at most two such components.*

For a proof, see [6].

In the following, we will denote by $K_{n,c}$ the filled Julia set and by $J_{n,c}$ the Julia set of $f_{n,c}$. We have

Proposition 2.2 *The following statements hold good:*
a) $K_{n,c}$ and $J_{n,c}$ are nonempty compact subsets of C;
b) $K_{n,c}$ and $J_{n,c}$ are completely invariant under $f_{n,c}$.

Let us consider $|z| \geq |c|$ and $|z|^{n-1} > 2$. Then, by the assumption $|z| \geq |c|$ and by the triangle inequality, after calculation we deduce

$$|f_{n,c}(z)| \geq |z|(|z|^{n-1} - 1).$$

Now, since $|z|^{n-1} > 2$, then there exists a sufficiently small $\lambda > 0$ such that $|z|^{n-1} - 1 > 1 + \lambda$. Consequently

$$|f_{n,c}(z)| > (1 + \lambda)|z|$$

and, in particular, $|f_{n,c}(z)| > |z|$. By applying the same argument repeatedly, we get

$$|f^k_{n,c}(z)| > (1 + \lambda)^k |z|,$$

hence the orbit of z under $f_{n,c}$ tends to infinity.

These ideas outline a proof of this result

Theorem 2.1 *If* $|z| \geq |c|$ *and* $|z|^{n-1} > 2$ *then* $|f^k_{n,c}(z)| \to \infty$ *as* $k \to \infty$.

Theorem 2.1 gives a sufficient condition for the orbit of z tends to infinity. We shall refer to this result as the escape criterion.

Now, we take into account that, for each integer $n > 1$, $f_{n,c}$ has a single critical point at $z = 0$. By the escape criterion we obtain

Corollary 2.1 *If* $|c|^{n-1} > 2$ *then the orbit of the critical point* $z = 0$ *escapes to infinity under* $f_{n,c}$.

The analytical approach given above suggests the following algorithm for computing approximations of filled Julia sets.

Algorithm 2.1 *Choose a maximum number of iterations* N, *and for any point* z *with* $|z| \leq |c|$ *compute the orbit of* z *under* $f_{n,c}$. *If, for some* $i \leq N$, $|f^i_{n,c}(z)| > max\{|c|, 2^{1/(n-1)}\}$ *then stop iterating and colour* z *white* ($z \notin K_{n,c}$). *If* $|f^i_{n,c}(z)| \leq max\{|c|, 2^{1/(n-1)}\}$ *for all* $i \leq N$, *then colour* z *black* ($z \in K_{n,c}$).

The preceding algorithm shows that the black points yield an approximation to the filled Julia set. The behaviour of Julia sets depends on the Mandelbrot set. We have, [10], the following

Definition 2.2 *The Mandelbrot set of* $f_{n,c}$ *consists of all* c *such that the filled Julia set* $K_{n,c}$ *is connected.*

We shall denote by $\mathcal{M}$ the Mandelbrot set. Mandelbrot, [10], gives an alternative characterization of $\mathcal{M}$ in terms of the iterates of $f_{n,c}$. This is

Theorem 2.2 *The Mandelbrot set* $\mathcal{M}$ *is given by*

$$\mathcal{M} = \{c \in C / f^k_{n,c}(0) \not\to \infty \text{ as } k \to \infty\}.$$

Using Theorem 2.2 we can give an algorithm for the Mandelbrot set to produce this set for the family $f_{n,c}$. These sets are called the *degree-n bifurcation sets*, [5].

Algorithm 2.2 *Choose a maximum number of iterations* N, *and for any point* c *in a grid compute the first* N *points on the orbit of* 0 *under* $f_{n,c}$. *If* $|f^i_{n,c}(0)| > 2$ *for some* $i \leq N$ *then stop iterating and colour* c *white* ($c \notin \mathcal{M}$). *If* $|f^i_{n,c}(0)| \leq 2$ *for all* $i \leq N$, *then colour* c *black* ($c \in \mathcal{M}$).

The black points yield an approximation to the Mandelbrot set, one of the most fascinating objects in Mathematics[1]. The Mandelbrot set for $n = 2$ can be contemplated, for example, in [10]. In Fig. 1 a simulated Mandelbrot set for $n = 3$ is given. The proposition in the following states some symmetry properties of these sets.

[1]The Mandelbrot set was discovered in 1980 by the famous scientist Benoit Mandelbrot

Proposition 2.3 *The degree-n bifurcation set, $n \geq 3$, is symmetric with respect to the rotation through angle $\frac{2\pi}{(n-1)}$.*

The connectivity of the filled Julia sets can be experimentally determined by iteration. In this respect, one can prove that $K_{n,c}$ change abruptly from one connected piece to many isolated points at certain special values of c. Also, these sets appear to be self-similar sets, [7], reminiscent of fractals.

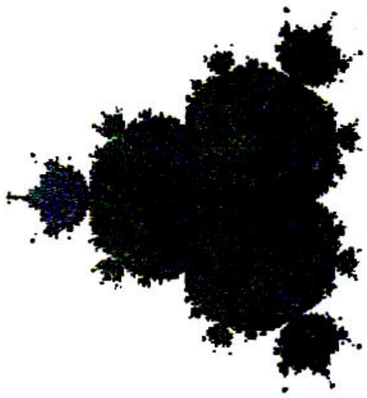

Fig. 1

Now, using Algorithm 2.1 it is straigthforward to write a computer program that plots just the boundary points, that is, the actual Julia set. The difference in this and the previous algorithm is that whenever a point that has a trapped orbit is located, say at pixel location (i,j), then the orbits for the points at the four neighboring locations, $(i,j-1)$, $(i,j+1)$, $(i-1,j)$ and $(i+1,j)$ are checked to see whenever any of them escape (Fig. 2). If at least one of them does escapes, then the point (i,j) is labeled as a member of the Julia set.

$$\begin{array}{ccccc} & & (i,j+1) & & \\ & & \uparrow & & \\ (i-1,j) & \longleftarrow & (i,j) & \longrightarrow & (i+1,j) \\ & & \downarrow & & \\ & & (i,j-1) & & \end{array}$$

Fig. 2

To cut down on computer memory requirements and still have access to outcomes of orbits already computed, we can store three rows (or columns) of outcomes, coded 1 for trapped points and 0 for escaping points. The values of the middle row are checked by the above criterion to see which are boundary points. Once this is done and the values are ploted, a new row is stored, and the oldest row is droped. The process is repeated until all of the interior rows have been processed.

The following computer program contains the main procedure designed using the basic ideas described above.

```
function f(x, y);
f = 0;
for n = 1 to 120
  begin
    x1 = x^3 - 3xy^2 + a; y1 = 3x^2y - y^3 + b;
    if x1^2 + y1^2 > 2^2 then f = 1; goto end;
    x = x1; y = y1;
  end;

INPUT: d, a, b;
begin
  x0 = -2;
  repeat
    y0 = -2;
    repeat
      x = x0; y = y0;
      if f(x, y) = 0 then
        if (f(x - d, y) = 1) or (f(x + d, y) = 1) or
        (f(x, y - d) = 1) or (f(x, y + d) = 1) then PLOT(x0, y0);
        y0 = y0 + d;
    until y0 > 2;
    x0 = x0 + d;
  until x0 > 2;
end.
```

Fig. 3 represents a Julia set for $n = 3$ using the above algorithm. What is interesting here is that we see spiraling into what looks like limit cycles. These cycles become smaller and finally become the vertices of the Julia polygons that enclose the components of the Fatou set.

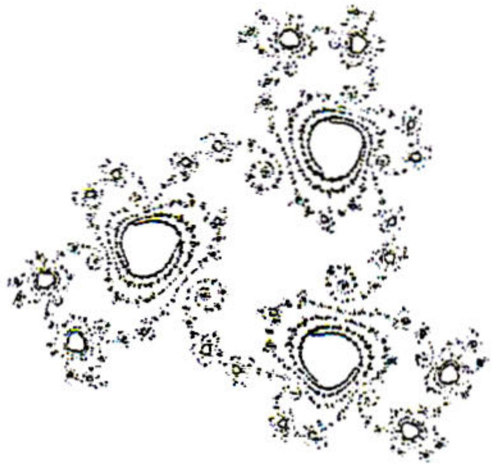

Fig. 3

3 Julia sets as repellors

The Julia set $J_{n,c}$ is either a Cantor set or a one-dimensional continuum. To deal with Julia sets as repellors we need the following

Definition 3.1 *A point z_0 is periodic if $f^k_{n,c}(z_0) = z_0$, for some positive integer k. The number $\lambda = (f^k_{n,c})'(z_0)$ is the eigenvalue of z_0.*

Let us now suppose $z_0 \notin J_{n,c}$ be a repelling periodic point with period k for $f_{n,c}$. Therefore, there exists $\lambda > 1$ such that $|(f^k_{n,c})'(z_0)| = \lambda$. Then there is a disk $D(z_0, r)$ centered at z_0 and whose radius is r, lying in $K_{n,c}$ such that no orbit of a point in this disk escapes to infinity. Since, for each ℓ, $f^{k\ell}_{n,c}$ is a polymonial, then for each $z \in D$ we obtain

$$|f^{k\ell}_{n,c}(z)| \leq \max\{|c|, 2^{1/(n-1)}\}.$$

If $M = \max\{|c|, 2^{1/(n-1)}\}$ the for each k we have

$$|(f^{k\ell}_{n,c})'(z_0)| < \frac{m}{r}.$$

But $|(f^{k\ell}_{n,c})'(z_0)| = \lambda^\ell \to \infty$ hence $z_0 \in J_{n,c}$.

We skeched a proof of this result

Theorem 3.1 *Any repelling periodic point for $f_{n,c}$ belongs to $J_{n,c}$.*

The reasons given above suggest another algorithm for computing Julia sets, not the filled Julia sets. This algorithm is reminiscent of the iterated functions systems already studied by Barnsley and Demko [1]. A result of applying this algorithm will be displayed in a forthcomming paper.

Fatou, [6], and Julia [9], give a criterion to determine, for given n, whether $f_c = f_{n,c}$ has a connected Julia set. This criterion is

Theorem 3.2 *The Julia set of f_c is totally disconnected if and only if*

$$\lim_{k\to\infty} f_c^k(0) = \infty.$$

Otherwise the Julia set is connected.

We conclude our paper with an interesting result of Rogers and Mayer, [14], related to the indecomposability of $J_{n,c}$. For this is necessary

Definition 3.2 *By a continuum, we mean a compact connected metric space. A continuum is indecomposable if it is not the union of two of its proper subcontinua.*

Theorem 3.3 *If the continuum J is the Julia set of some polynomial then the following statements are equivalent:*

1) *J is indecomposable;*

2) *is the union of a countable number of indecomposable continua.*

4 Software

All calculations were performed on a regular PC with coprocessor and an HP printer. The program, [13], was written in Turbo Pascal version 6.1 and needs arround 500 K. It takes between 5 and 50 minutes to produce an image. The program is available from the first author on a $5\frac{1}{4}$-inch floppy disk.

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Author's address:

M. Postolache and E. Ciobanu
Politehnica University of Bucharest
Dept. of Mathematics I and Dept. of Transport Techniques
Splaiul Independenței 313,
77206 Bucharest, Romania
E-mail: mihai@mathem.pub.ro