LINER CONNEECTIONS ON MODULES WITH DIFFERENTIALS

Paul Popescu and Marcela Popescu

Abstract

The aim of the paper is to give an abstract definition of a linear connection on modules with differentials over associative algebras and to study its properties.

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1 Introduction

This paper continues the ideas from some previous works [8, 9] and we keep all the definitions and the notations used there.

Consider an associative algebra $A$ over a field $k$ and $(A, M)$ a module, which can be a left, right or bi-module. We denote as $Z(A)$ the center of $A$.

For two left modules $(A', M')$ and $(A, M)$, a contravariant morphism of left module is a couple $(\varphi, \psi)$, where $\varphi: A \rightarrow A'$ is a morphism of algebras such that $\varphi(Z(A)) \subset Z(A')$ and $\psi: M' \rightarrow A' \otimes_{Z(A)} M$ is a morphism of left $A'$-module. We say that $\psi(m')$ is the $\psi$-decomposition of $m'$.

Contravariant morphisms of right module and bi-module are defined in an analogous way. According to [9, Theorem 1], the left modules (right modules, respectively bimodules) $(A, M)$ with $A$ an object from $\mathcal{A}$ and the contravariant morphisms of the corresponding module are the objects and the morphisms of a category $\mathcal{M}_L^A$ ($\mathcal{M}_R^A$, respectively $\mathcal{M}_B^A$).

A left module $(A, M)$ is a left module with arrow (l.m.w.a.) if a morphism of left module $p^M: M \rightarrow Der(A)$ is given, called an anchor. We denote $p^M(m)(a) = [m, a]_M$ for every $m \in M$ and $a \in A$. In an analogous way, the right module with arrow (r.m.w.a.), respectively bimodule with arrow (b.w.a.) are defined.
Let \((A', M')\) and \((A, M)\) be l.m.w.a.’s. A contravariant morphism of left module \((\varphi, \psi)\) is a morphism of l.m.w.a. if it is a contravariant morphism of left module and for every \(X' \in M'\) which has the \(\psi\)-decomposition

\[
\psi(X') = \sum a'_i \otimes_{Z(A)} X_i,
\]

and \(a \in A\), the condition \([X', \varphi(a)] = \sum a'_i \varphi([X_i, a])\) is fulfilled.

The morphism of r.m.w.a. and of b.m.w.a. can be defined in an analogous way. A preinfinitesimal left module (p.l.m.) is a l.m.w.a. \((A, M)\) together with a bracket \([\cdot, \cdot]_M : M \times M \to M\) which is \(k\)-bilinear, antisymmetric and

\[
[X, aY]_M = [X, a]_M Y + a [X, Y]_M, \quad (\forall) X, Y \in M, \; a \in A.
\]

Let \((A', M')\) and \((A, M)\) be p.l.m.’s. A contravariant morphism of l.m.w.a. \((\varphi, \psi)\) is a morphism of p.l.m. if it is a contravariant morphism of left module and for every \(x', y' \in M'\) which have the \(\psi\)-decompositions

\[
\psi(X') = \sum a'_i \otimes_{Z(A)} X_i, \quad \psi(Y') = \sum \alpha b'_\alpha \otimes_{Z(A)} Y_\alpha,
\]

then the following condition is fulfilled:

\[
\psi([X', Y']_M) = \sum \alpha [X', b'_\alpha]_M \otimes_{Z(A)} Y_\alpha - \sum \alpha [Y', a'_\alpha]_M \otimes_{Z(A)} X_i + \sum_{i, \alpha} a'_i b'_\alpha \otimes_{Z(A)} [X_i, Y_\alpha]_M.
\]

In an analogous way the morphism of p.l.m. and of p.b.m. can be defined. The definitions are correct; it can be checked up as in [6, Lemmas 4.1, 4.2].

Let \((A', M')\) and \((A, M)\) be p.l.m.’s and \((A', M') \xrightarrow{(\varphi, \psi)} (A, M)\) a morphism of l.m.w.a.. The curvature of \((\varphi, \psi)\) is the map \(K : M' \times M' \to A' \otimes_{Z(A)} M\) defined by

\[
K(X', Y') = \psi([X', Y']_M) - \sum \alpha [X', b'_\alpha]_M \otimes_{Z(A)} Y_\alpha + \sum_{i, \alpha} a'_i b'_\alpha \otimes_{Z(A)} [X_i, Y_\alpha]_M.
\]

It is clear that \((\varphi, \psi)\) is a morphism of preinfinitesimal module iff \(K\) vanish. The curvature of a morphism of r.m.w.a. (or b.m.w.a.) of two p.r.m.’s (respectively p.b.m.’s) can be defined in a similar way. It vanishes iff it is a morphism of p.r.m. (p.b.m. respectively). In the case of commutative algebras the definition agrees with [6, Proposition 4.1].

In the case of \(A = A'\), a morphism of left \(A\)-module \(\psi_0 : M' \to M\) induces a morphism \(\psi : M' \to A \otimes_{Z(A)} M, \; \psi(X') = 1_A \otimes_{Z(A)} \psi_0(X')\). If \((A, M')\) and \((A, M)\) are l.m.w.a.’s and \([X', \varphi(a)] = \varphi([\psi_0(X'), a])\), then we say that \(\psi\) is a strong morphism of \(A\)-l.m.w.a. It induces a morphism of l.m.w.a. as above. The curvature of a strong morphism of \(A\)-l.m.w.a. as above is \(K_0(X', Y') = \psi(X'), \psi(Y')\)
ψ ([X', Y']_M). An infinitesimal left module (i.l.m.) is a p.i.m. (A, M), such that the anchor \( p^M : M \rightarrow \text{Der}(A) \) is a morphism of p.i.m. with a vanishing curvature (i.e. \( p^M([X, Y]_M) - [p^M(X), p^M(Y)]_{\text{Der}(A)} = 0 \)). A left Lie pseudoalgebra (L.L.p.a.) is an i.l.m. which has the property \( \mathcal{J}(X, Y, Z) \overset{\text{def}}{=} [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \). \( \mathcal{J} \) is called the Jacobi map.

In an analogous way we can define the infinitesimal right module (i.r.m.), the infinitesimal bimodule (i.b.m.), the right Lie pseudoalgebra (r.L.p.a.) and the Lie bi-pseudoalgebra (L.b.p.a.).

All these modules, together with their morphisms, are the objects and the morphisms of some categories, called as in [6] categories of modules with differentials. Almost all the results from [6], stated for the associative and commutative algebras, can be extended with care for associative algebras.

### 2 Linear connections on modules with differentials

The linear connection defined here differs from that defined in [10] or [3], being closed to the linear connection defined in the classical differential geometry by the Koszul conditions.

**Definition 2.1** Let \( A \) be a associative \( k \)-algebra, \( (A, L) \) a left module and \( (A, M) \) a module with arrow.

A **linear left M-connection** on \( L \) is an \( A \)-module morphism

\[
\nabla : M \rightarrow \text{End}_A L,
\]

(1)
denoted as \( \nabla(X)(s) = \nabla_X s \), such that:

\[
\nabla(X)(u \cdot s) = [X, u]_M \cdot s + u \cdot \nabla(X)(s) , \quad X \in M, \quad s \in L, \quad u \in A.
\]

(2)

A **linear right M-connection** on \( L \) is an \( A \)-module morphism

\[
\nabla : M \rightarrow \text{End}_A L
\]

(3)
such that:

\[
\nabla(X)(s \cdot u) = s \cdot [X, u]_M + \nabla(X)(s) \cdot u , \quad X \in M, \quad s \in L, \quad u \in A.
\]

(4)

A **linear bilateral M-connection** on \( L \) is an \( A \)-module morphism

\[
\nabla : M \rightarrow \text{End}_A L
\]

(5)
such that:

\[
\nabla(X)(u \cdot s \cdot v) = [X, u]_M \cdot s \cdot v + \nabla(X)(s) \cdot u + u \cdot s \cdot [X, v]_M , \quad X \in M, \quad s \in L, \quad u \in A.
\]

(6)

We call a left, right or bilateral \( M \)-connection as a **linear \( M \)-connection**. We call as Koszul conditions the above conditions on \( \nabla \).
It is easy to see that if $\nabla^1$ and $\nabla^2$ are two linear $M$-connections on $L$, then $D = \nabla^1 - \nabla^2 : M \to \text{End}_AL$, for a left $M$-connection, $D = \nabla^1 - \nabla^2 : M \to \text{End}_AL$ for a right $M$-connection and $D = \nabla^1 - \nabla^2 : M \to \text{End}_AL$ for a bilateral $M$-connection. (The positions of the algebra denotes the kind of the module: left, right or bilateral) Conversely, given $L : M \to \text{End}_AL$ and $\nabla^1$ a linear $M$-connection on $L$, then $\nabla^2 = \nabla^1 + D$ is a linear left $M$-connection on $L$. Analogous statements are valid for left and bilateral case.

Consider now a preinfinitesimal module $(A, M)$. The map

$$D : M \times M \to \text{Der}A, \quad D(X, Y) = [p^MX, p^MY]_{\text{Der}A} - p^M[X, Y]_M$$

belongs to $\text{Hom}^2(M, \text{Der}(A))$, where $(A, \text{Der}A)$ is the Lie pseudoalgebra of the derivations on $A$. $D$ is an anchor for an module with arrow on $(A, M \times M)$ which is trivial iff $(A, M)$ is an infinitesimal module. In this particular case, if $\nabla : M \to \text{End}_AL$ is a linear left (right, bilateral) $M$-connection on the module $(A, L)$, then, denoting as

$$R(X, Y) = [\nabla X, \nabla Y]_{\text{End}_AL} - \nabla_{[X,Y]}_M,$$

we have $R(X, Y) \in \text{End}_AL (R(X, Y) \in \text{End}_AL$ and $R(X, Y) \in \text{End}_AL$ respectively).

If $(A, M)$ is a preinfinitesimal module, $\nabla$ is a left (right, bilateral) $M$-connection on $L$ and we define $R$ using the formula (8), then $R$ has the property:

$$R(X, Y)(u \cdot s) = [D(X, Y), u]_{M \times M} \cdot s + u \cdot R(X, Y)s, 
(\forall)(X, Y) \in M \times M, s \in L, u \in A,$$

thus $R$ is a linear left (right, bilateral) $M \times M$-connection on $L$.

Let $(A, L)$ be a module with arrow and $\nabla$ a linear $L$-connection on $L$. The formula

$$[X, Y]_L = \nabla_X Y - \nabla_Y X, \quad (\forall)X, Y \in L$$

defines a bracket on $L$, which makes $(A, L)$ a preinfinitesimal module.

**Definition 2.2** For a linear $L$-connection $\nabla$ on the preinfinitesimal module $(A, L)$, we say that

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_L, \quad (\forall)X, Y \in L$$

is the torsion of $\nabla$.

Notice that $T \in \text{Hom}^2(L, L)$ (left, right or bilateral, according to $L$) and the relation (9) holds true iff $T = 0$.

As remarked above, a linear $L$-connection on the module with arrow $(A, L)$ defines a bracket on $L$. In order to make an inverse construction, a supplementary structure is given usually on $L$. For example, the following construction is an extension of the Levi Civita connection on a (pseudo-)Riemannian manifold.

**Definition 2.3** We call a pseudo-Riemannian metric on the module $(A, L)$ an $A$-bilinear, symmetric and non-degenerate map $g : L \times L \to A$ (i.e. $(\forall)X \in L, g(X, Y) = 0, (\forall)Y \in L$ then $X = 0$).
Moreover, if $g$ is strict (i.e. $(\forall)X \in L$, $g(X, X) = 0$ implies $X = 0$), then we say that $g$ is a Riemannian metric.

If $(A, L_1)$ is a module with arrow and $\nabla$ is a linear $L_1$-connection on $L$, then we say that $\nabla$ is a metric connection if the following relation holds true:

$$[X, g(Y, Z)]_L = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), (\forall)X \in L_1, Y, Z \in L.$$

It is easy to see that a (pseudo-)Riemannian metric $g$ induces an injective morphism of $A$-module $\gamma \in \text{Hom}_A^1(L, L^*)$, where $L^*$ is the dual of $L$ related to $A$.

**Proposition 2.1** Let $(A, L)$ be a preinfinitesimal module, $g$ be a (pseudo) Riemannian metric on $L$ and suppose that $\gamma$ is isomorphism.

Then there is a unique linear $L$-connection on $L$ which is metric and has a vanishing torsion.

**Proof.** As the classical Levi Civita connection, the formula:

$$g(\nabla_X Y, Z) = [X, g(Y, Z)]_L + [Y, g(Z, X)]_L - [Z, g(X, Y)]_L + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X).$$

gives uniquely $\nabla$. □

Notice that some other classical constructions in the differential geometry can be generalized in an analogous manner.

Let $\psi: M_1 \rightarrow M_2$ be an $A$-module with arrow morphism and $\nabla^2$ be a linear $M_2$-connection on the module $(A, L)$. Then $\nabla_X^1 = \nabla^2_{\psi(X)}$ is a linear $M_1$-connection on $L$.

Notice that if a linear $\text{Der} A$-connection exists on the module $(A, L)$, then, for every module with arrow $(A, M)$, there is a linear $M$-connection on $L$. This observation can be extended as follows:

**Proposition 2.2** Let $(A', L') \xrightarrow{\varphi, \psi} (A, L)$ be a morphism of module with arrow, $(A, L_1)$ be a module and $\nabla : L \times L_1 \rightarrow L_1$ be a linear $L$-connection on $L_1$.

Then there is a linear $L'$-connection $\nabla$ on the module $(A', A' \otimes_A L_1)$.

**Proof.** We make the proof only for left connections. The cases of right and bilateral connections are analogous.

For every $X' \in L'$ such that

$$\psi(X') = \sum_i a'_i \otimes_A X_i \in A' \otimes_A L$$

and every $\sum_{\alpha} v'_\alpha \otimes_A z_\alpha \in A' \otimes_A L_1$, we define

$$\nabla_{X'}(\sum_{\alpha} v'_\alpha \otimes_A Z_\alpha) = \sum_{i, \alpha} (v'_{\alpha a_i}) \otimes_A \nabla_{X_i} Z_\alpha + \sum_{\alpha} [X', v'_\alpha]_{L'} \otimes_A Z_\alpha.$$

It is routine to prove that $\nabla$ does not depend on the tensor decompositions and to check the Koszul conditions (1) and (2). □
We say that the linear $L'$-connection $\nabla$ given by the Proposition above is $(\varphi, \psi)$-associated with $\nabla$.

Consider now a morphism of $A$-module with arrow $f : L \to L_1$, where $(A, L)$ is a preinfinitesimal module and $\nabla$ is a linear $L_1$-connection on $L$. If we define

$$T\nabla(X, Y) = \nabla f(X) Y - \nabla f(Y) X - [X, Y]_L \ , (\forall) X, Y \in L ,$$

then $T\nabla \in \text{Hom}_2^A(L, L)$ is called the $f$-torsion of $\nabla$, according to [3]. Taking $L = L_1$ and $f = \text{id}_L$ then $T\nabla$ is precisely the torsion of $\nabla$. Generally, $T\nabla$ is in fact the torsion of the linear $L$-connection $\nabla X Y = \nabla f(X) Y$, $(X, Y) \in L$.

Consider, moreover, a morphism of module $(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$ and denote

$$T\nabla \left( \sum_i u'_i \otimes_A X_i, \sum_i u'_i \otimes_A X_i \right) = \sum_{i,j} (u'_i u'_j) \otimes_A T\nabla(X_i, Y_j),$$

$$\forall \sum_i (u_i \otimes_A X_i), \sum_j (v_j \otimes_A Y_j) \in A' \otimes_A L.$$

It is easy to see that $T\nabla \in \text{Hom}_2^A(A' \otimes_A L, A' \otimes_A L)$ and the definition does not depend on the tensor decompositions.

The following result is an extension of [3, Proposition 1.14] in the case of preinfinitesimal modules. It is a characterization of the morphism of preinfinitesimal module without using explicitly the tensor decompositions. Actually it is good only for preinfinitesimal modules that admit linear connections.

**Proposition 2.3** Let $\nabla$ be a linear $L$-connection on the preinfinitesimal module $(A, L)$, $(A', L')$ be a preinfinitesimal module and $(A', L') \xrightarrow{(\varphi, \psi)} (A, L)$ be a morphism of module with arrow. Denote:

$$\tilde{\psi}(X', Y') = \psi([X', Y']_{L'}) - \nabla X' \psi(Y') + \nabla Y' \psi(X') + T\nabla(\psi(X'), \psi(Y')) ,$$

$(\forall) X', Y' \in L'$, where $\nabla$ is the $L'$-connection $(\varphi, \psi)$-associated with $\nabla$, given by Proposition 2.2. Then we have:

1. $\tilde{\psi} \in a_2(L, A' \otimes_A L)$;
2. The following assertions are equivalent:
   (a) $\psi$ is a morphism of preinfinitesimal module;
   (b) $\tilde{\psi} = 0$.

**Proof.** It suffices to prove that $\tilde{\psi} = K$, where $K$ is given by

$$K : L' \times L' \to A' \otimes_A L, \quad K(X', Y') = \psi([X', Y']_{L'}) - \chi(X', Y') , (\forall) X', Y' \in L'.$$
We make the proof only for left connections. The cases of right and bilateral connections are analogous. Indeed, we have:

\[ \tilde{\psi}(X', Y') = \psi([X', Y']_L) - \sum a'_i b'_j \otimes A \nabla X, Y_j - \sum [X', b'_j]_L \otimes A Y_j + \]
\[ \sum a'_i b'_j \otimes A \nabla X_i, Y_j + \sum [Y', a'_i]_L \otimes A X_i + \]
\[ \sum a'_i b'_j \otimes A (\nabla X_i Y_j - \nabla Y_j X_i - L(X_i, Y_j)) = K(X', Y'). \]

As in [3], using a linear \( L \)-connection \( \nabla \), the formula which gives the bracket of the pull-back of modules with differentials, defined in [8], can be easily written as:

\[ [X' \oplus C, Y' \oplus D]_{L^*} = [X', Y']_L \oplus \left( \nabla X'_i D - \nabla Y'_i C - T\nabla (C, D) \right) \]

\( (\forall) X' \oplus C, Y' \oplus D \in L^*. \)

References


Linear connections on modules with differentials


Authors’ address:

Marcela Popescu and Paul Popescu

*Department of Mathematics,*

*University of Craiova,*

*11, Al.I.Cuza St., Craiova, 1100, Romania.*

*e-mail paul@udjmath2.sfos.ro*