LINEAR CONNECTIONS ON MODULES WITH DIFFERENTIALS

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Abstract

The aim of the paper is to give an abstract definition of a linear connection on modules with differentials over associative algebras and to study its properties.

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1 Introduction

This paper continues the ideas from some previous works [8, 9] and we keep all the definitions and the notations used there.

Consider an associative algebra A over a field k and (A, M) a module, which can be a left, right or bi-module We denote as $Z(A)$ the center of A.

For two left modules (A', M') and (A, M), a *contravariant morphism of left module* is a couple (φ, ψ), where $\varphi : A \to A'$ is a morphism of algebras such that $\varphi(Z(A)) \subset Z(A')$ and $\psi : M' \to A' \otimes_{Z(A)} M$ is a morphism of left A'-module. We say that $\psi(m')$ is the ψ-decomposition of m'.

Contravariant morphisms of right module and *bi-module* are defined in an analogous way. According to [9, Theorem 1], the left modules (right modules, respectively bimodules) (A, M) with A an object from $\mathcal{A}$ and the contravariant morphisms of the corresponding module are the objects and the morphisms of a category $\mathcal{M}^l_A$ ($\mathcal{M}^r_A$, respectively $\mathcal{M}^b_A$).

A left module (A, M) is a *left module with arrow* (l.m.w.a.) if a morphism of left module $p^M : M \to Der(A)$ is given, called an *anchor*. We denote $p^M(m)(a) = [m, a]_M$ for every $m \in M$ and $a \in A$. In an analogous way, the right module with arrow (r.m.w.a.), respectively bimodule with arrow (b.w.a.) are defined.

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Let (A', M') and (A, M) be l.m.w.a.'s. A contravariant morphism of left module (φ, ψ) is a *morphism of l.m.w.a.* if it is a contravariant morphism of left module and for every $X' \in M'$ which has the ψ-decomposition

$$\psi(X') = \sum_i a'_i \otimes_{Z(A)} X_i,$$

and $a \in A$, the condition $[X', \varphi(a)] = \sum_i a'_i \varphi([X_i, a])$ is fulfilled.

The morphism of r.m.w.a. and of b.m.w.a. can be defined in an analogous way.

A *preinfinitesimal left module* (p.l.m.) is a l.m.w.a. (A, M) together with a bracket $[\cdot,\cdot]_M : M \times M \to M$ which is k-bilinear, antisymmetric and

$$[X, aY]_M = [X, a]_M Y + a [X, Y]_M \quad , \quad (\forall) X, Y \in M,\ a \in A.$$

Let (A', M') and (A, M) be p.l.m.'s. A contravariant morphism of l.m.w.a. (φ, ψ) is a *morphism of p.l.m.* if it is a contravariant morphism of left module and for every $x', y' \in M'$ which have the ψ-decompositions

$$\psi(X') = \sum_i a'_i \otimes_{Z(A)} X_i \quad , \quad \psi(Y') = \sum_\alpha b'_\alpha \otimes_{Z(A)} Y_\alpha,$$

then the following condition is fulfilled:

$$\psi([X', Y']_M) = \sum_\alpha [X', b'_\alpha]_M \otimes_{Z(A)} Y_\alpha - \sum_i [Y', a'_i]_M \otimes_{Z(A)} X_i + \sum_{i,\alpha} a'_i b'_\alpha \otimes_{Z(A)} [X_i, Y_\alpha]_M .$$

In an analogous way the morphism of p.l.m. and of p.b.m. can be defined. The definitions are correct; it can be checked up as in [6, Lemmas 4.1, 4.2].

Let (A', M') and (A, M) be p.l.m.'s and $(A', M') \overset{(\varphi,\psi)}{\longrightarrow} (A, M)$ a morphism of l.m.w.a.. The *curvature* of (φ, ψ) is the map $K : M' \times M' \to A' \otimes_{Z(A)} M$ defined by

$$K(X', Y') = \psi([X', Y']_M) - \sum_\alpha [X', b'_\alpha]_M \otimes_{Z(A)} Y_\alpha + \sum_i [Y', a'_i]_M \otimes_{Z(A)} X_i - \sum_{i,\alpha} a'_i b'_\alpha \otimes_{Z(A)} [X_i, Y_\alpha]_M .$$

It is clear that (φ, ψ) is a morphism of preinfinitesimal module iff K vanish. The curvature of a morphism of r.m.w.a. (or b.m.w.a.) of two p.r.m.'s (respectively p.b.m.'s) can be defined in a similar way. It vanishes iff it is a morphism of p.r.m. (p.b.m. respectively). In the case of commutative algebras the definition agrees with [6, Proposition 4.1].

In the case of $A = A'$, a morphism of left A-module $\psi_0 : M' \to M$ induces a morphism $\psi : M' \to A \otimes_{Z(A)} M$, $\psi(X') = 1_A \otimes_{Z(A)} \psi_0(X')$. If (A, M') and (A, M) are l.m.w.a.'s and $[X', \varphi(a)] = \varphi([\psi_0(X'), a])$, then we say that ψ is a strong morphism of A-l.m.w.a.. It induces a morphism of l.m.w.a. as above. The curvature of a strong morphism of A-l.m.w.a. as above is $K_0(X', Y') = [\psi(X'), \psi(Y')]_M -$

$\psi([X', Y']_{M'})$. An *infinitesimal left module* (i.l.m.) is a p.l.m. (A, M), such that the anchor $p^M : M \to Der(A)$ is a morphism of p.i.m. with a vanishing curvature (i.e. $p^M([X,Y]_M) - \left[p^M(X), p^M(Y)\right]_{Der(A)} = 0$). A *left Lie pseudoalgebra* (l.L.p.a.) is an i.l.m. which has the property $\mathcal{J}(X,Y,Z) \stackrel{def}{=} [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$. $\mathcal{J}$ is called the Jacobi map.

In an analogous way we can define the infinitesimal right module (i.r.m.), the infinitesimal bimodule (i.b.m.), the right Lie pseudoalgebra (r.L.p.a.) and the Lie bi-pseudoalgebra (L.b.p.a.).

All these modules, together with their morphisms, are the objects and the morphisms of some categories, called as in [6] *categories of modules with differentials*. Almost all the results from [6], stated for the associative and commutative algebras, can be extended with care for associative algebras.

2 Linear connections on modules with differentials

The linear connection defined here differs from that defined in [10] or [3], being closed to the linear connection defined in the classical differential geometry by the Koszul conditions.

Definition 2.1 Let A be a associative k-algebra, (A, L) a left module and (A, M) a module with arrow.

A *linear left M-connection* on L is an A-module morphism

$$\nabla : M \to End_k L, \tag{1}$$

denoted as $\nabla(X)(s) = \nabla_X s$, such that:

$$\nabla(X)(u \cdot s) = [X,u]_M \cdot s + u \cdot \nabla(X)(s)\ ,\ X \in M,\ s \in L,\ u \in A. \tag{2}$$

A *linear right M-connection* on L is an A-module morphism

$$\nabla : M \to End_k L \tag{3}$$

such that:

$$\nabla(X)(s \cdot u) = s \cdot [X,u]_M + \nabla(X)(s) \cdot u\ ,\ X \in M,\ s \in L,\ u \in A. \tag{4}$$

A *linear bilateral M-connection* on L is an A-module morphism

$$\nabla : M \to End_k L \tag{5}$$

such that:

$$\nabla(X)(u \cdot s \cdot v) = [X,u]_M \cdot s \cdot v + \nabla(X)(s) \cdot u + u \cdot s \cdot [X,v]_M\ , \tag{6}$$

$$X \in M,\ s \in L,\ u \in A. \tag{7}$$

We call a left, right or bilateral M-connection as a *linear M-connection*. We call as *Koszul conditions* the above conditions on ∇.

It is easy to see that if ∇^1 and ∇^2 are two linear M-connections on L , then $D = \nabla^1 - \nabla^2 : M \to End_A L$, for a left M-connection, $D = \nabla^1 - \nabla^2 : M \to End\, L_A$ for a right M-connection and $D = \nabla^1 - \nabla^2 : M \to End\,{}_A L_A$ for a bilateral M-connection.(The positions of the algebra denotes the kind of the module: left, right or bilateral) Conversely, given $L : M \to End_A L$ and ∇^1 a linear M-connection on L, then $\nabla^2 = \nabla^1 + D$ is a linear left M-connection on L. Analogous statements are valid for left and bilateral case.

Consider now a preinfinitesimal module (A, M). The map

$$\mathcal{D} : M \times M \to Der\ A\ ,\ \mathcal{D}(X, Y) = [p^M X, p^M Y]_{Der:A} - p^M [X, Y]_M$$

belongs to $Hom^2_A(M, Der(A))$, where $(A, Der\ A)$ is the Lie pseudoalgebra of the derivations on A. $\mathcal{D}$ is an anchor for an module with arrow on $(A, M \times M)$ which is trivial iff (A, M) is an infinitesimal module. In this particular case, if $\nabla : M \to End_k L$ is a linear left (right, bilateral) M-connection on the module (A, L), then, denoting as

$$R(X, Y) = [\nabla_X, \nabla_Y]_{End_k L} - \nabla_{[X,Y]_M}\ , \tag{8}$$

we have $R(X, Y) \in End_A L$ ($R(X, Y) \in End\, L_A$ and $R(X, Y) \in End_A L_A$ respectively).

If (A, M) is a preinfinitesimal module, ∇ is a left (right, bilateral) M-connection on L and we define R using the formula (8), then R has the property:

$$R(X, Y)(u \cdot s) = [\mathcal{D}(X, Y), u]_{M \times M} \cdot s + u \cdot R(X, Y)s,$$
$$(\forall)(X, Y) \in M \times M,\ s \in L,\ u \in A,$$

thus R is a linear left (right, bilateral) $M \times M$-connection on L .

Let (A, L) be a module with arrow and ∇ a linear L-connection on L. The formula

$$[X, Y]_L = \nabla_X Y - \nabla_Y X\ ,\ (\forall)X, Y \in L \tag{9}$$

defines a bracket on L, which makes (A, L) a preinfinitesimal module.

Definition 2.2 *For a linear L-connection ∇ on the preinfinitesimal module (A, L), we say that*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_L\ ,\ (\forall)X, Y \in L$$

is the torsion *of* ∇.

Notice that $T \in Hom^2_A(L, L)$ (left, right or bilateral, according to L) and the relation (9) holds true iff $T = 0$.

As remarked above, a linear L-connection on the module with arrow (A, L) defines a bracket on L. In order to make an inverse construction, a supplementary structure is given usually on L. For example, the following construction is an extension of the Levi Civita connection on a (pseudo-)Riemannian manifold.

Definition 2.3 *We call a* pseudo-Riemannian metric *on the module (A, L) an A-bilinear, symmetric and non-degenerate map $g : L \times L \to A$ (i.e. $(\forall)X \in L$, $g(X, Y) = 0$, $(\forall)Y \in L$ then $X = 0$).*

Moreover, if g is strict (i.e. $(\forall)X \in L,\ g(X,X)=0$ implies $X=0$), then we say that g is a Riemannian *metric.*

If (A,L_1) is a module with arrow and ∇ is a linear L_1-connection on L, then we say that ∇ is a metric connection if the following relation holds true:

$$[X,g(Y,Z)]_L = g(\nabla_X Y,Z)+g(Y,\nabla_X Z)\ ,\ (\forall)X\in L_1,\ Y,Z\in L.$$

It is easy to see that a (pseudo-)Riemannian metric g induces an injective morphism of A-module $\gamma \in Hom_A^1(L,L^*)$, where L^* is the dual of L related to A.

Proposition 2.1 *Let (A,L) be a preinfinitesimal module, g be a (pseudo) Riemannian metric on L and suppose that γ is isomorphism.*

Then there is a unique linear L-connection on L which is metric and has a vanishing torsion.

Proof. As the classical Levi Civita connection, the formula:

$$g(\nabla_X Y,Z)=[X,g(Y,Z)]_L+[Y,g(Z,X)]_L-[Z,g(X,Y)]_L+$$
$$g([X,Y],Z)+g([Z,X],Y)-g([Y,Z],X)\ .$$

gives uniquely ∇. □

Notice that some other classical constructions in the differential geometry can be generalized in an analogous manner.

Let $\psi : M_1 \to M_2$ be an A-module with arrow morphism and ∇^2 be a linear M_2-connection on the module (A,L). Then $\nabla^1_X=\nabla^2_{\psi(X)}$ is a linear M_1-connection on L .

Notice that if a linear $Der\ A$-connection exists on the module (A,L), then, for every module with arrow (A,M), there is a linear M-connection on L. This observation can be extended as follows:

Proposition 2.2 *Let $(A',L') \stackrel{(\varphi,\psi)}{\longrightarrow} (A,L)$ be a morphism of module with arrow, (A,L_1) be a module and $\nabla : L\times L_1 \to L_1$ be a linear L-connection on L_1.*

Then there is a linear L'-connection $\overline{\nabla}$ on the module $(A',A'\otimes_A L_1)$.

Proof. We make the proof only for left connections. The cases of right and bilateral connections are analogous.

For every $X'\in L'$ such that

$$\psi(X')=\sum_i a'_i \otimes_A X_i \in A'\otimes_A L$$

and every $\sum_\alpha v'_\alpha \otimes_A z_\alpha \in A'\otimes_A L_1$, we define

$$\overline{\nabla}_{X'}(\sum_\alpha v'_\alpha\otimes_A Z_\alpha)=\sum_{i,\alpha}(v'_{\alpha a_i})\otimes_A \nabla_{X_i}Z_\alpha+\sum_\alpha [X',v'_\alpha]_{L'}\otimes_A Z_\alpha.$$

It is routine to prove that $\overline{\nabla}$ does not depend on the tensor decompositions and to check the Koszul conditions (1) and (2). □

We say that the linear L'-connection $\overline{\nabla}$ given by the *Proposition* above is (φ, ψ)-*associated* with ∇ .

Consider now a morphism of $A-$module with arrow $f : L \to L_1$, where (A, L) is a preinfinitesimal module and ∇ is a linear L_1-connection on L. If we define

$$T_\nabla(X, Y) = \nabla_{f(X)}Y - \nabla_{f(Y)}X - [X, Y]_L \ \ , (\forall)\, X, Y \in L \ ,$$

then $T_\nabla \in Hom^2_A(L, L)$ is called the f-*torsion* of ∇, according to [3]. Taking $L = L_1$ and $f = id_L$ then T_∇ is precisely the torsion of ∇. Generally, T_∇ is in fact the torsion of the linear L-connection $\widetilde{\nabla}_X Y = \nabla_{f(X)}Y$, $(X, Y \in L)$, on L.

Consider, moreover, a morphism of module $(A', L') \overset{(\varphi,\psi)}{\to} (A, L)$ and denote

$$\overline{T}_\nabla \left(\sum_i u'_i \otimes_A X_i \ , \sum_i u'_i \otimes_A X_i \right) = \sum_{i,j} (u'_i v'_j) \otimes_A T_\nabla(X_i, Y_j), \tag{10}$$

$$(\forall) \sum_i (u_i \otimes_A X_i), \sum_j (v_j \otimes_A Y_j) \in A' \otimes_A L.$$

It is easy to see that $\overline{T}_\nabla \in Hom^2_{A'}(A' \otimes_A L, A' \otimes_A L\ )$ and the definition does not depend on the tensor decompositions.

The following result is an extension of [3, Proposition 1.14] in the case of pre-infinitesimal modules. It is a characterization of the morphism of preinfinitesimal module without using explicitly the tensor decompositions. Actually it is good only for preinfinitesimal modules that admit linear connections.

Proposition 2.3 *Let* ∇ *be a linear* L*-connection on the preinfinitesimal module* (A, L)*,* (A', L') *be a preinfinitesimal module and* $(A', L') \overset{(\varphi,\psi)}{\to} (A, L)$ *be a morphism of module with arrow. Denote:*

$$\widetilde{\psi}(X', Y') = \psi\left([X', Y']_{L'}\right) - \overline{\nabla}_{X'}(\psi(Y')) + \overline{\nabla}_{Y'}(\psi(X')) + \overline{T}_\nabla\left(\psi(X'), \psi(Y')\right),$$

$(\forall)X', Y' \in L'$ *, where* $\overline{\nabla}$ *is the* L'*-connection* (φ, ψ)*-associated with* ∇*, given by* Proposition 2.2. *Then we have:*

1. $\widetilde{\psi} \in a_2\left(L, A' \otimes_A L\right)$ *;*
2. *The following assertions are equivalent:*
   (a) ψ *is a morphism of preinfinitesimal module;*
   (b) $\widetilde{\psi} = 0$.

Proof. It suffices to prove that $\widetilde{\psi} = K$, where K is given by

$$K : L' \times L' \to A' \otimes_A L,$$

$$K(X', Y') = \psi([X', Y']_{L'}) - \chi(X', Y')\ ,\ (\forall)X', Y' \in L'.$$

We make the proof only for left connections. The cases of right and bilateral connections are analogous. Indeed, we have:

$$\widetilde{\psi}(X',Y') = \psi\left([X',Y']_{L'}\right) - \sum_{i,j} a'_i b'_j \otimes_A \nabla_{X_i} Y_j - \sum_j \left[X', b'_j\right]_{L'} \otimes_A Y_j +$$
$$\sum_{i,j} a'_i b'_j \otimes_A \nabla_{Y_j} X_i + \sum_i [Y', a'_i]_{L'} \otimes_A X_i +$$
$$\sum_{i,j} a'_i b'_j \otimes_A \left(\nabla_{X_i} Y_j - \nabla_{Y_j} X_i - L(X_i, Y_j)\right) = K(X',Y').\square$$

As in [3], using a linear L-connection $\overline{\nabla}$, the formula which gives the bracket of the pull-back of modules with:differentials, defined in [8], can be easily written as:

$$[X' \oplus C, Y' \oplus D]_{L^*} = [X',Y']_{L'} \oplus \left(\overline{\nabla}_{X'} D - \overline{\nabla}_{Y'} C - \overline{T}_{\nabla}(C,D)\right)$$

$$(\forall)\, X' \oplus C, Y' \oplus D \in L^*.$$

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