NOTES ON SUBGEODESIC AND GEODESIC MAPPINGS OF RIEMANNIAN MANIFOLDS

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Abstract

The present paper deals with subgeodesically and geodesically related Riemannian spaces. In the third section we study a subprojective transformation. In the last section, using properties of conharmonic, concircular transformations and tensors satisfying some pseudo-symmetry conditions we study subgeodesically related manifolds.

Key words: subgeodesic mappings, pseudo-symmetric spaces, conharmonic transformations, concircular transformations.

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1 Introduction

Let $V_n = (M, g)$ be a Riemannian manifold. It is said to be pseudo-symmetric ([6]) if

(1.1) at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

It is clear that any semi-symmetric manifold $R \cdot R = 0$ ([18]) is pseudo-symmetric. These notions arose during the study of totally umbilical subma-nifolds of semi-symmetric manifolds ([1]) as well as during the consideration of geodesic mappings ([3], [7], [8]).

In this paper we shall continue the study in this direction, considering subgeodesic mappings ([13], [16]) which are a generalization of the geodesic ma-ppings. We determine some properties of ξ^i-subgeodesically related spaces, the tensor of correspondence being $-g$, with conharmonically semi-symmetric spaces or pseudo-symmetric spaces. We study also the Weyl conformal curvature tensor produced by some subprojective transformations.

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2 Preliminaries

Let (M,g) be a Riemannian manifold, covered by a system of coordinates neibourhoods. We denote by g_{ij}, R_{ijkl}, R_{ij} and K the components of the Riemannian metric, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. We define the tensors $R(1), R\cdot R$ and $Q(g,R)$ by the formulae

$$R(1)_{ijkl} = \frac{K}{n(n-1)}(g_{ik}g_{jh} - g_{ij}g_{kh}), \tag{2.1}$$

$$(R\cdot R)_{hijklm} = -R^s_{hlm}R_{sijk} - R^s_{ilm}R_{hsjk} - R^s_{jlm}R_{hisl} - R^s_{klm}R_{hijs}, \tag{2.2}$$

$$\begin{aligned} Q(g,R)_{hijkl} = g_{mh}R_{lijk} - g_{hl}R_{mijk} + g_{mi}R_{hljk} - \\ - g_{il}R_{hmjk} + g_{jm}R_{lilk} - g_{jl}R_{himk} + g_{km}R_{hijl} - g_{kl}R_{hijm}, \end{aligned} \tag{2.3}$$

where $R^s_{hlm} = g^{sr}R_{rhlm}$.

The condition (1.1) holds on $U = \{x \in M \mid R \neq R(1) \text{ at x }\}$ if and only if

$$R\cdot R = LQ(g,R) \tag{*}$$

is satisfied on U, where L is a function.

We can also define the tensors $R\cdot g$ and $Q(g,B)$, B being a symmetric tensor field of type $(0,2)$.

$$(R\cdot g)_{jkri} = g_{jk,ir} - g_{jk,ri}, \tag{2.4}$$

$$Q(g,B)_{jkri} = g_{ij}B_{kr} - g_{rj}B_{ki} - g_{ki}B_{jr} - g_{rk}B_{ji}, \tag{2.5}$$

where the comma denotes covariant differentiation with respect to the metric g.

A $(0,4)$-tensor field T on M is said to be a generalized curvature tensor if

$$T_{ijkl} + T_{iklj} + T_{iljk} = 0, \tag{2.6}$$

$$T_{ijkl} = -T_{jikl}, \tag{2.7}$$

$$T_{ijkl} = T_{klij}. \tag{2.8}$$

Further, we define the tensor $T\cdot T$ analogous with $R\cdot R$ and the Weyl conformal curvature tensor $W(T)$ associated to T by the formula ([5])

$$\begin{aligned} (W(T))_{ijkl} = T_{ijlk} - \frac{1}{n-2}(g_{ik}T_{jl} - g_{il}T_{jk} - g_{jk}T_{il} - \\ - g_{jl}T_{ik}) + \frac{K(T)}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{kj}), \end{aligned} \tag{2.9}$$

where $T^s_{hlm} = g^{rs}T_{rhlm}, T_{il} = T^r_{irl}$ and $K(T)$ is the scalar curvature associated to T.

The conharmonic curvature tensor on $V_n = (M, g)$ is defined by ([11])

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}). \tag{2.10}$$

A manifold is said to be conharmonically semi-symmetric if $R \cdot C = 0$, the tensor $R \cdot C$ being defined by analogy with (2.2), replacing the Riemann tensor with the conharmonic tensor C.

Let A_n be a space with an affine, symmetric connection Γ^i_{jk} and ξ^i be the components of a vector field on A_n. We can associate the differential system of equations

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = a\frac{dx^i}{dt} + b\xi^i,$$

a and b being functions of t, which defines the ξ^i-subgeodesics.

Let $A_n(\Gamma^i_{jk})$ and $\overline{A}_n(\overline{\Gamma}^i_{jk})$ be two spaces. K.Yano introduced ([23]) the subprojective transformation of connections, which preserves the ξ^i-subgeodesics

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j + \phi_{jk}\xi^i, \tag{2.11}$$

where ψ_i and ϕ_{jk} are the components of a 1-form and of a symmetric tensor field of type $(0,2)$, respectively.

We say that two Riemannian spaces $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g})$ are ξ^i-subgeodesically related ([13], [16]), the tensor of correspondence being $-g_{ij}$, if we have Yano formulae

$$\overline{\left| \begin{matrix} i \\ jk \end{matrix} \right|} = \left| \begin{matrix} i \\ jk \end{matrix} \right| + \delta^i_j\psi_k + \delta^i_k\psi_j - g_{kj}\xi^i, \tag{2.12}$$

where $\overline{\left| \begin{matrix} i \\ jk \end{matrix} \right|}, \left| \begin{matrix} i \\ jk \end{matrix} \right|$ are the Christoffel symbols for $\overline{V}_n$ and V_n, respectively. ψ_i and ξ^i are the components of a 1-form and of a vector field, respectively.

This is equivalent with the existence of a diffeomorphism f between these two spaces which maps ξ^i-subgeodesics onto ξ^i-subgeodesics. f is called the subgeodesic mapping.

From (2.12) we have

$$(n+1)\psi_k - \xi_k = \frac{\partial}{\partial x^k}\ln\sqrt{\left|\frac{det(\overline{g}_{ij})}{det(g_{ij})}\right|},$$

where $\xi_k = g_{ks}\xi^s$. There exist the functions $\psi(x^1, \ldots, x^n), \xi(x^1, \ldots, x^n)$ ([13]) such that $\psi_k = \dfrac{\partial\psi}{\partial x^k}$ and $\xi_k = \dfrac{\partial\xi}{\partial x^k}$.

Spaces V_n and $\overline{V}_n$ are non-trivial ξ^i-subgeodesically related, the tensor of correspondence being $-g_{ij}$, if the components $\psi_i - \xi_i$ are non-zero.

If $\xi^i = 0$ then the Yano formulae become the Weyl formulae and V_n and $\overline{V}_n$ are geodesically related spaces ([2], [15], [21]).

3 Subprojective transformations

On a Riemannian manifold $V_n = (M, g)$ we consider the linear connection

$$\overline{\Gamma}^i_{jk} = \left| \begin{matrix} i \\ j\ k \end{matrix} \right| + \delta^i_j \Psi_k + \delta^i_k \Psi_j - g_{kj}\xi^i. \tag{3.1}$$

Using (3.1) we have

$$\begin{aligned} T_{ijkl} = R_{ijkl} - g_{ij}(A_{kl} - A_{lk}) + g_{il}A_{jk} - g_{ik}A_{jl} + \\ + g_{jk}B_{il} - g_{jl}B_{ik} - (g_{ik}g_{jl} - g_{il}g_{jk})\Psi_s\xi^s, \end{aligned} \tag{3.2}$$

where $T_{ijkl} = g_{is}\overline{R}^s_{jkl}, A_{ij} = \Psi_{i,j} - \Psi_i\Psi_j$, $B_{ij} = \xi_{i,j} - \xi_i\xi_j$
and the comma denotes covariant differentiation with respect to g.

Theorem 3.1. *T, defined by (3.2), is a generalized curvature tensor if and only if the relations*

$$B_{ij} = A_{ij} - \frac{1}{n} g_{ij}[T_r(A) - T_r(B)], A_{ij} = A_{ji}, \quad n \geq 3, \tag{3.3}$$

hold .

Proof. The condition (2.6) is identically satisfied. The relation (2.7) is equivalent to

$$\begin{aligned} g_{il}(A_{jk} - B_{jk}) + g_{jl}(A_{ik} - B_{ik}) - g_{jk}(A_{il} - B_{il}) - \\ - g_{ik}(A_{jl} - B_{jl}) = 2g_{ij}(A_{kl} - A_{lk}). \end{aligned} \tag{3.4}$$

Transvecting with g^{ij} and summing, (3.4) leads to

$$B_{kl} - B_{lk} = (n+1)(A_{kl} - A_{lk}). \tag{3.5}$$

The relation (2.8) is equivalent with

$$\begin{aligned} g_{ij}(A_{kl} - A_{lk}) + g_{ik}(A_{jl} - A_{lj}) + g_{jl}(B_{ik} - B_{ki}) - \\ - g_{il}(A_{jk} - B_{kj}) + g_{jk}(A_{li} - B_{il}) = g_{kl}(A_{ij} - A_{ji}). \end{aligned} \tag{3.6}$$

Transvecting with g^{ik} in (3.6), we have

$$B_{jl} = \frac{n+2}{n} A_{jl} - \frac{2}{n} A_{lj} - \frac{1}{n} g_{jl}[T_r(A) - T_r(B)]. \tag{3.7}$$

Using (3.6) and (3.7), we obtain (3.2), for $n \geq 3$.

If the conditions (3.2) are satisfied, then T verifies the relations (2.6), (2.7) and (2.8).

Theorem 3.2. *If T, defined by (3.2), is a generalized curvature tensor on $V_n = (M, g)$, $n \geq 3$, then the Weyl curvature tensor of V_n coincides with the Weyl curvature tensor associated with T.*

Proof. Substituting the relation (3.3) in (3.2) we get

$$\begin{aligned} T_{ijkl} = {} & R_{ijkl} + g_{il}A_{jk} - g_{ik}A_{jl} + g_{jk}A_{il} - g_{jl}A_{ik} + \\ & +(g_{ik}g_{jl} - g_{il}g_{jk})\left[\frac{1}{n}(T_r(A) - T_r(B)) - \Psi_s\xi^s\right]. \end{aligned} \tag{3.8}$$

In view of (3.8), we obtain

$$\begin{aligned} T_{jl} = {} & R_{jl} - (n-2)A_{jl} - g_{jl}T_r(A) + \\ & +(n-1)g_{jl}\left[\frac{1}{n}(T_r(A) - T_r(B)) - \Psi_s\xi^s\right]. \end{aligned} \tag{3.9}$$

Using (3.9) we have

$$\begin{aligned} K(T) = {} & K - (n-2)T_r(A) - nT_r(A) + \\ & +n(n-1)\left[\frac{1}{n}(T_r(A) - T_r(B)) - \Psi_s\xi^s\right]. \end{aligned} \tag{3.10}$$

The relations (3.9) and (3.10) lead to

$$A_{jl} = \frac{R_{jl} - T_{jl}}{n-2} + \frac{K(T) - K}{n(n-2)}g_{jl} + \frac{1}{n}T_r(A)g_{jl}. \tag{3.11}$$

Substituting (3.10) and (3.11) in (3.8), we obtain $W = W(T)$.

Proposition 3.1. *Let T, defined by (3.2), be a generalized curvature tensor on the Riemannian manifold $V_n = (M, g)$, $n \geq 3$. If the Weyl conformal curvature tensor W verifies the relation*

$$\tau(X)W(Y, Z) + \tau(Y)W(Z, X) + \tau(Z)W(X, Y) = 0, \tag{3.12}$$

where τ is a 1-form, then the condition $W(T) \cdot W(T) = 0$ holds at x, where $\tau \neq 0$ at x, $W(T)$ being the Weyl conformal curvature tensor associated with T.

Proof. If the relation (3.12) is satisfied, then the condition $W \cdot W = 0$ holds at x, where $\tau \neq 0$ at x ([5]).

Using the previous theorem we obtain our assertion.

4 Subgeodesically and geodesically related Riemannian spaces

Theorem 4.1. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be two ξ^i-subgeodesically related Riemannian spaces, the tensor of correspondence being $-g$, such that $\xi_{hk} = 0$, where* $\xi_{hk} = \xi_{h,k} - \xi_h\xi_k + \frac{1}{2}\xi_i\xi^i g_{hk}$.

If $\overline{V}_n$ is a conharmonically semi-symmetric space and with irreducible curvature tensor, then V_n is an Einstein space.

Proof. Let $\widetilde{V}_n = (M, \widetilde{g} = e^{2\xi} g)$. V_n and $\widetilde{V}_n$ being conformally related, we have

$$\widetilde{\left|\begin{matrix} i \\ j\ k \end{matrix}\right|} = \left|\begin{matrix} i \\ j\ k \end{matrix}\right| + \delta^i_j \xi_k + \delta^i_k \xi_j - g_{jk}\xi^i. \tag{4.1}$$

Using (2.12) and (4.1) we get $\overline{\left|\begin{matrix} i \\ j\ k \end{matrix}\right|} = \widetilde{\left|\begin{matrix} i \\ j\ k \end{matrix}\right|} + \delta^i_j \omega_k + \delta^i_k \omega_j$, where $\omega_i = \Psi_i - \xi_i$.

Hence $\overline{V}_n$ and $\widetilde{V}_n$ are geodesically related spaces. In the same way as in the proofs of the theorems 1-3 of [10], $\widetilde{V}_n$ is an Einstein space.

A necessary and sufficient condition that an Einstein space be transformed into an Einstein space by a conharmonic transformation is that $\xi_{ij} = 0$ ([11]).

Because $\xi_{ij} = 0$, $g \to \widetilde{g} = e^{2\xi} g$ is a conharmonic transformation i.e. $\xi^p_p = g^{ij}\xi_{ij} = 0$. Applying the previous property, V_n will be an Einstein space.

Theorem 4.2. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be two ξ^i - subgeodesically related Riemannian spaces, the tensor of correspondence being $-g$, such that $B = \frac{1}{n} T_r(B) g$, where B is a tensor field of type (0, 2), having the components $B_{rs} = \xi_{r,s} - \xi_r \xi_s$.*

If V_n is a pseudo-symmetric space, then $\overline{V}_n$ is a pseudo-symmetric space.

Proof. If $B = \frac{1}{n} T_r(B) g$, then $g \to \widetilde{g} = e^{2\xi} g$ is a concircular transformation. If V_n is a pseudo-symmetric space, then $\widetilde{V}_n$ is a pseudo-symmetric space ([4]).

$\overline{V}_n$ and $\widetilde{V}_n$ being geodesically related manifolds, the space $\overline{V}_n$ will be pseudo-symetric ([3], [12]).

Theorem 4.3. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g})$, $n \geq 3$, be two non-trivial ξ^i-subgeodesically related Riemannian spaces, the tensor of correspondence being $-g$.*

Then $\overline{R} \cdot g = Q(g, F)$, where F is a symmetric tensor field of type (0,2), having the components

$$F_{ij} = \xi_{i;j} - \Psi_{i;j} - (\xi_i - \Psi_i)(\xi_j - \Psi_j), \tag{4.2}$$

where ";" denotes covariant differentiation with respect to the metric $\overline{g}$.

Proof. Using (2.12) we get

$$\begin{aligned} g_{jk;ir} = {} & -2\Psi_{i;r} g_{jk} - (\Psi_{j;r} - \xi_{j;r}) g_{ik} - (\Psi_{k;r} - \xi_{k;r}) g_{ij} - \\ & -2\Psi_i \left[-2\Psi_r g_{jk} - (\Psi_j - \xi_j) g_{rk} - (\Psi_k - \xi_k) g_{rj}\right] - \\ & -(\Psi_j - \xi_j)\left[-2\Psi_r g_{ik} - (\Psi_i - \xi_i) g_{rk} - (\Psi_k - \xi_k) g_{ir}\right] - \\ & -(\Psi_k - \xi_k)\left[-2\Psi_r g_{ij} - (\Psi_j - \xi_j) g_{ri} - (\Psi_i - \xi_i) g_{rj}\right]. \end{aligned}$$

Hence

$$(\overline{R} \cdot g)_{jkri} = g_{jk;ir} - g_{jk;ri} = Q(g, F)_{jkri},$$

where $F_{ij} = \xi_{i;j} - \Psi_{i;j} - (\xi_i - \Psi_i)(\xi_j - \Psi_j)$.

Proposition 4.1. Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be two non-trivial ξ^i-subgeodesically related Riemannian spaces, the tensor of correspondence being $-g$.

We suppose that $\overline{V}_n$ is a pseudo-symmetric space and the function $\overline{L}$, which satisfies the condition $(*)$ on $\overline{U}$, is constant.

If $F_{ij} = fg_{ij} + h\overline{g}_{ij}$, where $f, h \in \mathcal{F}(M), h$ being non-nulle and F_{ij} defined by (4.2), then the relation :

$$(\overline{L} + h)\left[g - \frac{1}{n}(\overline{g}^{ij} g_{ij})\overline{g}\right] = 0 \tag{4.3}$$

holds on $\overline{U}$.

Proof. We will apply the property ([7]) :

Let A and D be symmetric tensors of type (0,2) on a pseudo-symmetric manifold $V_n = (M, g)$, having a constant function L, which satisfies the condition $(*)$ on U.

If the relation $R \cdot A = Q(g, D)$ holds on U, then the relation $E - \frac{1}{n}T_r(E)g = 0$ is satisfied on U, where $E = D - LA$.

Because $F = fg + h\overline{g}$, using the theorem 4.3, we have $\overline{R} \cdot g = Q(\overline{g}, -hg)$.

In this case, the tensor $E = -hg - \overline{L}g$ satisfies on $\overline{U}$ the relation

$$E - \frac{1}{n}(\overline{g}^{ij} E_{ij})\overline{g} = 0.$$

This condition is equivalent with (4.3).

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