LIE GROUPS APPLICATIONS TO MINIMAL SURFACES PDE

Nicoleta Bîlă

Abstract

Using the theory of symmetry groups of differential equations (P.J.Olver [15]) we determine the symmetry group of the minimal surfaces equation (1) (presented in Theorem 2). Several group-invariant solutions of the equation are given.

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Consider the surfaces which are given in nonparametric form, that is, as the graph of a function $u = f(x, y)$ on some domain D of R^2. The mean curvature H of the surface is given by

$$H = \frac{u_{xx}(1+u_y^2) + u_{yy}(1+u_x^2) - 2u_x u_y u_{xy}}{2(1+u_x^2+u_y^2)^{\frac{3}{2}}}.$$

The equation $H = 0$ is equivalent to the nonlinear second order partial differential equation

$$u_{xx}(1+u_y^2) + u_{yy}(1+u_x^2) - 2u_x u_y u_{xy} = 0, \tag{1}$$

the so-called *minimal surfaces equation.*

The function $u^{(2)} : D \to U^{(2)}$ is called *the second prolongation of the function u,* where $U^{(2)} = U \times U_1 \times U_2 \cong R^6$ is the Cartesian product space, whose coordinates represent the derivatives of function u of all orders from 0 to 2,

$$u^{(2)} = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}).$$

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The total space $D \times U^{(2)}$ whose coordinates represent the independent variables, the dependent variable and the derivatives of the dependent variable up to order 2 is called *the second order jet space* of the underlying space $D \times U$. Let us consider

$$F(x, y, u^{(2)}) = u_{xx}(1+u_y^2) + u_{yy}(1+u_x^2) - 2u_x u_y u_{xy}.$$

Thus the equation (1) can be identified with the linear subvariety S in $D \times U^{(2)}$ determined by the vanishing of the function F:

$$S = \{(x, y, u^{(2)}) \in D \times U^{(2)} | F(x, y, u^{(2)}) = 0\}.$$

The equation

$$F(x, y, u^{(2)}) = 0,$$

is said to be of *maximal rank* if the Jacobian

$$J_F(x, y, u^{(2)}) = (F_x, F_y; F_u; F_{u_x}, F_{u_y}; F_{u_{xx}}, F_{u_{xy}}, F_{u_{yy}}),$$

satisfies the condition

(2) $$\text{rank } J_F = 1 \quad \text{whenever} \quad F(x, y, u^{(2)}) = 0.$$

For equation (1), we have

$$F_x = 0, \quad F_y = 0, \quad F_u = 0, \quad F_{u_x} = 2u_x u_{yy} - 2u_y u_{xy},$$
$$F_{u_y} = 2u_y u_{xx} - 2u_x u_{xy}, \quad F_{u_{xx}} = 1+u_y^2, \quad F_{u_{xy}} = -2u_x u_y, \quad F_{u_{yy}} = 1+u_x^2$$

and thus the Jacobian is

$$J_F(x, y, u^{(2)}) = (0, 0; 0; 2u_x u_{yy} - 2u_y u_{xy}, 2u_y u_{xx} - 2u_x u_{xy};$$
$$1+u_y^2, -2u_x u_y, 1+u_x^2).$$

and it satisfies the condition (2).

On the other hand, a *symmetry group* of a PDE is a local group of transformations G acting on an open set M of the space of independent and dependent variables, with the property: if $u = f(x, y)$ is a solution of the equation then $v = g \cdot f(x, y)$ is a solution for any $g \in G$ also. The computational procedure for finding the symmetry group uses the following *infinitesimal criterion of invariance.*

Theorem 1. *Let $F(x, y, u^{(2)}) = 0$ be a differential equation of maximal rank defined over an open set $M \subset D \times U$. If G is a local group of transformations acting on M and*

(3) $$pr^{(2)}X[F(x, y, u^{(2)})] = 0 \quad \textit{whenever} \quad F(x, y, u^{(2)}) = 0,$$

for every infinitesimal generator X of G, then G is a symmetry group of the PDE.

Consider the vector field

(4) $$X = \zeta(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}$$

on $M \in D \times U$.

The first prolongation of X is the vector field

$$\mathrm{pr}^{(1)}X = X + \Phi^x \frac{\partial}{\partial u_x} + \Phi^y \frac{\partial}{\partial u_y}, \tag{5}$$

where

$$\Phi^x = \phi_x + (\phi_u - \zeta_x)u_x - \eta_x u_y - \zeta_u u_x^2 - \eta_u u_x u_y$$

and

$$\Phi^y = \phi_y - \zeta_y u_x + (\phi_u - \eta_y)u_y - \zeta_u u_x u_y - \eta_u u_y^2.$$

The second prolongation of X is the vector field

$$\mathrm{pr}^{(2)}X = pr^{(1)}X + \Phi^{xx}\frac{\partial}{\partial u_{xx}} + \Phi^{xy}\frac{\partial}{\partial u_{xy}} + \Phi^{yy}\frac{\partial}{\partial u_{yy}}, \tag{6}$$

where

$$\begin{aligned}
\Phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \zeta_{xx})u_x - \eta_{xx}u_y + (\phi_{uu} - 2\zeta_{xu})u_x^2 - \\
&- 2\eta_{xu}u_x u_y - \zeta_{uu}u_x^3 - \eta_{uu}u_x^2 u_y + (\phi_u - 2\zeta_x)u_{xx} - 2\eta_x u_{xy} - \\
&- 3\zeta_u u_x u_{xx} - \eta_u u_y u_{xx} - 2\eta_u u_x u_{xy},
\end{aligned}$$

$$\begin{aligned}
\Phi^{xy} &= \phi_{xy} + (\phi_{uy} - \zeta_{xy})u_x + (\phi_{ux} - \eta_{xy})u_y - \zeta_{uy}u_x^2 + (\phi_{uu} - \zeta_{ux} - \\
&- \eta_{uy})u_x u_y - \eta_{ux}u_y^2 - \zeta_y u_{xx} + (\phi_u - \zeta_x - \eta_y)u_{xy} - \eta_x u_{yy} - \\
&- \zeta_u u_y u_{xx} - 2\eta_u u_y u_{xy} - 2\zeta_u u_x u_{xy} - \eta_u u_x u_{yy} - \zeta_{uu}u_x^2 u_y - \eta_{uu}u_x u_y^2, \\
\Phi^{yy} &= \phi_{yy} + (2\phi_{uy} - \eta_{yy})u_y - \zeta_{yy}u_x + (\phi_{uu} - 2\eta_{uy})u_y^2 - 2\zeta_{uy}u_x u_y - \\
&- \eta_{uu}u_y^3 - \zeta_{uu}u_x u_y^2 + (\phi_u - 2\eta_y)u_{yy} - 2\zeta_y u_{xy} - 3\eta_u u_y u_{yy} - \\
&- \zeta_u u_x u_{yy} - 2\zeta_u u_y u_{xy}.
\end{aligned}$$

For equation (1) the condition (3) becomes

$$\begin{aligned}
&\Phi^x(2u_x u_{yy} - 2u_y u_{xy}) + \Phi^y(2u_y u_{xx} - 2u_x u_{xy}) + \Phi^{xx}(1 + u_y^2) - \\
&-2u_x u_y \Phi^{xy} + \Phi^{yy}(1 + u_x^2) = 0.
\end{aligned} \tag{7}$$

Substituing the functions Φ^x, Φ^y, Φ^{xx}, Φ^{xy} and Φ^{yy} defined by (5), and (6) and eliminating any dependencies among the derivatives of the u's caused by the equation (1) itself, we find:

$$\begin{aligned}
&\phi_{xx} + \phi_{yy} + (2\phi_{xu} - \zeta_{xx} - \zeta_{yy})u_x + (2\phi_{uy} - \eta_{yy} - \eta_{xx})u_y + (\phi_{yy} + \\
&+\phi_{uu} - 2\zeta_{xu})u_x^2 + (\phi_{xx} + \phi_{uu} - 2\eta_{yu})u_y^2 - 2(\phi_{xy} + \eta_{xu} + \zeta_{yu})u_x u_y - \\
&-(\zeta_{yy} + \zeta_{uu})u_x^3 - (\eta_{xx} + \eta_{uu})u_y^3 + (2\eta_{xy} - \zeta_{xx} - \zeta_{uu})u_x u_y^2 + (2\zeta_{xy} - \\
&-\eta_{yy} - \eta_{uu})u_x^2 u_y + 2(\eta_y - \phi_u)u_{xx} + 2(\zeta_x - \phi_u)u_{yy} - 2(\zeta_y + \eta_x)u_{xy} + 2(\phi_y +
\end{aligned}$$

$$+\eta_u)u_y u_{xx} + 2(\zeta_u + \phi_x)u_x u_{yy} - 2(\phi_y + \eta_u)u_x u_{xy} - 2(\zeta_u + \phi_x)u_y u_{xy} = 0.$$

Now we can equate the coefficients of the remaining unconstrained partial derivatives of u to zero. This will result in a large number of PDEs for the coefficient functions ζ, η, and ϕ of the infinitesimal operator, called *the defining equations* for the symmetry group of the given equation:

$$(9)\qquad \begin{array}{cc}
\phi_{xx} + \phi_{yy} = 0, & \zeta_{xx} + \zeta_{yy} = 2\phi_{xu}, \\
\eta_{xx} + \eta_{yy} = 2\phi_{yu}, & \phi_{yy} + \phi_{uu} = 2\zeta_{xu}, \\
\phi_{xx} + \phi_{uu} = 2\eta_{yu}, & \phi_{xy} + \eta_{xu} + \zeta_{yu} = 0, \\
\zeta_{yy} + \zeta_{uu} = 0, & \eta_{xx} + \eta_{uu} = 0, \\
\zeta_{xx} + \zeta_{uu} = 2\eta_{xy}, & \eta_{yy} + \eta_{uu} = 2\zeta_{xy}, \\
\eta_y = \phi_u, & \phi_u = \zeta_x, \\
\zeta_y = -\eta_x, & \phi_y = -\eta_u, \\
\phi_x = -\zeta_u. &
\end{array}$$

By integration, we find the following solutions

$$\begin{aligned}
\zeta(x,y,u) &= C_7 x - C_4 y + C_6 u + C_1, \\
\eta(x,y,u) &= C_4 x + C_7 y - C_5 u + C_2, \\
\phi(x,y,u) &= -C_6 x + C_5 y + C_7 u + C_3,
\end{aligned}$$

with $C_1, ..., C_7 \in R$, and the vector field X is given by:

$$(10)\qquad X = C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y} + C_3\frac{\partial}{\partial u} + +C_4\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) +$$

$$+C_5\left(-u\frac{\partial}{\partial y} + y\frac{\partial}{\partial u}\right) + C_6\left(-x\frac{\partial}{\partial u} + u\frac{\partial}{\partial x}\right) + C_7\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}\right).$$

Proposition 1. *Let a partial differential equation of the maximal rank defined by over $M \subset D \times U$. The set of all infinitesimal symmetries of the equation form a Lie algebra of vector fields on M . Moreover, if this Lie algebra is finite-dimensional, the symmetry group of the equation is a local Lie group of transformations acting on M.*

We get the following

Theorem 2. *The Lie algebra of infinitesimal symmetries of the minimal surfaces equation is spanned by seven vector fields:*

$$(11)\qquad X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial u},$$

$$X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\ \ X_5 = -u\frac{\partial}{\partial y} + y\frac{\partial}{\partial u},$$

$$X_6 = -x\frac{\partial}{\partial u} + u\frac{\partial}{\partial x},$$

$$X_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.$$

Since each one-parameter subgroup G_i generated by X_i is a symmetry group, every solution $u = f(x, y)$ is changed into the following solutions

$$u^{(1)} = f(x - \varepsilon, y),$$

$$u^{(2)} = f(x, y - \varepsilon),$$

$$u^{(3)} = f(x, y) + \varepsilon,$$

$$u^{(4)} = f(x \cos\varepsilon - y \sin\varepsilon, x \sin\varepsilon + y \cos\varepsilon),$$

$$x \sin\varepsilon + u^{(5)} \cos\varepsilon = f(x \cos\varepsilon - u^{(5)} \sin\varepsilon, y),$$

$$y \sin\varepsilon + u^{(6)} \cos\varepsilon = f(x, y \cos\varepsilon - u^{(6)} \sin\varepsilon),$$

$$u^{(7)} = e^{\varepsilon} f(e^{-\varepsilon} x, e^{-\varepsilon} y),$$

where ε is a real number.

For each s-parameter subgroup H of the full symmetry group G of the equation correspond a family of group-invariant solutions. Thus, a classification of these solutions is by using an *optimal system of group-invariant solutions* from which every other solution can be derived.

Proposition 2. *If $u = f(x, y)$ is an H-invariant solution to the equation and $g \in G$ is an any other group element, then the transformed function $v = \tilde{f}(x, y) = g \cdot f(x, y)$ is a $\tilde{H}$-invariant solution, where $\tilde{H} = gHg^{-1}$ is the conjugate subgroup to under g.*

The problem of classifying group-invariant solutions reduces to the problem of classifying subgroups of the full symmetry group G under conjugation and this is equivalent with the classifying subalgebras of the Lie algebra g of the group G.

Thus, we compute the adjoint representation $Ad\ G$ of the underlying Lie group G, by using the Lie series :

$$Ad(exp(\varepsilon X)Y) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adX)^n(Y) = Y - \varepsilon[X, Y] + \frac{\varepsilon^2}{2}[X, [X, Y]] - \ldots$$

and we construct the table:

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4 - \varepsilon X_2$
X_2	X_1	X_2	X_3	$X_4 + \varepsilon X_1$
X_3	X_1	X_2	X_3	X_4
X_4	$X_1 \cos\varepsilon + X_2 \sin\varepsilon$	$X_2 \cos\varepsilon - X_1 \sin\varepsilon$	X_3	X_4
X_5	X_1	$X_2 \cos\varepsilon + X_3 \sin\varepsilon$	$X_3 \cos\varepsilon - X_2 \sin\varepsilon$	$X_4 \cos\varepsilon - X_6 \sin\varepsilon$
X_6	$X_1 \cos\varepsilon - X_3 \sin\varepsilon$	X_2	$X_3 \cos\varepsilon + X_1 \sin\varepsilon$	$X_4 \cos\varepsilon + X_5 \sin\varepsilon$
X_7	$e^{\varepsilon} X_1$	$e^{\varepsilon} X_2$	$e^{\varepsilon} X_3$	X_4

Ad	X_5	X_6	X_7
X_1	X_5	$X_6+\varepsilon X_3$	$X_7-\varepsilon X_1$
X_2	$X_5-\varepsilon X_3$	X_6	$X_7-\varepsilon X_2$
X_3	$X_5+\varepsilon X_2$	$X_6-\varepsilon X_1$	$X_7-\varepsilon X_3$
X_4	$X_5\cos\varepsilon+X_6\sin\varepsilon$	$X_6\cos\varepsilon-X_5\sin\varepsilon$	X_7
X_5	X_5	$X_6\cos\varepsilon+X_4\sin\varepsilon$	X_7
X_6	$X_5\cos\varepsilon-X_4\sin\varepsilon$	X_6	X_7
X_7	X_5	X_6	X_7

with the (i,j)-th entry indicating $Ad(exp(\varepsilon X_i))X_j$.

For one-dimensional subalgebras, this classification problem is esentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in g. The method consists in taking a general element X and subjecting it to various adjoint transformations so as to "simplify" it as much as possible.

For equation (1) we find an optimal system of one-dimensional subalgebras spanned by

$$X_1,\ X_2,\ X_7, \tag{12.1}$$

$$X_4,\ X_5,\ X_6, \tag{12.2}$$

$$X_3+X_4,\ X_1+X_5,\ X_2+X_6, \tag{12.3}$$

$$X_4+X_7,\ \ X_5+X_7,\ \ X_6+X_7 \tag{12.4}$$

For each one-parameter subgroup there will be a coresponding class of group-invariant solutions which determined from a reduced ODE, whose form depends on the particular subgroup.

(12.1) If we consider $X_7 = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}$ then the global invariants of this group are $C_1=\frac{y}{x}$, $C_2=\frac{u}{x}$. Thus, we have $u=xh(\frac{y}{x})$.

By substituting in the equation (1) we get

$$u = ax+by,\ \ a,b\in R. \tag{13}$$

Also for X_1, and X_2 we find planes.

(12.2) If we consider $X_4 = -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$ then the global invariants are $C_1 = \sqrt{x^2+y^2}$, $C_2=u$, so that a group-invariant solution has the form $u=h(\sqrt{x^2+y^2})$.

Consider

$$r=\sqrt{x^2+y^2},\ \ \theta=\text{arctg}(\frac{y}{x})$$

and by substitution in the equation (1) rewritten in these coordinates

$$u_{rr}\left(1+\frac{1}{r^2}u_\theta^2\right)+\frac{1}{r^2}u_{\theta\theta}(1+u_r^2)-\frac{2}{r^2}u_ru_\theta u_{r\theta}+\frac{1}{r}u_r\left(1+u_r^2+\frac{2}{r^2}u_\theta^2\right)=0,$$

the following equation $h'' = -\frac{h'}{r} - \frac{h'^3}{r}$ hold good.

1. For $h' = 0$ it follows $u = k$ and the minimal surface is a plane.

2. For $h' \neq 0$ it results the Bernoulli equation $g' = -\frac{1}{r}g - \frac{1}{r}g^3$, $g = h'$ with the general solution $g = \frac{a}{\sqrt{r^2 - a^2}}$, $a > 0$ and we find $h = a\ln(r + \sqrt{r^2 - a^2}) + b$. So, the minimal surface is the catenoid

$$u = a\ln(r + \sqrt{r^2 - a^2}) + b, \quad a > 0, \ b \in R \tag{14}$$

the only minimal surface of rotation (Theorem Meusnier [7]).

For X_5 we find also the catenoid (with y-axis of rotation):

$$u = \sqrt{a^2 \cosh^2\left(\frac{x}{a} + b\right) - y^2}, \ ; \ a, b \in R^*. \tag{15}$$

For X_6, we find the catenoid with the x-axis of rotation.

(12.3) Consider

$$X = X_3 + X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial u}.$$

The global invariants of this group are $C_1 = \sqrt{x^2 + y^2}$, $C_2 = u - \operatorname{arctg}\frac{y}{x}$. Thus $u = \theta + h(r)$. By substituting in the equation (1) we get

$$h''\left(1 + \frac{1}{r^2}\right) + \frac{1}{r}h'\left(1 + h'^2 + \frac{2}{r^2}\right) = 0.$$

1. If $h' = 0$ it follows

$$u = \operatorname{arctg}\frac{y}{x} + a, \quad a \in R \tag{16}$$

and we find the helicoid.

2. Suppose $h' \neq 0$ and denote $h' = \frac{g}{r}$. We get the differential equation $g'(r^2 + 1) + \frac{g}{r}(g^2 + 1) = 0$, with the general solution $g = \sqrt{\frac{r^2+1}{(a^2-1)r^2-1}}$; $a > 1$, and

$$h(r) = b\ln(\sqrt{r^2 + 1} \pm \sqrt{r^2 - b^2}) + \operatorname{arctg}(\pm\frac{1}{b}\sqrt{\frac{r^2 - b^2}{r^2 + 1}}) - b\ln\sqrt{r^2 + b^2} + c,$$

where $b = \frac{1}{\sqrt{a^2 - 1}}$, $c \in R$. In this case, we obtain the Scherk's second surface

$$u = \operatorname{arctg}\theta + h(r) \tag{17}$$

which is a helicoidal surface ([6], p.144).

(12.4) Consider

$$X = X_4 + X_7 = (x - y)\frac{\partial}{\partial x} + (x + y)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.$$

We find the global invariants $C_1 = \theta - \ln r$, $C_2 = \frac{u}{r}$.Thus, we have$u = rh(\theta - \ln r)$. By substituting in the equation (1), the following differential equation

$$h''(2 + h^2) + h - 2h' - h'^3 + (h - h')^3 = 0$$

hold good. In this case, the finding of the solution is more difficult and it is going to be carefully studied in the future. We are also going to study the classification of the s-subalgebra, for $s > 1$. Using this theory we must obtain all the solutions of equation (1).

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Author's address:

Nicoleta Bîlă
Politehnica University of Bucharest
Departament of Mathematics I
Splaiul Independenţei 313
77206 Bucharest Romania
email: nbila@mathem.pub.ro