FINSLER CONNECTION IN THE HIGHER ORDER GEOMETRY

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Abstract

The first part of this work is a natural extension of a recent paper by M. Anastasiei ([1]). In §1 the Finsler connections are defined by local components. In §2 a Finsler connection appears as a pair \((N, \nabla)\), where \(N\) is a nonlinear connection on the jet bundle of order \(k\), \(Osc^k M\) and \(\nabla\) is a linear connection in the pull-back bundle of the tangent bundle by the projection map \(\pi^k : Osc^k M \rightarrow M\). Next, the notion of linear connection of Cartan type is generalized and two examples are given.

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1 Introduction

Let \(M\) be a real, smooth manifold of dimension \(n\) and \((Osc^k M, \pi^k, M)\) its \(k\)-osculator bundle, \(k \in \mathbb{N}^*\) (the jets bundle of order \(k\) of the manifold \(M\)). Then \(Osc^k M\) is a real, smooth manifold of dimension \(n(k + 1)\). We set \(E = Osc^k M\). For \(k = 1\), \(Osc^k M\) can be identified, in a canonical way, with the tangent bundle \(T M\).

Let \((x^i)\) be the local coordinates in a local chart \(U \subset M\). The local coordinates on \((\pi^k)^{-1}(U) \subset Osc^k M\) will be denoted by \((x^i, y^{(1)}_i, \ldots, y^{(k)}_i)\).

For each \(u \in E\), let \(\{\frac{\partial}{\partial x^i} \bigg|_u, \frac{\partial}{\partial y^{(1)}_i} \bigg|_u, \ldots, \frac{\partial}{\partial y^{(k)}_i} \bigg|_u\}\) be the natural basis of the tangent spaces \(T_u E\).

As \((\pi^k)_* : (TE, \tau_E, E) \rightarrow (TM, \tau, M)\) is a \(\pi^k\) morphism of vector bundles, its results that its kernel is a vector subbundle of the bundle \((TE, \tau_E, E)\). This will be denoted by \(V_1 E\) and will be called the vertical subbundle of the \(TE\). The fibers of \(V_1 E\) determine an integrable distribution \(V_1 : u \in E \mapsto V_1(u) \subset T_u E\) that has the dimension \(kn\), called vertical distribution.
For each \( u \in E \) we consider the linear mapping \( J_u : T_u E \to T_u E \) defined in the natural basis as follows:

\[
\begin{align*}
J_u(\frac{\partial}{\partial x^i} | u) &= \frac{\partial}{\partial y^{(i)} | u}, \\
J_u(\frac{\partial}{\partial y^j | u}) &= 0,
\end{align*}
\]

and extended by linearity. The map \( J : TE \to TE \) is called \( k \)-tangent structure. Consider \( V_k = J^k(V_1) \) and the map \( V_k : u \in E \to V_k(u) \) is an integrable, \( n \)-dimensional distribution.

Let be \( (\pi^k)^*(TM) = \{(u, X) \in Osc^k M \times TM, \pi^k(u) = \tau(X)\} \) and the map \( (\pi^k)^*(\tau) : (\pi^k)^*(TM) \to Osc^k M \) defined by \( (\pi^k)^*(\tau)(u, X) = u \). That is, \( ((\pi^k)^*(TM), (\pi^k)^*(\tau), Osc^k M) \) is a vector bundle over \( E \), called the pull-back bundle of the tangent bundle \( (TM, \tau, M) \) by the map \( \pi^k \). This vector bundle is isomorphic with the vertical bundle \( (V_k E, \tau E|V_k E, E) \).

A section in this bundle is locally of the form

\[
S : u = (x, y^{(1)}, \ldots, y^{(k)}) \mapsto S'(x, y^{(1)}, \ldots, y^{(k)}) \frac{\partial}{\partial x^i}
\]

with \( \frac{\partial}{\partial x^i} \) the natural basis in \( T_{\pi^k(u)} M \). This will be called \( \pi^k \)-vector field of \( TM \) and will be identified to a \( d \)-vector field on \( E \), which has the components \( (S') \). The \( \mathcal{F}(E) \)-module of the \( \pi^k \)-vector fields will be denoted by \( \Gamma(\pi^k(TM)) \).

There exists a remarkable \( \pi^k \)-vector field on \( TM: C : u \mapsto (u, \pi^k_1(u)) \) and it can be identified to the Liouville vector field

\[
(1.1) \quad 1^2 = y^{(1)} \frac{\partial}{\partial y^{(1)}} + 2 y^{(2)} \frac{\partial}{\partial y^{(2)}} + \cdots + ky^{(k)} \frac{\partial}{\partial y^{(k)}}.
\]

The vector fields \( \Gamma, \Gamma_1, \ldots, \Gamma_k \) are called Liouville vector fields of the \( k \)-osculator bundle.

A nonlinear connection in the \( k \)-osculator bundle is a subbundle \( NE \), of the tangent bundle of \( E \) such that the Whitney sum

\[
(1.2) \quad TE = NE \oplus V_1 E,
\]

holds. A nonlinear connection induces a regular distribution \( N \) on \( E \) of dimension \( n \), which is supplementary to the vertical distribution \( V_1 \). For each \( u \in E \) the map \( \pi^k_{\ast, u}|N(u) : N(u) \to T_{\pi^k(u)} M = ((\pi^k)^*(TM))_u \) is an isomorphism of linear spaces. It inverse, which is denoted by \( (l_k)_u \), will be called horizontal lift. Using the notation

\[
\frac{\delta}{\delta x^k} | u = (l_k)_u \frac{\partial}{\partial x^k}
\]

we obtain a basis for \( N(u) \). This is expressed in the natural basis of the tangent space \( T_u E \) as follows:

\[
(1.3) \quad \frac{\delta}{\delta x^k} | u = \frac{\partial}{\partial x^k} | u - N^j_{(1)} (u) \frac{\partial}{\partial y^{(j)} | u} - \cdots - N^j_{(k)} (u) \frac{\partial}{\partial y^{(k)}} | u.
\]

The functions \( (N^j_{(1)}, \ldots, N^j_{(k)}) \) are called the coefficients of the nonlinear connection \( N \).

We use the following notations:

\[
\frac{\delta}{\delta y^{(i)}} = J^\ast \left( \frac{\delta}{\delta x^k} \right). \quad \text{If} \quad N_0 = N, N_1 = J(N_0), \ldots, N_{k-1} =
\]
For each 

\[ C \]

expressed in the adapted basis as follows:

\[ \{ \frac{\delta}{\delta x^i} |_{u, \delta y^{(1)i}} |_{u, \delta y^{(k-1)i}} |_{u}, \frac{\partial}{\partial y^{(k)i}} |_{u} \} \]

is a non-holonomic basis for \( T_u E \) that is adapted to the decomposition (1.4). The dual basis of the basis (1.5) is expressed in the dual natural basis using a system of functions \( ((1.5) \text{ of coordinates are modified like as the coefficients of a linear connection on } F) \) which are called the dual coefficients of the nonlinear connection \( N \).

The Liouville vector fields are expressed in the adapted basis as follows:

\[ \Gamma^1 = z^{(1)i} \frac{\partial}{\partial y^{(1)i}}, \]
\[ \Gamma^2 = z^{(1)i} \frac{\delta}{\delta y^{(1)i}} + 2z^{(2)i} \frac{\partial}{\partial y^{(1)i}}, \]
\[ \Gamma^k = z^{(1)i} \frac{\delta}{\delta y^{(1)i}} + 2z^{(2)i} \frac{\delta}{\delta y^{(2)i}} + \ldots + k z^{(k)i} \frac{\partial}{\partial y^{(k)i}}, \]

where

\[ z^{(1)i} = y^{(1)i}, 2z^{(2)i} = 2y^{(2)i} + M_j^{(1)} y^{(1)j}, \ldots, \]
\[ k z^{(k)i} = k y^{(k)i} + (k - 1) M_j^{(1)} y^{(k-1)j} + \ldots + M_j^{(1)} y^{(1)j}. \]

Let \( ((\pi^k)^*(TM))^{(k)} \) the Whitney sum of the pull-back bundle on itself of \( k \)-times. We call connection map on the \( k \)-osculator bundle \( (2,3) \) a \( \pi^k \) epimorphism of vector bundles \( K = (1,2,3, \ldots) : T E \rightarrow ((\pi^k)^*(TM))^{(k)} \) that satisfies:

\[ (k) K \circ J^a = K^{(k)} \text{, } \forall a \in \{1,2,\ldots,k-1\}, \]
\[ (k) K \circ J^k = \pi^k. \]

In the paper [3] we have proved that every nonlinear connection \( N \) is the kernel of a connection map.

**Definition 1.1** A Finsler connection on the \( k \)-osculator bundle is a system \( F \Gamma = (N, F^{(m)}_{ij}, C^{(m)}_{ij}, \alpha = (1,2)) \), where \( N \) is a nonlinear connection on the \( k \)-osculator bundle, the functions \( F^{(m)}_{ij} \) are defined on every domain of local chart and which at a change of coordinates are modified like as the coefficients of a linear connection on \( M \) and \( C^{(m)}_{ij} \) are the local components of a \( d \)-tensor field of type (1,2).

To a Finsler connection \( F \Gamma \) we associate a linear connection \( D \) on \( E \), which is expressed in the adapted basis as follows:

\[ D \frac{\delta}{\delta x^i} = F^{m}_{ij} \frac{\delta}{\delta x^j}, D \frac{\delta}{\delta y^{(\alpha)i}} = F^{m}_{ij} \frac{\delta}{\delta y^{(\alpha)j}}, \]
\[ D \frac{\delta}{\delta y^{(k)i}} = C^{m}_{ij} \frac{\delta}{\delta y^{(k)j}}. \]
The linear connection $D$ is called $N$-linear connection. It preserves by parallelism the distributions $N_0, N_1, \ldots, N_{k-1}, V_k$ and the $k$-tangent structure $J$ is parallel with respect to it.

For a Finsler connection $\Gamma$ we denote by $D^j_i$ the $h$-tensor fields of deflection and by $d^j_i$ the $v_\alpha$-tensor fields of deflection. These are given by:

\begin{align}
D^j_i & = D^j_i \frac{\delta}{\delta y^i} + 2 D^j_i \frac{\delta}{\delta y^{(\alpha)}} + \cdots + k \frac{\delta}{\delta y^{(k)}} \frac{\partial}{\partial y^i} \\
d^j_i & = d^j_i \frac{\delta}{\delta y^i} + 2 d^j_i \frac{\delta}{\delta y^{(\alpha)}} + \cdots + k \frac{\delta}{\delta y^{(k)}} \frac{\partial}{\partial y^i}.
\end{align}

## 2 Finsler Connection on the $k$-osculator bundle.

### Characterizations.

**Theorem 2.1** There exists a one to one correspondence between the set of Finsler connections $\Gamma$ and the set of pairs $(\alpha, \Gamma)$, with $\alpha$ a nonlinear connection on $E$ and $\Gamma$ a linear connection in the pull-back bundle $(\pi^k)^*(TM)$.

**Proof.** If $\Gamma$ is specified by $(\alpha, \Gamma)$, we define $\Gamma : \chi(E) \times (\pi^k)^*(TM)$.

Conversely, let $(\alpha, \Gamma)$ be a pair like in hypothesis. In the natural basis $\Gamma$ takes the form:

\begin{align}
\nabla \Gamma & = \Gamma^j_i \frac{\partial}{\partial x^i} \\
\nabla & = \Gamma^j_i \frac{\partial}{\partial x^i}.
\end{align}

If we consider the system of functions (defined on every neighborhood coordinates):

\begin{align*}
F^j_i & = \Gamma^j_i \Gamma^0 \Gamma^1 \Gamma^2 \cdots \Gamma^k \\
C^j_i & = \Gamma^j_i \Gamma^0 \Gamma^1 \Gamma^2 \cdots \Gamma^k,
\end{align*}

by a direct calculation we obtain that $\Gamma = (\alpha, \Gamma)$ is a Finsler connection on $E$. In the adapted basis the linear connection $\nabla$ is expressed as (2.1), where $F^j_i$ and $C^j_i$ are given by the previous formula. Concluding, the correspondence $\Gamma \leftrightarrow (\alpha, \Gamma)$ is one to one. $\square$. 

Definition 2.1 A linear connection $\nabla$ in the pull-back bundle $(\pi^k)^*(TM)$ is said to be regular if the subspace $H(u) = \{X_u \in T_u E, \nabla_{X_u} C = 0\}$ of $T_u E$ is supplementary to $V_k(u)$ for every $u \in E$.

Proposition 2.1 Every regular connection $\nabla$ in the pull-back bundle $(\pi^k)^*(TM)$ induces a nonlinear connection $N$ on $E$.

Proof. As $\nabla$ is a regular connection we obtain a distribution $H : u \in E \rightarrow H(u) = \{X_u \in T_u E, \nabla_{X_u} C = 0\}$ of dimension $kn$ supplementary to the distribution $V_k$ and so $T_u E = H(u) \oplus V_k(u), \forall u \in E$. We denote by $\psi_k$ the vertical projector induced by the previous decomposition. For every $u \in E$ the map $(l_{\psi_k})_u : ((\pi^k)^*(TM))_u \rightarrow V_k(u)$ defined by $(l_{\psi_k})_u \left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} |_u$ and extended by linearity is an isomorphism of vector spaces. We denote by $K_u : V_k(u) \rightarrow ((\pi^k)^*(TM))_u$ the inverse map of $(l_{\psi_k})_u$ and we extend it to $T_u E$ by $K_u := K_u \circ \psi_k$. In this way we obtain a morphism of vector bundles with the base $E, K : T E \rightarrow (\pi^k)^*(TM)$ for which $H = \text{Ker} K$. Let $(k-1)$ $(k)$ $(k)$ $(k)$ $(k)$ $(k)$ $K = K \circ J, \cdots, K = K \circ J^{k-1}$. The map $K = (K, \cdots, K) : T E \rightarrow ((\pi^k)^*(TM))_u$ is a connection map. Its kernel is a nonlinear connection $N$.

Next, we give a characterization for the regular connections.

Theorem 2.2 There exists a one to one correspondence between the set of regular connections $\nabla$ in the pull-back bundle and the set of Finsler connections $F \Gamma$ satisfying:

\[
(*) \quad (1) \quad (1) \quad (k-1) \quad (k) \quad (k) \quad (k) \quad (k) \quad (k) \quad (k) \quad (k) \quad (k) \quad (k) \quad (k)

\text{det} (d^i_j) \neq 0.

Proof. Let $\nabla$ be a regular connection. According to the definition and the previous proposition we obtain a $kn$-dimensional distribution $H$ and a connection map $K = (K, \cdots, K)$. Let $N = \text{Ker} K$. For the pair $(N, \nabla)$ we have according to the Theorem 2.1. a Finsler connection $F \Gamma$. We prove that for this the conditions $(*)$ and $(**)$ hold.

For $X_u \in T_u E$ we have:

\[
(2.3) \quad \nabla_{X_u} C = (X^i D^i_j + (K X)^i_j d^i_j + \cdots + (K X)^i_j d^i_j + (K X)^i_j d^i_j) \left(\frac{\partial}{\partial x^i}\right) |_u.
\]

The condition $H(u) \subset \text{Ker} K_u$ assures that $(*)$ is true and $H(u) \cap V_k(u) = \{0\}$ assures that $(**)$ hold.

Conversely, let $F \Gamma = (N, F^m \alpha, C^m_{ij}, \alpha = \text{I} K)$, be a Finsler connection satisfying $(*)$ and $(**)$). According to (2.3), the condition $(*)$ implies that if $X_u \in H_u$ then $X_u \in \text{Ker} K_u$ and so $H_u \subset \text{Ker} K_u$. The condition $(**)$ assures that $H(u) \cap V_k(u) = \{0\}$ and so $H_u = K_u$. 

Finsler connection
**Definition 2.2** Let \( N \) be a linear connection on \( E \). An \( N \)-linear connection on \( E \) is said to be of *Cartan type* if:

\[
(2.4) \quad N(u) = \{X_u \in T_u E, D_{X_u} \Gamma^\alpha = 0 \ \forall \alpha = 1, k\} \ \forall u \in E.
\]

**Theorem 2.3** Let \( N \) be a nonlinear connection on \( E \) and \( D \) an \( N \)-linear connection. Then \( D \) is a linear connection of Cartan type, if and only if

\[
(2.5) \quad \frac{D_j^\alpha}{(\alpha)} = 0 \text{ and } \det(D_j^\alpha) \neq 0.
\]

**Proof.** Let \( X_u \in T_u E \). According to (1.9) we obtain the following formulae:

\[
D_{X_u} \Gamma^\alpha = (X^t D_j^\alpha + (K X)^t d_j^\alpha + \cdots + (K X)^t d_j^\alpha) \frac{\partial}{\partial y^{(k)x\alpha}} |u = \nonumber\]

\[
= A^j \frac{\partial}{\partial y^{(k)x\alpha}} |u,
\]

\[
D_{X_u} \Gamma^\alpha = A^j \frac{\delta}{\delta y^{(k)x\alpha}} |u + 2(X^t D_j^\alpha + (K X)^t d_j^\alpha + \cdots + (K X)^t d_j^\alpha) \frac{\partial}{\partial y^{(k)x\alpha}} |u + \nonumber\]

\[
+ (K X)^t d_j^\alpha + \cdots + (K X)^t d_j^\alpha) \frac{\partial}{\partial y^{(k)x\alpha}} |u = \nonumber\]

\[
= A^j \frac{\delta}{\delta y^{(k)x\alpha}} |u + 2A^j \frac{\delta}{\delta y^{(k)x\alpha}} |u + \cdots + (k-1) A^j \frac{\delta}{\delta y^{(k)x\alpha}} |u + \nonumber\]

\[
+ k(X D_j^\alpha + (K X)^t d_j^\alpha + \cdots + (K X)^t d_j^\alpha) \frac{\partial}{\partial y^{(k)x\alpha}} |u = \nonumber\]

\[
= A^j \frac{\delta}{\delta y^{(k)x\alpha}} |u + 2A^j \frac{\delta}{\delta y^{(k)x\alpha}} |u + \cdots + k A^j \frac{\delta}{\delta y^{(k)x\alpha}} |u.
\]

The conditions \( D_{X_u} \Gamma^\alpha = 0, \ \forall \alpha = 1, k \) are equivalent with the equations \( A^j = 0 \ \forall \alpha = 1, k \) and these are equivalent with:

\[
(2.6) \quad X^t (D_j^\alpha \cdots D_j^\alpha) + ((K X)^t d_j^\alpha \cdots (K X)^t d_j^\alpha) = 0.
\]

Let \( N \) be a nonlinear connection and \( D \) be an \( N \)-linear connection which satisfies (2.3). We have that \( X \in N \) if and only if \( K X = (K X = \cdots = (K X = 0. \) Then (2.5) becomes \( X^t (D_j^\alpha \cdots D_j^\alpha) = 0 \) from where we obtain that the \( h \)-tensor fields of deflections are vanishing. If we assume that \( \det(D_j^\alpha) = 0 \) then exists \( X_u \in T_u E \) with \( K_u X_u \neq 0 \) such that (2.5) holds. So, we have \( X_u \notin N(u) \) with \( D_{X_u} \Gamma^\alpha = 0, \ \forall \alpha = 1, k \).

This means that \( N(u) \subset \{X_u \in T_u E, D_{X_u} \Gamma^\alpha = 0 \ \forall \alpha = 1, k\} \). □
3 Examples of linear connections of Cartan type.

Let \((M, g)\) be a Riemann manifold, \(\gamma^i_{jk}\) the local coefficients of the Levi-Civita connection.

**Theorem 3.1** There exists a unique nonlinear connection \(N\) on \(E\) such that \(F^\Gamma = (N, F^m_{ij} = \gamma^m_{ij} \circ \pi^k, C^m_{ij} = 0)\) is a linear connection of Cartan type on \(E\).

**Proof.** We prove that on every domain of local chart we can define a system of functions \((N^1_j, \ldots, N^k_j)\) which is uniquely determined by the conditions (2.5) of the
(1)

Theorem 2.3. By a change of coordinates these functions will change according to the
(1)

rule which allows us to say that and these functions are the coefficients of a nonlinear connection
(1)

\(N\), \(F\) given by this theorem is
(1)

such that
(1)

\(\gamma^i_{jk}\) and \(z^{(3)i}\) defined by (1.7).
(1)

Next, in the same manner we obtain \((N^1_j, \ldots, N^k_j), (M^i_j, \ldots, M^i_j)\) and \(z^{(5)i}, \ldots, z^{(k)i}\).
(1)

Finally, with the previous functions we define
(1)

\(N^i_j (x, y^{(1)}, \ldots, y^{(k)}) = (\partial \partial x^p - N^p_j - \partial y^q \partial y^{(1)p} - \partial y^q \partial y^{(2)p} - \partial y^q \partial y^{(3)p} + \gamma^i_{jk}(x)z^{(k)m}.
(1)

By a direct calculation we can verify that at a change of coordinates on \(E\) the system of functions \((N^1_j, \ldots, N^k_j)\) satisfies (4.9) from [6] and so these are the local coefficients of a nonlinear connection on \(E\). The Finsler connection \(F^\Gamma\) given by this theorem is of Cartan type because:

- a) the \(h\)-tensors of deflections are vanishing (by the construction of the functions
(1)

\(N^i_j\) we have that \(D_j^i\) are vanishing);
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b) \( \text{det}(d_{ij}^{\alpha\beta}) \neq 0 \) (for \( \alpha > \beta \) we have \( d_{ij}^{\alpha\beta} = 0 \), \( d_{ij}^{\alpha\alpha} = \delta_i^j \) and so \( \text{det}(d_{ij}^{\alpha\beta}) = 1 \)). □

Let \( F^n = (M, F) \) be a Finsler space and \((N_i^j, F_{ij}^m, C_{ij}^m)\) the Cartan connection of the Finsler space \( F^n \) ([5, p.113]).

**Theorem 3.2** There exists a unique nonlinear connection \( N \) on \( E \) such that \( \Gamma = (N, F_{ij}^m := F_{ij}^m \circ \pi_1^k, C_{ij}^m = C_{ij}^m \circ \pi_1^k, C_{ij}^m = 0 \; \alpha \geq 2) \) is a linear connection of Cartan type on \( E \).

**Proof.** The proof follows the previous theorem line. Thus, on every domain of local chart, we define the systems of functions \((N_i^j(1), \ldots, N_i^j(k))\) and \((M_i^j(1), \ldots, M_i^j(k))\) which will be the coefficients and the dual coefficients, respectively, of a nonlinear connection \( N \). From the vanishing of the first \( h \)-tensor of deflection we obtain \( N_i^j(x, y^{(1)}, \ldots, y^{(k)}) = \)

\( N_i^j(1) \circ \pi_1^k \). To determine the next coefficients a method like in the previous theorem is used. □

**References**


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