THE SUPERTRACE ON Q-TYPE SUPERMATRICES

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Abstract

Certain aspects of the Grassmannified structures are studied in the paper. The linear invariants, generalizing the trace function, are considered for algebras of Q-type superematrices.

1 INTRODUCTION

In recent years theoretical physicists have justifiably been excited by the prospect that so-called supersymmetric theories of elementary particles may play a significant role in the unification of the forces of nature. The latter aim has been a driving force in physics for a long time and, indeed, was a preoccupation of Einstein during much of his life.

The mathematical forms which support supersymmetry are obtained from the familiar ones of mathematical physics , for example real and complex functions, vectors, matrices, manifolds, Lie groups, by a Grassmann-algebraic version of complexification. By this we mean the replacement of real or complex numbers with elements from a Grassmann algebra. The new structures, for which the prefix "super" is often used, incorporate, in a sense, both commuting and anticommuting variables or parameters. A useful source of the definitions and properties of these super-objects is the book of deWitt [1] and the book of Berezin [2].

In this paper we explore certain aspects of one of these Grassmannified structures, namely the linear invariants, generalizing the trace function, for algebras of Q-type superematrices. First we recall from [2], [3] , some basic results on Grassmann algebras.

Denote by Λ_p, p finite, the Grassmann algebra on p mutually anticommuting generators $\{\theta_1, \theta_2, \dots, \theta_p\}$. The underlying field of scalars, to which we have no need to refer again, is either the real or complex numbers. The subscript p , defining the

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number of independent generators, will usually be omitted if it is understood from the context. The subspace $\Lambda_{\bar{0}}$ (respectively $\Lambda_{\bar{1}}$) is the even (respectively uneven or odd) subspace, consisting of linear combinations of products of an even (respectively odd) number of generators. This provides a decomposition of Λ into a direct sum of two subspaces each of dimension 2^{p-1}. Evidently $\Lambda_{\bar{0}}$, $\Lambda_{\bar{1}}$ and Λ itself are $\Lambda_{\bar{0}}$ -modules with respect to the action of pre-multiplication. Furthermore, elements of $\Lambda_{\bar{0}}$ commute with elements of Λ, and elements of $\Lambda_{\bar{1}}$ anticommute among themselves.

There are two further subspaces Λ_{num} and Λ_{nil} which are both subalgebras, and which deserve mention. Λ_{num} simply consists of scalar multiplies of the identity, its elements being termed "numeric". Λ_{nil} consists of linear combinations of products of the generators, and its elements are clearly nilpotent. As subspaces, Λ_{num} and Λ_{nil} have dimension 1 and 2^{p-1} respectively. We are thus able to speak not only of the even and odd parts of an element of Λ, but also of its numeric and nilpotent parts.

These definitions can be extended to $p = \infty$ if we consider Λ_∞ to be formal linear combinations of finite products of elements of a countably infinite number of independent anticommuting generators. For a more restrictive definition of Λ_∞ which can be given the structure of a Banach algebra and allows one to do calculus, see the paper of Rogers [4].

A supermatrix over F has the block from

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1}$$

where the entries of the submatrices A, B, C and D belong to F. We denote by $M(m,n;F)$ the F-algebra of (m,n)-supermatrices, where the submatrices A and D, have sizes $m \times m$ and $n \times n$ respectively. $M(m,n;F)$ is a Z_2-graded associative algebra. It becomes a Lie superalgebra by defining the superbracket operation by

$$[M,N\} = MN - (-1)^{|M||N|}NM \tag{2}$$

for all homogeneous elements M and N and then it is denoted by $gl(m,n;F)$.

We denote by $Q(m;F)$ the subalgebra $M(m,m;F)$ consisting of supermatrices of the form

$$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix}. \tag{3}$$

It is a Lie superalgebra with superbracket operation defined for homogeneous elements as in (2).

The supertrace function on $gl(m,n;F)$ is the map

$$str : gl(m,n;F) \mapsto F\ , \text{ with } str\ M = tr\, A - tr\, D, \tag{4}$$

for any supermatrix M given by (1) , where tr is the usual trace function, namely the sum of diagonal elements. The key property of supertrace is that it vanishes on all super-commutators, that is:

$$str\,([M,N\}) = 0, \tag{5}$$

for all $M, N \in gl(m,n;F)$.

The supertrace function on $Q(m;F)$ is the map

$$str : Q(m;F) \mapsto F\,,\ \text{ with } str\,M = tr\,B, \tag{6}$$

for any supermatrix M given by (3), where tr is the usual trace function.

The Grassmannification of the full matrix algebra $M(m;F)$ is the associative superalgebra $M(m;\Lambda) = \Lambda \otimes M(m;F)$. Its even part $\Lambda_{\bar{0}} \otimes M(m;F)$ consists of $m \times m$ matrices with entries in $\Lambda_{\bar{0}}$, whereas its odd part $\Lambda_{\bar{1}} \otimes M(m;F)$ consists of $m \times m$ matrices with entries in $\Lambda_{\bar{1}}$. The superbracket operation is defined for homogeneous elements P and Q, by

$$[P,Q\} = PQ - (-1)^{|P||Q|}QP, \tag{7}$$

where the grade function $|.|$ takes value 0 or 1 according as the argument is even or odd. Then $M(m;\Lambda)$ becomes a Lie superalgebra denoted by $gl(m;\Lambda)$.

The Grassmannification of the Lie superalgebra $M(m,n;F)$ is the full supermatrix superalgebra over Λ given by the tensor product $\Lambda \otimes M(m,n;F)$. Its even part is the so-called " Grassmann envelope " of the Lie superalgebra $\Lambda \otimes M(m,n;F)$. It is a Lie algebra by virtue of being the even component of a Lie superalgebra and it is denoted by $M(m,n;\Lambda)$. The arbitrary element in $M(m,n;\Lambda)$ is an even (m,n)-supermatrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{8}$$

where the entries of A and D (respectively B and C) belong to $\Lambda_{\bar{0}}$ (respectively $\Lambda_{\bar{1}}$).

The supertrace function on $M(m,n;\Lambda)$ is the map $str : M(m,n;\Lambda) \mapsto \Lambda$, defined by

$$str\,M = tr\,A - tr\,D, \tag{9}$$

for any supermatrix M given by (8). It is Λ-linear and $\Lambda_{\bar{0}}$-linear and vanishes on all commutators.

The Grassmannification of the Lie superalgebra $Q(m;F)$ is the tensor product $\Lambda \otimes Q(m;F)$. It has the structure if a graded Lie (left) Λ-module (see [[4]]), i.e. it is:

(i) a Lie superalgebra with supercommutator defined by the linear extension of the formula

$$[\alpha \otimes M, \beta \otimes N\} = (-1)^{|\beta||M|}\alpha\beta \otimes [M,N\}\,, \tag{10}$$

for all homogeneous elements $\alpha, \beta \in \Lambda_{\bar{0}} \cup \Lambda_{\bar{1}}$ and for all homogeneous supermatrices M, N in $M(m,n;F)_{\bar{0}} \cup M(m,n;F)_{\bar{1}}$, where the grade function $|\ |$ takes value 0 or 1 according as the argument is even or odd.

(ii) a graded left Λ-module in the usual sense with operation defined by

$$\alpha \cdot (\beta \otimes M) = \alpha\beta \otimes M, \tag{11}$$

for all $\alpha, \beta \in \Lambda$ and $M \in Q(m;F)$

(iii)

$$\left[\alpha\tilde{M}, \tilde{N}\right\} = \alpha \left[\tilde{M}, \tilde{N}\right\}, \tag{12}$$

for all $\alpha \in \Lambda$ and $\tilde{M}, \tilde{N}$ in $\Lambda \otimes Q(m;F)$.

We note that the structure of a graded Lie (left) Λ-module can also be given on the tensor product $\Lambda \otimes M(m, n; F)$ with operations defined as in (10) and (11).

The even component of $\Lambda \otimes Q(m; F)$, denoted by $Q(m; \Lambda)$ is a Lie algebra with arbitrary element

$$M = \left[\begin{array}{cc} A & B \\ B & A \end{array} \right], \tag{13}$$

where the entries of A (B respectively) belong to $\Lambda_{\bar{0}}$ ($\Lambda_{\bar{1}}$ respectively).

The known supertrace on $M(m, n; \Lambda)$ is identically zero on $Q(m; \Lambda)$. There are, however, known invariants defined on $Q(m; \Lambda)$, as the *queertrace* (see [2]), qtr : $Q(m; \Lambda) \mapsto \Lambda$ defined by

$$qtr \left(\left[\begin{array}{cc} A & B \\ B & A \end{array} \right] \right) = tr\, B \tag{14}$$

and the so-called ω - *supertrace* (see [5]), $str_\omega : Q(m; \Lambda) \mapsto \Lambda$ defined by

$$str_\omega \left(\left[\begin{array}{cc} A & B \\ B & A \end{array} \right] \right) = \omega tr\, B, \tag{15}$$

where $\omega \in \Lambda_{\bar{1}}$ is some fixed odd element. The factor ω is there to ensure the image is in $\Lambda_{\bar{0}}$, although this is not important.

Later, we prove that the supertrace function on $Q(m; F)$ is the unique, up to a scalar multiplier, F - linear function vanishing on all supercommutators. Moreover, we prove that the queertrace on $Q(m; \Lambda)$ is also unique, up to a Grassmann multiplier, as a Λ - linear function vanishing on all commutators.

The paper is set as follows. In section 2 we state and give a proof of the uniqueness of the supertrace function for Q-type supermatrices over F. The extension of supertrace to the Lie algebra $Q(m; \Lambda)$ is explored in section 3, the main result being given in Theorem 3.1.

2 THE SUPERTRACE ON $Q(m; F)$

The Lie superalgebra $Q(m; F)$ of Q -type supermatrices over the field F, consists of supermatrices of the form (3), while the superbracket operation is defined by

$$[M, N\} = MN - (-1)^{|M||N|} NM, \tag{16}$$

for homogeneous M, N in $Q(m; F)$.

A basis of $Q(m; F)$ consist of

$$X_{ij} = \left[\begin{array}{cc} \varepsilon_{ij} & 0 \\ 0 & \varepsilon_{ij} \end{array} \right], \; Y_{ij} = \left[\begin{array}{cc} 0 & \varepsilon_{ij} \\ \varepsilon_{ij} & 0 \end{array} \right], \tag{17}$$

for $1 \leq i, j \leq m$. Its elements satisfy the following nonzero commutation relations

$$[X_{ij}, X_{kl}\} = \delta_{jk} X_{il} - \delta_{il} X_{kj}, \tag{18}$$

$$[X_{ij}, Y_{kl}\} = \delta_{jk} Y_{il} - \delta_{il} Y_{kj}, \tag{19}$$

$$[X_{ij}, Y_{kl}\} = \delta_{jk} X_{il} - \delta_{il} X_{kj}. \tag{20}$$

The ordinary supertrace vanishes identically on $Q(m;F)$. We can therefore ask whether there is a non-trivial functional on $Q(m;F)$ which vanish on all supercommutators. Thus we have:

Theorem 2.1 *Let $f : Q(m;F) \mapsto F$ be a map such that:*

(i) *it is F-linear;*

(ii) *it vanishes on all supercommutators.*

Then, for every M in $Q(m;F)$ given by (3),

$$f(M) = \lambda tr\, B, \tag{21}$$

where λ is independent of M.

Proof. From the commutation relations (18) - (20), for specific values of the indices, we find:

$$\begin{aligned} X_{ij} &= [X_{ii}, X_{ij}\}\,, i \neq j, && (22)\\ X_{ii} - X_{jj} &= [X_{ij}, X_{ji}\}\,, && (23)\\ X_{ii} &= \frac{1}{2}[Y_{ii}, Y_{ii}\}\,, i \neq j, && (24)\\ Y_{ij} &= [X_{ii}, Y_{ij}\}\,, i \neq j, && (25)\\ Y_{ii} - Y_{jj} &= [X_{ij}, Y_{ji}\}\,. && (26) \end{aligned}$$

From the invariance condition of the theorem and the F-linearity of f we find:

$$\begin{aligned} f(X_{ij}) &= 0, \quad 1 \leq i,j \leq m, && (27)\\ f(Y_{ij}) &= 0, \quad 1 \leq i,j \leq m, i \neq j, && (28)\\ f(Y_{ii}) &= f(Y_{ij}), \quad 1 \leq i,j \leq m. && (29) \end{aligned}$$

We can write arbitrary supermatrix M in $Q(m;F)$ as a linear combination of the elements of the XY-basis. Using F-linearity of f and the identities (27) - (29) we find

$$f(M) = \varphi(Y_{ii}) tr\, B = \lambda tr\, B,$$

where $\lambda = f(Y_{ii})$ in F is fixed and independent of M.

3 THE SUPERTRACE ON $Q(m;\Lambda)$

The Lie algebra $Q(m;\Lambda)$ is the even component of the tensor product $\Lambda \otimes Q(m;F)$ which has the structure of a graded Lie (left) Λ - module. Its arbitrary element is given by (13). A free-basis for $\Lambda \otimes Q(m;F)$ is given by

$$\tilde{X}_{ij} = 1 \otimes X_{ij}\,, \quad \tilde{Y}_{ij} = 1 \otimes Y_{ij}\,,\ 1 \leq i,j \leq m, \tag{30}$$

where X_{ij}, Y_{ij} , $1 \leq i, j \leq m$ is the XY-basis of $Q(m; F)$ given by (17), and satisfies the same set of commutation relations (18) - (20) as the XY-basis.
Let now $f : Q(m; \Lambda) \mapsto \Lambda$ be a Λ-linear function vanishing on all commutators in $Q(m; \Lambda)$. Working as in Theorem 2.1, we find that

$$\begin{aligned} f(\tilde{X}_{ij}) &= 0, \ 1 \leq i, j \leq m, && (31)\\ f(\tilde{Y}_{ij}) &= 0, \ 1 \leq i, j \leq m, \ i \neq j, && (32)\\ f(\tilde{Y}_{ii}) &= f(\tilde{Y}_{jj}), \ 1 \leq i, j \leq m. && (33) \end{aligned}$$

The arbitrary supermatrix M in $Q(m; \Lambda)$ can be written as a linear combination of the elements of the $\tilde{X}\tilde{Y}$-basis. Using Λ-linearity and the identities (31) - (33) we find:

$$f(M) = f(\tilde{Y}_{ii}) tr\, B = \alpha tr\, B, \tag{34}$$

where $\alpha = f(\tilde{Y}_{ii}) \in \Lambda$ is a fixed Grassmann number independent of M.

Thus we have proved the following:

Theorem 3.1 *Let $f : Q(m; \Lambda) \mapsto \Lambda$ be a Λ-linear function vanishing on all commutators. Then, for every supermatrix M given by (13), we have*

$$f(M) = \alpha tr\, B\,, \tag{35}$$

where $\alpha \in \Lambda$ is a fixed and independent of M.

4 CONCLUSION

We have proved that

(i) the known supertrace function on the Lie superalgebra $Q(m; F)$ of the so-called Q-type matrices is unique, up to a scalar multiplier, among the F-linear functions vanishing on all supercommutators.

(ii) the more general function of the form $f : Q(m; \Lambda) \mapsto \Lambda$, which is Λ-linear and vanishes on all commutators is a Grassmann multiple of the trace of the off-diagonal block of the supermatrix M in $Q(m; \Lambda)$. It is a form which contains as special cases both known supertraces, the queertrace and the ω-supertrace. Thus, the known supertrace on $Q(m; \Lambda)$ is essentially unique.

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