ON H-STRUCTURE MANIFOLDS II

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Abstract

On this communication we prove that every $2m$- dimensional connected NK-manifold of pontwise constant holomorphic sectional curvature is an Einstein manifold.

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1 Introduction

Let F be a $(1,1)$ tensor field on a C^∞ n-dimensional differentiable manifold M, such that

$$F^2(X) = a^2 X, \tag{1}$$

where a is a real or a purely imaginary number and X an arbitrary vector field on M. Clearly, F is an endomorphism of the tangent space $T_p(M)$, for every point $p \in M$. F gives a differentiable structure on M called GF-structure defined by (1).

If $a = 0$, we have an almost tangent structure.

If $a \neq 0$, we have a π-structure which is known from G. Legrand in [4], (1956).

Especially if $a^2 = 1$, we have an almost product structure, if $a^2 = -1$ we have the well known almost complex structure, $(J^2X = -X)$.

If the above mentioned GF-structure F is equipped with a Hermitian metric g such that

$$g(FX, FY) + a^2 g(X, Y) = 0,$$

for any vector fields X, Y on M, then (g, F) is called H-structure and M is said to be H-structure manifold. Many authors studied H-structure manifolds: K. L. Duggal in [1] and [2] was of the first ones.

If the structure tensor F is parallel (i.e. $(\nabla_X F)\, Y = 0$, where ∇ is the Riemannian connection), then M is called K-manifold.

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If the structure tensor F satisfies the condition $(\nabla_X F)\,Y = 0$, for arbitrary vector field X on M, then M is called nearly K-manifold (briefly NK-manifold).

In the present communication we deal with some $2m$- dimensional H-structure manifolds. In the second section we shall give some useful preliminaries. In the last section we shall give the main result of the present communication, which are referred in [1].

2 Preliminaries

Let Φ be $(0,2)$-tensor on a $2m$-dimensional H-structure manifold M such that

$$\Phi(X,Y) = g(FX,Y) = -g(X,FY). \tag{2}$$

Using (2) we can prove:

$$\Phi(X,Y) + \Phi(Y,X) = 0, \tag{3}$$

$$\Phi(FX,FY) + a^2\Phi(X,Y) = 0, \tag{4}$$

$$(\nabla_X\Phi)\,(Y,Z) + (\nabla_X\Phi)\,(Z,Y) = 0, \tag{5}$$

$$(\nabla_X\Phi)\,(FY,FZ) = a^2\,(\nabla_X\Phi)\,(Y,Z). \tag{6}$$

Denoting by $(W,X,Y,Z) = g((\nabla_W F)\,X, (\nabla_Y F)\,Z)$, the properties (3) and (4) give:

$$\begin{aligned} (W,X,Y,Z) &= (Y,Z,W,X), \\ (W,FX,Y,FZ) &= -a^2(W,X,Y,Z), \\ (W,FX,Y,Z) &= -(W,X,\ Y,\ FZ). \end{aligned} \tag{7}$$

It is well known that the curvature tensor R is defined by:

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X,\nabla_Y]\,Z.$$

We denote by

$$R(W,X,Y,Z) = g(R_{WX}Y,Z),$$

for arbitrary vector fields W, X, Y and Z on M.

It is also known that the holomorphic sectional curvature $H(x)$ is defined by

$$H(x) = R(x,Fx,x,Fx)/g(x,x)g(Fx,Fx),$$

for $x \in T_P(M)$, $(p \in M)$.

Let $\{E_1,\dots,E_m,E_{m+1},\dots,E_{2m}\}$ be an orthonormal frame field such that:

$$E_{m+i} = \sqrt{-1}E_i/a,\ (i = 1,\dots,m)\,.$$

We denote by r and r^* the Ricci tensor and the Ricci $*$tensor on M, respectively. The Ricci $*$tensor r^* is defined by

$$r^*(x,y) = trace\ of\ (z \to R(Fz,x)Fy),$$

for $x,y,z \in T_p(M)$, $p \in M$.

3 Main results

In the present section we shall state the main results of the present communication.

Theorem 1 *Let M be an H-structure manifold of pointwise constant holomorphic sectional curvature $c(p)$. Then*

$$4a^2c(p)[-2\Phi(x,y)Fw-\Phi(x,w)Fy+\Phi(y,w)Fx+a^2g(x,w)y-a^2g(y,w)x]=$$
$$=-3a^4R(w,x)y+3FR(Fw,Fx)Fy+a^2R(Fw,Fx)y-a^2FR(w,x)Fy-$$
$$a^2R(Fw,x)Fy-3a^2FR(Fw,x)y+3a^2R(w,Fx)Fy+a^2FR(w,Fx)y+$$
$$+3a^4R(w,y)x-3FR(Fw,Fy)Fx-a^2R(Fw,Fy)x+a^2FR(w,y)Fx+$$
$$+a^2R(Fw,y)Fx+3a^2FR(Fw,y)x-3a^2R(w,Fy)Fx-a^2FR(w,Fy)x.$$

We now state some lemmas.

Lemma 1 *If M is an H-structure manifold and $\{E_i\}$ is an orthonormal frame field, for every vector field X we have:*

$$\sum_{i=1}^{2m}\left[R\left(X,FE_i\right)FE_i+a^2R(X,E_i)E_i\right]=0,$$

$$\sum_{i=1}^{2m}\left[R\left(X,E_i\right)FE_i+a^2R(X,FE_i)E_i\right]=0.$$

Lemma 2 *Let M be a H-structure manifold. Then for arbitrary vector fields X, Y on M we have:*

$$r(X,Y)=r(Y,X),$$
$$r^*(FX,FY)=-a^2r^*(Y,X),$$
$$r(FX,Y)=-r^2(FY,X).$$

If s and s^* are the scalar and the $*$scalar curvatures of M respectively, then we have:

Proposition 1 *If M be a $2m$-dimensional H-structure manifold of pointwise constant holomorphic sectional curvature $c(p)$, then for arbitrary vector fields X, Y on M, we have:*

$$a^2r(X,Y)-r(FX,FY)-3\left[r^*(X,Y)+r^*(Y,X)\right]=$$
$$=4(m+1)c(p)a^2g(X,Y),$$
$$a^2s-3s=4m(m+1)a^2c(p).$$

The results of the theorem 1 and the proposition 1 have been obtained by G. B. Rizza for $a=-1$,([5]).

For every NK-manifold we have:

$$r(X,Y)=(a^2-3)^{-1}\sum_{i=1}^{2m}\left(X,E_i,Y,E_i\right),$$

$$r(FX, FY) = -a^2 r\,(X, Y)\,,$$
$$r^*(X, Y) = r^*(Y, X),$$
$$r^*\,(X, Y) = (a^2 - 3)^{-1} a^2 r\,(X, Y)\,.$$

Using the above relations we can obtain the following:

Theorem 2 *If M is a $2m$-dimensional connected NK-manifold of pointwise constant holomorphic sectional curvature, then M is an Einstein manifold.*

References

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