AFFINE LIE ALGEBRAS AND HYPERSTRUCTURES

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Abstract

Hyperstructures are used in order to organize affine Lie Algebras. More precisely, the H_v-structures are used in the principal vertex operator construction of the Affine Kac-Moody Lie Algebras.

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Some hyperstructures have already been used as organized devices in several branches of mathematics. For example, the irreducible characters of finite groups form canonical hypergroups. New classes and generalizations of hyperstructures give more opportunities towards this direction. A generalization of the classic hyperstructures is the class of the H_v-structures where the equality in several axioms is replaced by the non-empty intersection. In this paper we present a way how the affine Kac-Moody Lie algebras can be viewed as H_v -structures.

1 THE H_v-STRUCTURES

A hyperoperation $(\cdot)$ defined on the set H is called *weak associative*, we write WASS, if $(xy)z \cap x(yz) \neq \emptyset$ for all $x, y, z \in H$. $(\cdot)$ is called *weak commutative*, we write WCO, if $xy \cap yx \neq \emptyset$ for all $x, y \in H$. In the same sense the other basic properties can be replaced by the weak ones, i.e. the equality is replaced by the non-empty intersection. The new hyperstructures introduced in [7] are called H_v-structures. One can generalize the classical hyperstructures and several properties can be obtained, see [1], [3], [6], [7]. The motivating example is the quotient of a structure by an equivalence relation. The H_v-group (H_v-ring, H_v-vector space) is the hyperstructure which satisfies the group (ring, vector space, respectively) like axioms.

Every H_v-structure "hides" a corresponding structure. This structure is obtained from the H_v -structure by quotient out by the fundamental relation β^*, γ^* or ε^*.

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Therefore, if H is a H_v -group (H_v-ring, H_v-vector space) then H/β^* is a group (H/γ^* is a ring, H/ε^* is a vector space, resp.). The above corresponding structures are the fundamental ones. The fundamental relation β^* in a H_v-group $(H,\cdot)$ equivalently can be defined as follows:

An element $a \in H$ is called β equivalent to the element $b \in H$ if there exists a finite set of elements $\{z_1, \ldots, z_n\}$ of H such that $\{a,b\} \in z_1 \cdots z_n$. Then the transitive closure of β is the β^*.

In a similar way the γ^* is defined in H_v-rings and the ε^* is defined in H_v-vector spaces. Using this analytic construction one can also define analogous fundamental relations in weak and partial hyperoperations. This is the ones used in this paper and we denote it by β_b^*.

2 ON AFFINE LIE ALGEBRAS

Recall the basic construction for the affine Lie algebras given by Kac in [2].

Let $\mathbf{g}'(A)$ be an affine Lie algebra of type $X_N^{(r)}$ corresponding to the finite-dimensional Lie algebra $\mathbf{g}$ of type X_N . Consider the elements E_i , F_i , H_i $(i = 0, \ldots, \ell)$, which are the Chevalley generators, such that the relations

$$degE_i = -degF_i = 1, \quad degH = 0 \; (i = 1, ..., \ell)$$

define a $Z/h^{(r)}Z$-gradation $\mathbf{g} = \bigoplus_i \mathbf{g}_i(1;r)$ called the r-principal gradation of $\mathbf{g}$. Note that $h^{(r)} = r \sum_{i=0}^{\ell} a_1$ is the Coxeter number of $\mathbf{g}$, where a_i be the labels of the diagram of the affine matrix $X_N^{(r)}$. Take the r-cyclic element of $\mathbf{g}$, $E = \sum_{i=0}^{\ell} E_i$ and denote by $S^{(r)}$ the centralizer of E in $\mathbf{g}$. It is graded with respect to the r-principal gradation

$$S^{(r)} = \bigoplus_{i \in Z/h^{(r)}Z} S_j^{(r)}$$

and the relation $\dim g_j(1;r) = \ell + \dim S_j^{(r)}$ $(j \in Z/h^{(r)Z})$ is valid.
Although there is no general way to normalize the basis of $S^{(r)}$ we can fix a normalized, with respect to the standard invariant form, basis of $S^{(r)}$. Denote by $T_{i,j}$ $(i = 1, \ldots, t_j)$ $(t_j = \dim S_j^{(r)})$ the homogeneous components of degree j.

Finally, consider a set of square matrices A_α $(\alpha = 0, ..., \ell)$ such that the homogeneous components $A_{\alpha,j}$ of them together with the homogeneous components $T_{i,j}$ $(i = 1, \ldots, t_j)$ $(t_j = \dim S_j^{(r)})$ of the $S^{(r)}$, should form a basis of $\mathbf{g}$.

The above realization of the basic representation is called the *principal vertex operator construction*. In the following we write down the explicit formulas for the principal vertex operator construction of the affine algebra $g'(A)$ of type $A_{n-1}^{(1)}$ $(n \geq 2)$ and $D_n^{(1)}$ $(n \geq 4)$, see respectively [2] and [4], [5].

3 THE CASE $A_{n-1}^{(1)}$ $(n \geq 2)$

Let E_{ij} $(i, j = 1, \ldots, n)$ denote the $n \times n$ matrix which is 1 in the i, j-entry and 0 everywhere else. Take the elements

$$\left.\begin{array}{ll} E_0 = E_{n1}, & E_i = E_{i,i+1} \\ F_0 = E_{n1}, & F_i = E_{i+1,i} \\ H_0 = E_{nn} - E_{11}, & H_i = E_{i,i} - E_{i+1,i+1} \end{array}\right\} \quad (i = 1, \ldots, n-1)\,.$$

The 1-principal Z/nZ-gradation of $\mathbf{g}$ is given by setting $\deg E_{ij} = j - i$ for $i \neq j$ and $\deg D = 0$ for a traceless diagonal matrix D.
Set

$$E = \sum_{i=0}^{n-1} E_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and let S be the centalizer of E in $\mathbf{g}$. Then a basis of S is

$$T_j = E^j \ \ (j = 1, \ldots, n-1) \quad \textit{and} \quad \deg T_j = j \, mod \, n\,.$$

For two distinct n-th roots of unity, ε and η, define $n \times n$ matrices $A_{(\varepsilon,\eta)} = (\varepsilon^i \eta^{-j})_{i,j=1}^n$. Let $A_{(\varepsilon,\eta),j}$ be the homogeneous components of the $A_{(\varepsilon,\eta)}$. The T_k together with the homogeneous components of $A_{(\varepsilon,1)}$ form a basis of $\mathbf{g}$. We denote the elements of $A_{(\varepsilon,1)}$ by $A_{\alpha,j}$.

4 THE CASE $D_n^{(1)}$ $(n \geq 4)$

Consider the $2n \times 2n$ matrices then take the Chevalley generators given, for $i = 1, \ldots, n-1$, as follows

$$\left[\begin{array}{l} E_0 = E_{2n-1,1} - E_{2n,2}, \ E_i = E_{i,i+1} - E_{2n-1,2n-i+1}\ , \\ E_n = E_{n-1,n+1} - E_{n,n+2}\ , \\ \\ F_0 = E_{1,2n-1} - E_{2,2n}\ , \ F_i = E_{i+1,1} - E_{2n-i+1,2n-i}\ , \\ F_n = E_{n+1,n-i} - E_{n+2,n}\ , \\ \\ H_0 = E_{2n,2n} + E_{2n-1,2n-1} - E_{22} - E_{11}\ , \\ H_i = E_{2n-i,2n-i} + E_{ii} - E_{2n-i+1,2n-i+1} - E_{i+1,i+1}\ , \\ H_n = E_{nn} + E_{n-1,n-1} - E_{n+2,n+2} - E_{n+1,n+1}\ . \end{array}\right.$$

Denote by $\underline{k}$ the number k if $k \leq n$ and $k-1$ if $k > n$ for every $k \in Z$. Moreover, let $h = 2(n-1)$ be the Coxeter number. The 1-principal Z/hZ-gradation of $\mathbf{g}$ is given by setting

$$\deg E_{ij} = \big(\underline{j} - \underline{i}\big) \, mod \, h\ .$$

In a similar way, as in the above case $A_{n-1}^{(1)}$ $(n \geq 2)$, we obtain a of the canonical part of $\mathbf{g}$ (see [4]). Furthermore, a normalized basis of the centralizer of E is given as follows:

For $n = 2k$:

$$T_s = \frac{1}{\sqrt{h}} E^{2s-1} \quad for\ s = 1, \ldots, k-1;$$

$$T_s = {}^t T_{n+1-s} = \frac{1}{\sqrt{h}} E^{2s-3} \quad for\ s = k+2, \ldots, n;$$

$$T_k = \frac{1}{\sqrt{2}} E^{n-1} + \frac{i}{4} E_0,\ T_{k+1} = \frac{1}{\sqrt{2h}} E^{n-1} - \frac{i}{4} E_0,\ where\ i^2 = -1.$$

For $n = 2k+1$:

$$T_s = \frac{1}{\sqrt{h}} E^{2s-1} \quad for\ s = 1, \ldots, k; \quad T_{k+1} = \frac{1}{2\sqrt{2}} E_0;$$

$$T_s = -{}^t T_{n+1-s} = \frac{1}{\sqrt{h}} E^{2s-3} \quad for\ s = k+2, \ldots, n.$$

5 TWO HYPEROPERATIONS

Using the notation of the above sections 3 and 4, we define the following two "normalizing" hyperproducts $(\odot)$ and $(*)$ on $A_{\alpha,j}$ and $S_j^{(r)}$ as follows:

$$\left[\begin{array}{l} A_{\alpha,i} \odot A_{\beta,j} = \left\{A_{\alpha,i} A_{\beta,j}, T_{1,i+j}, \ldots, T_{t_{i+j},i+j}\right\}, \\ A_{\alpha,i} \odot T_{p,j} = \left\{A_{\alpha,i} T_{p,j}, T_{1,i+j}, \ldots, T_{t_{i+j},i+j}\right\}, \\ T_{p,j} \odot A_{\alpha,i} = \left\{T_{p,j} A_{\alpha,i}, T_{1,i+j}, \ldots, T_{t_{i+j},i+j}\right\}, \\ T_{p,j} \odot T_{q,i} = \left\{T_{1,i+j}, \ldots, T_{t_{i+j},i+j}\right\}, \end{array}\right.$$

where $A_{\alpha,i},\ A_{\beta,j} \in \{A_{\alpha,i} \mid \alpha = 0, \ldots, \ell,\ i \in Z/h^{(r)}Z\}$
and $T_{p,j},\ T_{q,i} \in \{T_{i,j} \mid i = 1, \ldots, t_j,\ t_j = \dim S_j^{(r)}\}$.
Note that if $S_j^{(r)} = \emptyset$ for some j, then $(\odot)$ is partial hyperoperation.

For the second hyperproduct $(*)$ we pick up and fix an element $A_{\sigma,i}$ of every degree i.

$$Y_{\alpha,i} * Y_{\beta,j} = Y_{\alpha,i} \odot Y_{\beta,j} \cup \{A_{\sigma,i+j}\},$$

where $Y_{\alpha,i},\ Y_{\beta,j}$are elements of the basis of degree i and j respectively.

In the case $A_{n-1}^{(1)}$ $(n \geq 2)$ we have $\dim S_j^{(r)} = 1$ for all $j = 1, \ldots, n-1$ thus, the hyperoperation $\odot$ is partial only in the case $T_{p,j} \odot T_{q,i}$, where $i + j = 0\, mod\, n$.

In the case $D_n^{(1)}$ $(n \geq 4)$ we have the following:

(a) If $n = 2k+1$ then $\dim S_j^{(r)} = 1$, $j \in \{1, 3, ..., n-2, n-1, n, ..., 2n-3\}$ and in the rest cases $\dim S_j^{(r)} = 0$,

(b) If $n = 2k$ then $\dim S_j^{(r)} = 1$, $j \in \{1, 3, ..., n-3, n+1, n+3, ..., 2n-3\}$ and $\dim S_{n-1}^{(r)} = 2$ and in the rest cases $\dim S_j^{(r)} = 1$.

Theorem 1 *The hyperoperations* ($\odot$) *and* ($*$) *are WASS in* $\mathbf{g}$.

Proof. We have

$$(A_{\alpha,i} \odot A_{\beta,j}) \odot A_{\gamma,k} = \left\{A_{\alpha,i}A_{\beta,j}, T_{1,i+j}, \ldots, T_{t_{i+j},i+j}\right\} \odot A_{\gamma,k} =$$

$$\left\{A_{\alpha,i}A_{\beta,j}A_{\gamma,k}, T_{1,i+j}A_{\gamma,k}, \ldots, T_{t_{i+j},i+j}A_{\gamma,k}, T_{1,i+j+k}, \ldots, T_{t_{i+j+k},i+j+k}\right\}$$

and on the other hand

$$A_{\alpha,i} \odot (A_{\beta,j} \odot A_{\gamma,k}) = A_{\alpha,i} \odot \left\{A_{\beta,j}A_{\gamma,k}, T_{1,j+k}, \ldots, T_{t_{j+k},j+k}\right\} =$$

$$\left\{A_{\alpha,i}A_{\beta,j}A_{\gamma,k}, A_{\alpha,i}T_{1,j+k}, \ldots, A_{\alpha,i}T_{t_{i+j},i+j}T_{1,i+j+k}, \ldots, T_{t_{i+j+k},i+j+k}\right\}.$$

Therefore ($\odot$) is WASS.
The hyperoperation ($*$) is also WASS since it is greater than ($\odot$).□

The element E shifts the gradation by 1 in both hyperoperations ($\odot$) and ($*$). Therefore "orbits" of the elements of the basis, of degree 1, are obtained. These "orbits" are normalized in the sense that in the same degree they have the same length. This normalization is transferred in every non-zero element of $\mathbf{g}$.

Theorem 2 *Let* β_b^* *be the fundamental relation obtained by using elements on one only level. Then the set of fundamental classes is isomorphic to* $Z/h^{(r)}Z$.

Proof. Take $S_{j_0}^{(r)}$ such that $\dim S_{j_0}^{(r)} \neq 0$. Then, the product which has degree j_0 contains all the elements of $S_{j_0}^{(r)}$. This means that the β_b^*-class of every element of degree j_0 is the subspace of degree j_0. For the rest degrees one has simply to shift the degree by using the element E. □

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