ŢIŢEICA INDICATRIX AND FIGURATRIX

C. Udrişte, O. Şandru, C. Dumitrescu and A. Zlătescu

Abstract

Our paper starts from the remark that the indicatrices of some Finsler spaces, which appear in the study of cristals, are Ţiţeica hypersurfaces.

It is proved that the indicatrices of a Finsler space are Ţiţeica hypersurfaces if the function $a(x, y) = \det(a_{ij}(x, y))$ is constant (with respect to y) on every indicatrix, where $a_{ij}(x, y)$ is the metric d-tensor.

The dual of a Finsler space is a Miron space. Figuratrices of a Miron space are Ţiţeica hypersurfaces if the function $g(x, p) = \det(g^{ij}(x, p))$ is constant (with respect to p) on every figuratrix, where $g^{ij}(x, p)$ is the fundamental d-tensor of the Miron space.

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Preliminaries

Let $F^n = (R^n, \mathcal{F})$ be a Finsler space. By x we shall denote the elements of the R^n space, and by $y_x = (y^1, ..., y^n)_x$ the canonical coordinate system of the tangent space R^n_x at every point $x \in R^n$. Under these conditions we know that the fundamental Finsler function $\mathcal{F}(x, y)$ is a $1(p)$ - homogeneous function, with respect to y.

The constant level set $\mathcal{I}_x = \{y \in R^n_x / \mathcal{F}^2(x, y) = 1\}$ is a hypersurface of the Euclidean space R^n_x, because

$$\nabla \mathcal{F}^2 = 2\mathcal{F}\left(\frac{\partial \mathcal{F}}{\partial y^1}, \ldots, \frac{\partial \mathcal{F}}{\partial y^n}\right) \neq 0, \quad \text{on} \quad \mathcal{I}_x.$$

Of course, if $\nabla \mathcal{F}^2 = 0$ at a point $y \in \mathcal{I}_x$, then the Hessian of the function $\mathcal{F}^2$ would not be nondegenerate on $R^n_x \setminus \{0\}$. The hypersurface $\mathcal{I}_x$ is called the *indicatrix* of the Finsler space F^n. We orient this indicatrix by choosing the unit normal vector field $N = -\nabla \mathcal{F}^2 / \|\nabla \mathcal{F}^2\|$.

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By $T_y\mathcal{I}_x$ we shall denote the tangent space to the indicatrix $\mathcal{I}_x$ at the point $y = (y^1, ..., y^n)$. As $T_y\mathcal{I}_x$ is an $(n-1)$-dimensional subspace of R^n_x, it will be called the *tangent hyperplane* to $\mathcal{I}_x$ at the point y.

The Weingarten mapping $S : T_y\mathcal{I}_x \to T_y\mathcal{I}_x$ defined by $S(v) = -D_vN$, $v \in T_y\mathcal{I}_x$, where D_v represents the derivative with respect to the vector v, is a linear transformation.

The Gauss curvature $K(y)$ of the hypersurface $\mathcal{I}_x$ at the point $y \in \mathcal{I}_x$ is usually defined, as being the determinant of Weingarten mapping S. For proving that the given definition of the Gauss curvature is correct, we underline the fact that the determinant of S is independent of the base in which the matrix of S is expressed.

Lemma 1 [2]. *Consider V a Euclidean and n-dimensional vector space, $W \subset V$ an $(n-1)$-dimensional vector subspace of V and $S : W \to W$ a linear transformation.*

If the scalar product $< u, S(u) >$, $u, v \in W$ is defined by the matrix $A = (a_{ij})$, that means $< u, S(v) >= {}^tuAv$, then the determinant of S is given by

$$K = -\begin{vmatrix} A & N \\ {}^tN & 0 \end{vmatrix},$$

where N is the normal versor of the hyperplane W.

Proof. Consider that $\{X_1, X_2, ..., X_{n-1}\}$ is a basis of the space W so that $\{X_1, ..., X_{n-1}, N\}$ is an orthonormal basis of the space V. Let us denote by $B = (b^{\alpha\beta})_{\alpha,\beta=\overline{1,n-1}}$ the $(n-1)\times(n-1)$ matrix of S with respect to the basis $\{X_1, ..., X_{n-1}\}$. Obviously $B = (< X_\alpha, S(X_\beta) >)$.

Denoting by X the orthogonal matrix $(X_1, X_2, ..., X_{n-1}, N)$ of type $n \times n$, we define the matrices $\tilde{A}$ and $\tilde{X}$ of the type $(n+1) \times (n+1)$ by

$$\tilde{A} = \begin{pmatrix} A & N \\ {}^tN & 0 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

Because

$${}^t\tilde{X}\tilde{A}\tilde{X} = \begin{pmatrix} {}^tXAX & {}^tXAN & 0 \\ {}^tNAX & {}^tNAN & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we find

$$\tilde{A} = -\det({}^tXAX) = -\det(< X_\alpha, S(X_\beta) >) =$$
$$= -\det(b^{\alpha\beta}) = -\det S.$$

Lemma 2. *For every $u = (u^i)$, $v = (v^i) \in T_y\mathcal{I}_x$, we have the formula*

$$< u, S(v) >= a_{ij}u^iv^j / \left(\sum_{k=1}^n l_k^2\right)^{\frac{1}{2}}.$$

Proof. We denote

Obviously $\mathcal{I}_x = \{y \in R^n_x / f(x,y) = 0\}$. On $\mathcal{I}_x$ we find $\nabla f = (l_i)$. Therefore $N = -\nabla f / \|\nabla f\|$. Under these circumstances we have successively

$$< u, S(v) >=< u, -D_v N >=< u, D_v(\nabla f / \|f\|) > .$$

But

$$D_v = v^j \frac{\partial}{\partial y^i}, \quad \text{so} \quad D_v \left(\frac{f_i}{\|\nabla f\|} \right) = v^j \frac{f_{ij}}{\|\nabla f\|} + f_i D_v \frac{1}{\|\nabla f\|}.$$

Considering $u \in T_y \mathcal{I}_x$ we have $< u, \nabla f >= 0$, that means

$$< u, D_v \frac{\nabla f}{\|\nabla f\|} >= \frac{f_{ij} u^i v^j}{\|\nabla f\|} = \frac{a_{ij} u^i v^j}{(\sum l_k^2)^{\frac{1}{2}}}.$$

1 Ţiţeica - Finsler Indicatrix

Let $F^n = (R^n, \mathcal{F})$ be a Finsler space and $\mathcal{I}_x$ be the indicatrix of F^n at x.

Theorem 1. *The Gauss curvature $K(y)$ of the hypersurface $\mathcal{I}_x$ at the point y has the expression*

$$K(y) = \frac{a}{(\sum_{k=1}^n l_k^2)^{\frac{n+1}{2}}},$$

where

$$a_{ij} = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}, \quad a = \det(a_{ij}), \quad l_k = \frac{\partial \mathcal{F}}{\partial y^k}.$$

Proof. Since $N = -\nabla \mathcal{F}^2 / \|\nabla \mathcal{F}^2\| = -\nabla \mathcal{F} / \|\nabla \mathcal{F}\|$, the components of N are

$$l_i / \left(\sum_{k=1}^n (l_k)^2 \right)^{\frac{1}{2}}.$$

Consider $S : T_y \mathcal{I}_x \to T_y \mathcal{I}_x$ as the Weingarten mapping of the tangent hyperplane $T_y \mathcal{I}_x$.

According to lemma 2, the scalar product $< u, S(v) >$, $u, v \in T_y \mathcal{I}_x$, is defined by the matrix

$$\left(a_{ij} / \left(\sum_{k=1}^n (l_k)^2 \right)^{\frac{1}{2}} \right).$$

But the Gauss curvature $K(y)$ of the indicatrix $\mathcal{I}_x$ at the point $y \in \mathcal{I}_x$ is the determinant of the Weingarten mapping S, so applying Lemma 1, we obtain

$$K(y) = - \begin{vmatrix} a_{ij} / \left(\sum_{k=1}^n (l_k)^2 \right)^{\frac{1}{2}} & -l_i / \left(\sum_{k=1}^n (l_k)^2 \right)^{\frac{1}{2}} \\ -l_j / \left(\sum_{k=1}^n (l_k)^2 \right)^{\frac{1}{2}} & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} a_{ij} & l_i \\ l_j & 0 \end{vmatrix} / \left(\sum_k (l_k^2) \right)^{\frac{n+1}{2}} .$$

Because $\begin{vmatrix} a_{ij} & l_i \\ l_j & 0 \end{vmatrix} = -a$, we find

$$K(y) = a / \left(\sum_{k=1}^{n} (l_k)^2 \right)^{\frac{n+1}{2}} .$$

Theorem 2. *Indicatrices of a Finsler space $F^n = (R^n, \mathcal{F})$ are Ţiţeica hypersurfaces if for every $x \in R^n$ there exists $C(x)$ so that $a(x, y) = C(x)$, $y \in R^n_x$.*

Proof. Consider a Finsler space $F^n = (R^n, \mathcal{F})$ and its indicatrix $\mathcal{I}_x$ at the point x; the tangent hyperplane $T_y\mathcal{I}_x$ at the point $y = (y^1, ..., y^n) \in \mathcal{I}_x$ has the equation $l_i(Y^i - y^i) = 0$. The distance d from the origin of R^n to the hyperplane $T_y\mathcal{I}_x$ is

$$d = |l_i y^i| / \left(\sum_{k=1}^{n} (l_k)^2 \right)^{\frac{1}{2}} = 1 / \left(\sum_{k=1}^{n} (l_k)^2 \right)^{\frac{1}{2}} .$$

By definition, the indicatrix $\mathcal{I}_x$ of the space F^n at the point $x \in R^n$ is a Ţiţeica hypersurface [1], [7] if and only if $K(y)/d^{n+1} = \text{const}$, where $K(y)$ is the Gauss curvature at the arbitrary point $y \in \mathcal{I}_x$, and d is the distance from the origin of the R^n to the tangent hyperplane $T_y\mathcal{I}_x$.

So a necessary and sufficient condition that $\mathcal{I}_x$ to be a Ţiţeica hypersurface is $a(x, y) = C(x)$, $y \in \mathcal{I}_x$, where $C(x)$ is a constant with respect to y that depends on x. To develop further this idea see [3], pages 154-156.

Theorem 3 [10]. *Let g be a non-singular $n \times n$ symmetrical matrix, ξ a column vector, η a line vector such that ${}^t\eta = g\xi$, $\eta\xi = 1$.*

We consider the non-zero matrix

$$h = \alpha g + \beta \ {}^t\eta\eta, \quad \alpha, \beta \in R.$$

Then $\det h = \alpha^{n-1}(\alpha + \beta) \det g$.

Example 1. The indicatrices of any Riemann space $(R^n, a_{ij}(x))$ are Ţiţeica hypersurfaces. Indeed, in this case the function $a = \det(a_{ij}(x))$ depends only upon x.

Example 2. Consider $f : R^n \to R$ a differentiable and strictly positive function, and

$$\mathcal{F}(x, y) = f(x) \sqrt[n]{|y^1, ..., y^n|}, \ x \in R^n, \ y \in R^n_x.$$

The indicatrices of the Finsler space $F^n = (R^n, \mathcal{F})$ are Ţiţeica hypersurfaces.

Indeed, for any $y \in \mathcal{I}_x$, where x is an arbitrary point in R^n, we have

$$a_{ij} = \frac{1}{n|y^i y^j|} \left(\frac{2}{n} - \delta_{ij} \right),$$

and hence $\det(a_{ij}) = (-1)^{n-1} f^{2n}(x) n^{-n}$.

Remark 1. The preceding two examples are according to the following result due to Ţiţeica [7] and ordered by Gh. Th. Gheorghiu in [1]: "For the hyperquadrics and the hypersurface $x^1x^2...x^n = 1$ of the Euclidean space R^n, the invariant K/d^{n+1} is constant".

Example 3. Consider $f : R^n \to R$ a differentiable and strictly positive function and consider

$$\mathcal{F}(x,y) = f(x)|y^1|^{\varepsilon^1}...|y^n|^{\varepsilon^n}, \quad \text{where} \quad \sum_{k=1}^{n} \varepsilon^k = 1.$$

For any $y \in \mathcal{I}_x$, where x is an arbitrary point in R^n, we have

$$a_{ij} = \frac{\varepsilon^j}{y^i y^j}(2\varepsilon^i - \delta_{ij}).$$

If we define

$$\eta_i = \frac{\varepsilon^i}{y^i}, \quad g_{ij} = \delta_{ij}\frac{\varepsilon^j}{y^i y^j}, \quad \alpha = -1, \quad \beta = 2.$$

then according to Theorem 3 we obtain

$$\det a_{ij} = (-1)^{n-1}\frac{\varepsilon^1...\varepsilon^n}{|y^1...y^n|^2}.$$

If $\varepsilon^1 = ... = \varepsilon^n = 1/n$, then $\det(a_{ij})$ is constant with respect to y (see example 2) and hence $\mathcal{I}_x$ are Ţiţeica hypersurfaces. For contrary, $\det(a_{ij})$ depends explicitely on y.

Example 4. Consider $f : R^n \to R$ a differentiable and strictly positive function and

$$\mathcal{F}(x,y) = f(x)\sqrt[m]{|\varepsilon^1(y^1)^m + ... + \varepsilon^n(y^n)^m|}, \quad x \in R^n, \quad y \in R^n_x,$$
$$\varepsilon^k \neq 0, \quad k = \overline{1,n}.$$

For any $y \in \mathcal{I}_x$, where x is an arbitrary point in R^n, we have

$$a_{ij} = (2-m)\varepsilon^i\varepsilon^j(y^iy^j)^{m-2}f^{2m}(x) \pm \delta_{ij}(m-1)\varepsilon^j(y^j)^{m-2}f^m(x)$$

for

$$\varepsilon^1(y^1)^m + ... + \varepsilon^n(y^n)^m > 0 \quad \text{or} \quad < 0.$$

If we define

$$\eta_i = \varepsilon^i(y^i)^{m-1}, \; g_{ij} = \delta_{ij}\varepsilon^j(y^j)^{m-2}f^m(x), \; \alpha = m-1, \; \beta = (2-m)f^{2m}(x),$$

then, according to Theorem 3 we obtain

$$\det(a_{ij}) = (m-1)^{n-1}[m-1+(2-m)f^{2m}(x)]f^{mn}(x)\varepsilon^1...\varepsilon^n(y^1...y^n)^{m-2}$$

for

$$\varepsilon^1(y^1)^m + ... + \varepsilon^n(y^n)^m > 0,$$

and

$$\det(a_{ij}) = (-1)^n(m-1)^{n-1}[m-1+(2-m)f^{2m}(x)]f^{mn}(x)\varepsilon^1...\varepsilon^n(y^1...y^n)^{m-2}$$

for

$$\varepsilon^1(y^1)^m + ... + \varepsilon^n(y^n)^m < 0.$$

If $m = 2$, then $\det(a_{ij})$ is constant with respect to y and then $\mathcal{I}_x$ are Ţiţeica hypersurfaces.

If $m \neq 2$, then $\det(a_{ij})$ depends explicitely on y.

Remark 2. The Finsler spaces having the fundamental functions according to those in the 3-rd and 4-th examples, are $S3$ - like spaces (see also H. Shimada [8]).

2 Ţiţeica - Miron Figuratrix

Consider $M^n = (R^n, \mathcal{M})$ a Miron space [4]. By $p_x = (p_1, ..., p_n)_x$ we shall denote the canonical coordinate system of the cotangent space R_x^{*n} at each point $x \in R^n$. Under these conditions the fundamental function $\mathcal{M}(x,p)$ of the Miron space is 1(p)-homogeneous with respect to p. The constant level set $\mathcal{F}_x = \{p \in R_x^{*n} / \mathcal{M}^2(x,p) = 1\}$ is a hypersurface of the Euclidean space R_x^{*n}, because $\nabla\mathcal{M}^2 \neq 0$ on $\mathcal{F}_x$. Indeed, the function $\mathcal{M}^2(x,p)$ has a nondegenerate Hessian on $R_x^{*n} \setminus \{0\}$, so $\nabla\mathcal{M}^2 \neq 0$ on $\mathcal{F}_x$. The hypersurface $\mathcal{F}_x$ is called *figuratrix* at x of the Miron space M^n.

We orient the figuratrix $\mathcal{F}_x$ by choosing the unit normal vector field

$$U = -\nabla\mathcal{M}^2 / \|\nabla\mathcal{M}^2\|,$$

and we shall denote by $T_p\mathcal{F}_x$ the space tangent to $\mathcal{F}_x$ at a point $p = (p_1, ..., p_n) \in \mathcal{F}_x$. As $T_p\mathcal{F}_x$ is an $(n-1)$-dimensional subspace of the space R_x^{*n}, the set $T_p\mathcal{F}_x$ may be called the *hyperplane* tangent to $\mathcal{F}_x$ at the point $p \in \mathcal{F}_x$.

We define the Weingarten mapping $S^* : T_p\mathcal{F}_x \to T_p\mathcal{F}_x$, $S^*(v) = -D_vU$, $v \in T_p\mathcal{F}_x$, where D_v represents the derivative with respect to the vector v. The Gauss curvature $K^*(p)$ of the hypersurface $\mathcal{F}_x$ at the point $p \in \mathcal{F}_x$ is defined as being the determinant of the Weingarten mapping.

By definition, the figuratrix $\mathcal{F}_x$ of the space M^n at the point $x \in R^n$ is a Ţiţeica hypersurface if and only if $K^*(p)/d^{n+1} = \text{const}$ (with respect to p), where $K^*(p)$ is the Gauss curvature on the hypersurface at the arbitrary point $p \in \mathcal{F}_x$ and d is the distance from the origin of the space R^n to the tangent hyperplane $T_p\mathcal{F}_x$.

Similarly we obtain

Theorem 3. *The Gauss curvature $K^*(p)$ of the hypersurface $\mathcal{F}_x$ at the point $p \in \mathcal{F}_x$ has the expression*

$$K^*(p) = g \Big/ \left(\sum_{k=1}^{n} (l_k)^2\right)^{(n+1)/2},$$

where

$$g^{ij} = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j\mathcal{M}^2,\ g = \det(g^{ij}),\ l^i = \dot{\partial}^i\mathcal{M},\ \dot{\partial}^k = \partial/\partial_{p_k},\ k = 1, 2, ..., n.$$

Theorem 4. *Figuratrices of a Miron space $M^n = (R^n, \mathcal{M})$ are Ţiţeica hypersurfaces if for any $x \in R^n$ there exists $C(x)$ so that $g(x,p) = C(x)$.*

Theorem 5. *Figuratrices of the Riemann space $M^n = (R^n, g^{ij}(x))$ are Ţiţeica hypersurfaces.*

In the following examples, f is a differentiable and strictly positive function from R^n to R.

Example 6. The figuratrices of a Miron space $M^n = (R^n, \mathcal{M})$ having the fundamental function of the following form

$$\mathcal{M}(x,p) = f(x)\sqrt[n]{|p_1 p_2 ... p_n|},\ x \in R^n,\ p \in R_x^{*n},$$

are Ţiţeica hypersurfaces.

Example 7. The figuratrices of a Miron space $M^n = (R^n, \mathcal{M})$ having the fundamental function of the form

$$\mathcal{M}(x,p) = f(x)|p_1|^{\varepsilon^1} ... |p_n|^{\varepsilon^n},$$

where $\sum_{k=1}^{n} = 1$, are Ţiţeica hypersurfaces if and only if $\varepsilon^1 = ... = \varepsilon^n = \frac{1}{n}$.

Example 8. The figuratrices of a Miron space $M^n = (R^n, \mathcal{M})$ having the fundamental function of the form

$$\mathcal{M}(x,p) = f(x)\sqrt[m]{|\varepsilon^1 p_1^m + ... + \varepsilon^n p_n^m|},$$

are Ţiţeica hypersurfaces if and only if $m = 2$.

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Author's address:

C. Udrişte, O Şandru, C. Dumitrescu and A. Zlătescu
Politehnica University of Bucharest
Department of Mathematics I
Splaiul Independenţei 313,
77206 Bucharest, Romania
E-mail: udriste@mathem.pub.ro