EQUILIBRIUM SETS OF MAGNETIC FIELDS AROUND ELECTRIC CIRCUITS

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Abstract

§2 and §3 determine the equilibrium points sets of the differential system which describes the field lines of the magnetic field generated by two coplanar filiform electrical circuits of right angle type disposed such that:

1) they are negative oriented, or one negative and the other positive oriented;

2) the edges of the angles are two by two parallel;

3) the supporting straight lines of the edges determine a rectangle.

These equilibrium points sets are algebraic curves (either void set, pair of points, part of an ellipse or part of a quartic) fixed by the coordinates of the right angles vertices. §4 describes the scalar potentials and their level surfaces at height zero. §5 presents Runge-Kutta approximations of some magnetic lines, since in our cases the analytic expressions are not known yet.

Mathematics Subject Classification: 78A25, 34C35, 58F25.
Key Words: magnetic lines, equilibrium points, magnetic scalar potential, Runge-Kutta approximations.

1 Introduction

We are mainly concerned with morphology of the magnetic fields produced by piece-wise rectilinear configurations suitable selected by Sabba Ştefănescu for problems in Electrotechnics, Geophysics and Plasma Physics (realising windings with special properties, guidance of charge particles in the terrestrial magnetic field, building of nuclear reactors, confinement of thermonuclear plasma, etc.).

Consider the magnetic field $\bar{H} = H_x\bar{i} + H_y\bar{j} + H_z\bar{k}$ generated around a configuration Γ of piecewise rectilinear wires, defined on $A = R^3\setminus\Gamma$. The relations $\text{curl}\bar{H} = 0$, $\text{div}\bar{H} = 0$ imply $\Delta\bar{H} = 0$ and hence $\bar{H}$ is an harmonic vector field. The Cauchy problem

$$(1)\qquad \begin{array}{l} \frac{dx}{dt} = H_x; \quad \frac{dy}{dt} = H_y; \quad \frac{dz}{dt} = H_z \\ x(0) = x_0; \quad y(0) = y_0; \quad z(0) = z_0 \end{array}$$

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describes a field (magnetic) line. Let $\varphi: A_1 \subset A \rightarrow R$ be the potential of $\bar{H}$, i.e., $\bar{H} = \text{grad}\varphi$ on $A_1 \subset A$. The field lines of $\bar{H}$ contain the gradient lines of φ. Since φ is bounded, being essentially the "arctan function", a maximal field line through $(x_0, y_0, z_0) \in A$ is defined on R. So, (1) defines a global dynamical system $T : R \times A \rightarrow R^3$, of class C^∞, [16].

The equilibrium points of the above differential system are the solutions of the algebraic system

$$(*) \qquad H_x = 0, \quad H_y = 0, \quad H_z = 0.$$

These equilibrium points, are *unstable*. They are critical points of the potential function φ since locally $\bar{H} = grad\varphi$. Since φ is a harmonic function, its critical points are of the saddle type only.

2 Magnetic Field Around Planar Wires of Right Angle Type, One Negative and the Other Positive Oriented

In this section, we consider the wires configuration presented in Fig. 1 (pair of coplanar right angles, opposite oriented) wandered through by unitary currents which have the senses indicated by arrows. The components of magnetic field generated around this configuration of currents, at the point $M(x, y, z)$, are given in the following theorem (see Biot-Savart-Laplace formula, Sabba Stefănescu works [8-11] and [12], [16-19]).

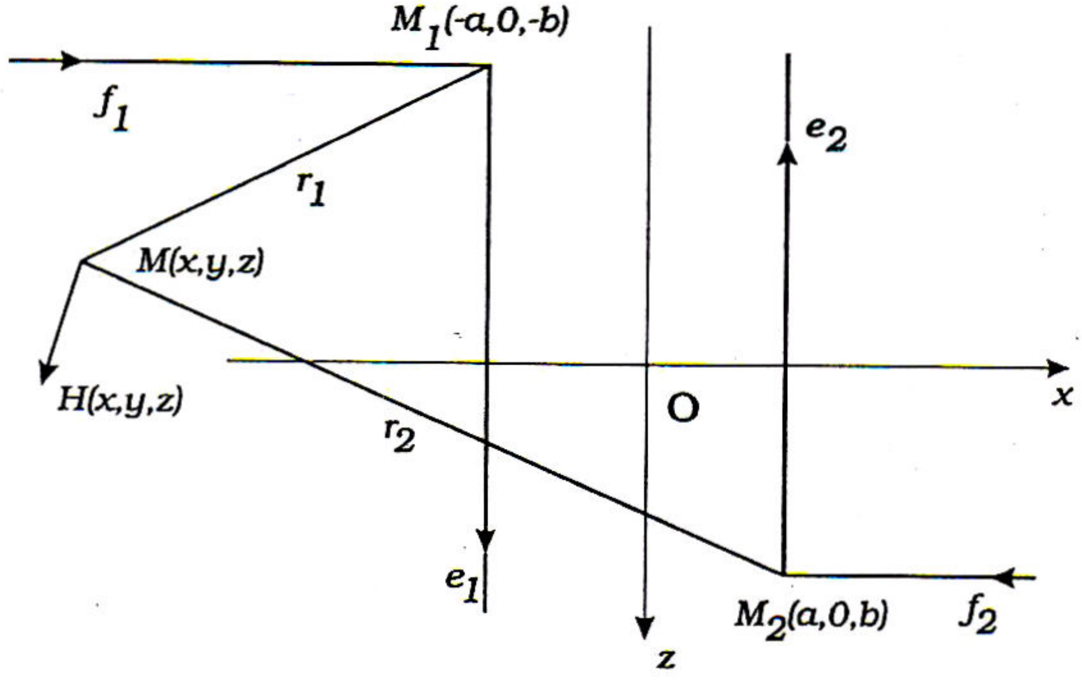

Fig. 1

Theorem 2.1. *The magnetic field $\bar{H}$ generated around the configuration from Fig. 1 has the components*

$$H_x = \frac{-y}{r_1(r_1 - z - b)} + \frac{y}{r_2(r_2 + z - b)}$$
$$H_y = \frac{x + a}{r_1(r_1 - z - b)} - \frac{z + b}{r_1(r_1 + x + a)} - \frac{x - a}{r_2(r_2 + z - b)} + \frac{z - b}{r_2(r_2 - x + a)}$$
$$H_z = \frac{y}{r_1(r_1 + x + a)} - \frac{y}{r_2(r_2 - x + a)},$$

where $r_1^2 = (x+a)^2 = y^2 + (z+b)^2, \quad r_2^2 = (x-a)^2 + y^2 + (z-b)^2$.

The study of the existence of the components $H_x,\ H_y,\ H_z$ leads to the fact that the field $\bar{H}$ is defined on $A = R^3 \backslash (e_1 \cup f_1 \cup e_2 \cup f_2)$. Here $e_1,\ f_1,\ e_2,\ f_2$ are the straight semilines defined as follows

$$f_1 \colon x \leq -a;\ y = 0;\ z = -b, \quad e_1 \colon x = -a;\ y = 0;\ z \geq -b$$
$$f_2 \colon x \geq a;\ y = 0;\ z = b, \quad e_2 \colon x = a;\ y = 0;\ z \leq b.$$

Remark 2.2. *The kinematic system attached to the magnetic field in Theorem 2.1 cannot be reduced to a Hamiltonian system in two variables, but it can be transformed into a Hamiltonian system in R^6, via the induced geometric dynamics [20], [21].*

We take into account the magnetic field given in Theorem 2.1 and we look for the equilibrium positions.

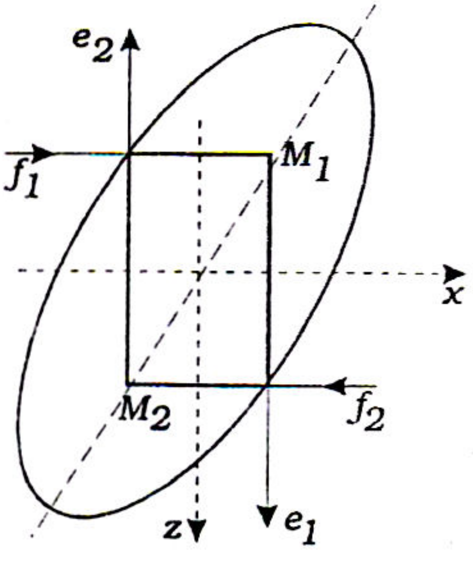

Fig. 2

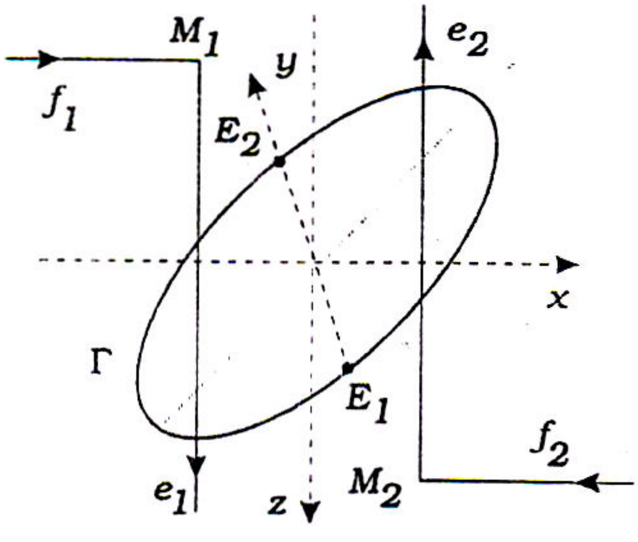

Fig. 3

Theorem 2.3. *Suppose that a and b do not vanish simultaneously.* 1) *If $a \geq 0,\ b > 0$, then the set of equilibrium points is the ellipse (Fig. 2)*

$$E \colon y = 0,\ r_1 + r_2 = 2(b - a)$$

without the points $(a,0,-b)$ and $(-a,0,b)$. 2) If $ab>0$, $a\neq b$, then the set of equilibrium points consists of two points (Fig. 3)

$$E_{1,2}=(0;\ \frac{\pm\sqrt{2ab(a^2+b^2)}}{b-a};0).$$

3) Otherwise, the set of equilibrium points is void.

Proof. We remark that the system $(*)$ is equivalent to the following two algebraic systems:

$$(2)\qquad \begin{aligned} &y=0\\ &\frac{x+a}{r_1(r_1-z-b)}-\frac{z+b}{r_1(r_1+x+a)}-\frac{x-a}{r_2(r_2+z-b)}+\frac{z-b}{r_2(r_2-x+a)}=0\end{aligned}$$

and

$$(3)\qquad \begin{aligned} \frac{1}{r_1(r_1-z-b)}&=\frac{1}{r_2(r_2+z-b)}\\ \frac{a}{r_1(r_1-z-b)}&=\frac{b}{r_2(r_2-x+a)}\\ \frac{1}{r_1(r_1+x+a)}&=\frac{1}{r_2(r_2-x+a)}.\end{aligned}$$

We shall show that these two systems have not solutions simultaneously.

The system (2) is equivalent to

$$(2')\qquad y=0,\ H_y(x,0,z)=0$$

and the solutions of this system generates equilibrium positions located in the xOz-plane. To obtain the solutions of (2') we put $H_y(x,0,z)$ in the follwing form

$$H_y(x,0,z)=\frac{2(r_1+r_2+2a-2b)}{[r_1+x+a-(z+b)][r_2+(z-b)-(x+a)]},$$

hence (2') is equivalent to

$$(2")\qquad y=0,\ r_1+r_2=2(b-a).$$

Taking into account the relation $r_1+r_2\geq M_1M_2=2\sqrt{a^2+b^2}$ we deduce that for $a\leq 0$, $b>0$ the system (2') represents an ellipse of equilibrium points with the foci M_1 and M_2. From this ellipse we remove the points $(a,0,-b)$ and $(-a,0,b)$ wich are not in the definition domain.

The system (3) leads us at equilibrium points located on Oy-axis. If $a\neq 0$ and $b\neq 0$, then (3) is equivalent to

$$(3')\qquad \begin{aligned} &r_2+z-b=\tfrac{a}{b}(r_2-x+a)\\ &r_1+z-b=\tfrac{a}{b}(r_1-x+a)\\ &(z+b)r_1+(z-b)r_2=4ax+4bz.\end{aligned}$$

If $a=b$, then (3') has not solutions. If $a\neq b$, then (3') becomes

$$r_1(b-a) = ax + bz + a^2 + b^2$$
$$r_2(b-a) = -ax - bz + a^a + b^2$$
$$(b-2a)x = (2b+a)z.$$

If $b < a$ this system has not solutions. If $b > a$ the system of the two first equations can be replaced by the intersection

$$\Gamma: \quad x^2 + y^2 + z^2 = \frac{2ab(a^2+b^2)}{(b-a)^2}, \qquad ax + bz = 0.$$

Hence, if $ab > 0$, Γ is a circle. If $ab < 0$, Γ is void.

The system (3′) is equivalent to

$$x^2 + y^2 + z^2 = \frac{2ab(a^2+b^2)}{(b-a)^2}$$
$$ax + bz = 0$$
$$(b-2a)x - (2b+a)z = 0.$$

In the case $a \neq 0$ and $b \neq 0$ the two planes determine the straight line $x = 0$, $z = 0$. Consequently, we have two equilibrium points

$$E_{1,2} = (0; \frac{\pm\sqrt{2ab(a^2+b^2)}}{b-a}; 0).$$

located on Oy–axis. For $ab < 0$ we have not equilibrium positions.

3 Magnetic Field Around Planar Wires of Right Angle Type, Both Negative Oriented

In this section, we consider the wires configuration presented in Fig. 4 (pair of coplanar right angles, the same orientation) wandered through by unitary currents which have the senses indicated by the arrows. The components of magnetic field generated around this configuration of currents, at the point $M(x, y, z)$ are given in the following theorem [12], [16]-[19].

Theorem 3.1. *The magnetic field $\bar{H}$ generated around the configuration from Fig. 4 has the components*

$$H_x = \frac{-y}{r_1(r_1 - z - b)} - \frac{y}{r_2(r_2 + z - b)}$$
$$H_y = \frac{x+a}{r_1(r_1 - z - b)} - \frac{z+b}{r_1(r_1 + x + a)} + \frac{x-a}{r_2(r_2 + z - b)} - \frac{z-b}{r_2(r_2 - x + a)}$$
$$H_z = \frac{y}{r_1(r_1 + x + a)} + \frac{y}{r_2(r_2 - x + a)}.$$

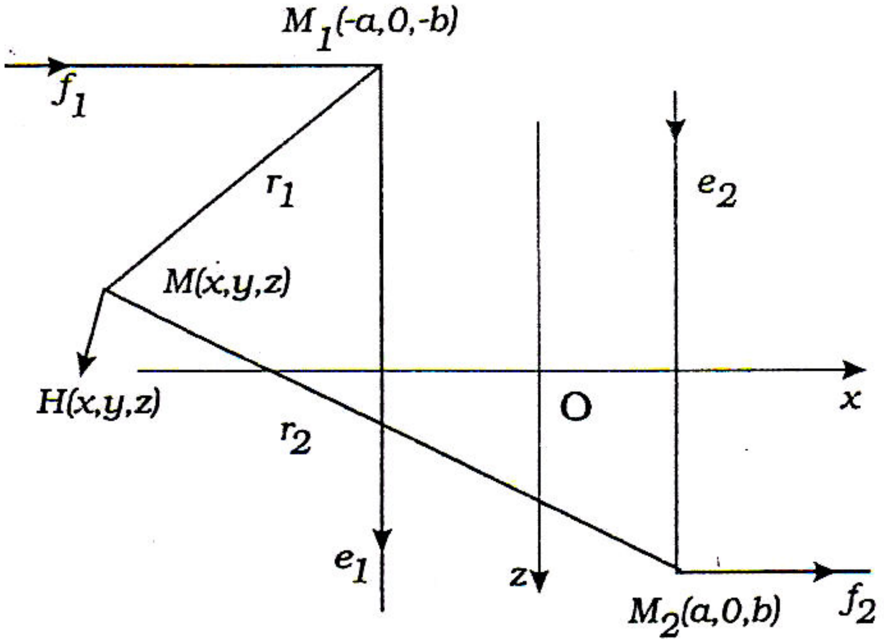

Fig. 4

The domain of definition for this field is also the open set A, which is given in §1.

Remark 3.2. *The kinematic system attached to the magnetic field in Theorem 3.1 cannot be reduced to a Hamiltonian system in two variables, but it can be transformed into a Hamiltonian system in R^6, via the induced geometric dynamics [20], [21].*

Theorem 3.3. *Let $\bar{H}$ be the magnetic field in the Theorem 3.1. The set of equilibrium points is the curve (Fig. 5)*

$$\Gamma_1: \quad y = 0;\ r_2 - r_1 = 2(x - z).$$

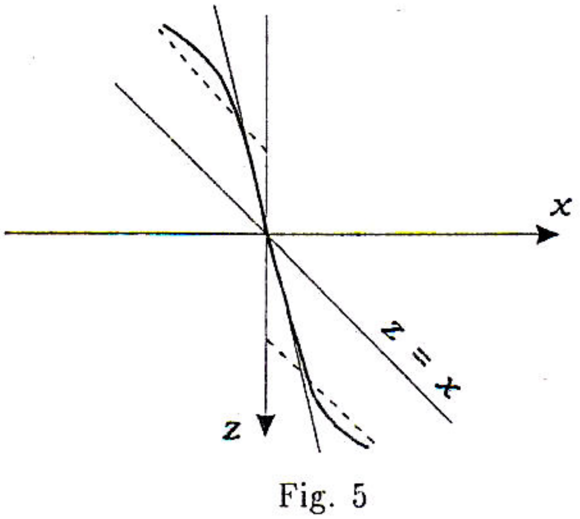

Fig. 5

Proof. The equilibrium positions (constant field lines) are solutions of the systems

$$(4) \quad \begin{aligned} & y = 0 \\ & \frac{x+a}{r_1(r_1-z-b)} - \frac{z+b}{r_1(r_1+x+a)} + \frac{x-a}{r_2(r_2+z-b)} - \frac{z-b}{r_2(r_2-x+a)} = 0 \end{aligned}$$

and

$$\begin{cases} \frac{1}{r_1(r_1-z-b)} + \frac{1}{r_2(r_2+z-b)} = 0 \\ \frac{a}{r_1(r_1-z-b)} + \frac{b}{r_2(r_2-x+a)} = 0 \\ \frac{1}{r_1(r_1+x+a)} + \frac{1}{r_2(r_2-x+a)} = 0. \end{cases} \tag{5}$$

Because $r_1 - z - b > 0$ and $r_2 + z - b > 0$, (5) represents the void set. The system (4) is equivalent to

$$y = 0, \quad H_y(x,0,z) = 0 \tag{4'}$$

and its solutions are equilibrium points belonging to the xOz–plane. Because

$$r^2_{1|y=0} = (x+a)^2 + (z+b)^2 \qquad r^2_{2|y=0} = (x-a)^2 + (z-b)^2$$

we have, after laborious calculations

$$H_y(x,0,z) = \frac{2(r_2 - r_1 + 2z - 2x)}{[r_1 + x + a - (z+b)][r_2 + (z-b) - (x+a)]}.$$

Hence (4′) is equivalent to Γ_1 given above, which represents the set of equilibrium positions (Fig.5). The study and the shape of the curve Γ_1 where realized in [18] (for $a = 1,\ b = 2$).

4 Scalar Potentials of Magnetic Field Around Two Planar Filiform Electrical Circuits of Right Angle Type [18]

First we consider two coplanar filiform electrical circuits of right angle type with currents of opposite sense (Fig. 1). The scalar potential φ of the magnetic field $\bar{H}$ generated around this configuration is defined by

$$\varphi(x,y,z) = -2\arctan\frac{y(r_1 + r_2 + 2a - 2b)}{y^2 - [r_1 + x + a - (z+b)][r_2 + (z-b) - (x+a)]},$$

where $(x,y,z) \in (R^3 \setminus xOz) \cup \{f_1 \cup e_1 \cup f_1 \cup e_2\}$.

The field lines of $\bar{H}$ are orthogonal to the equipotential surfaces

$$\sum\nolimits_c^{(1)} : \varphi(x,y,z) = C.$$

The equilibrium positions of $\bar{H}$ are critical points of φ, and also zeros of φ. The set of zeros of φ (the set of vanishing potential) contains also the wires and the set described by the equation

$$r_1 + r_2 = 2(b-a). \tag{6}$$

The set (6) would be nonvoid only in the case $b \geq a$. For $b = 0$ this set contains of two points, namely $\{M_1, M_2\}$. For $b > a$, we remark that $r_1 + r_2 \geq M_1M_2 = 2\sqrt{a^2+b^2}$ and the conditions $2(b-a) \geq \sqrt{a^2+b^2}$, $b > a$ are equivalent to $a \leq 0$, $b > 0$. The relations $r_1 + r_2 = 2(b-a)$, $a \leq 0$, $b > 0$ describe an ellipsoid with the foci M_1 and M_2, without the ellipse from the plane $y = 0$, because this plane is not included in the domain of φ. This ellipsoid cuts the axis Oy at the points $(0, \pm\sqrt{-2ab}, 0)$. For $b > a$ and $ab > 0$, the set (6) is void since $b - a < \sqrt{a^2+b^2}$.

Second, we consider two coplanar filiform electrical circuits of right angle type with currents of same sense (Fig. 4). The scalar potential φ of the magnetic field $\bar{H}$ generated around this configuration is defined by

$$\varphi(x,y,z) = 2\arctan\frac{y(r_2 - r_1 - 2x + 2z)}{y^2 - [r_1 + x + a - (z+b)][r_2 - (x-a) + (z-b)]},$$

where $(x,y,z) \in (R^3\backslash xOz) \cup \{f_1 \cup e_1 \cup f_2 \cup e_2\}$.

The equipotential surfaces

$$\sum\nolimits_c^{(2)} : \varphi(x,y,z) = C$$

are normal to the magnetic lines. The equilibrium positions of $\bar{H}$ are critical points of φ, and also zeros of φ. The set of zeros of φ contains also the wires and the set of equation

$$r_2 - r_1 = 2(x - z). \tag{7}$$

The surface (7) is symmetrical with respect to the plane xOz and with respect to the axis Oy. Moreover, it contains the axis Oy. The surface (7) is included in the surface described by the rational equation

$$(a^2 + b^2 + y^2 + 2xz)(x-z)^2 = (ax + bz)^2.$$

5 Computational Results

Let us consider again the magnetic fields described by Theorems 2.1 and 3.1. Since the parametric equations or the Cartesian equations of the corresponding magnetic lines are not known till now, we reproduce some Runge-Kutta approximations obtained using a PC.

In the case $a = 1$ and $b = 2$, Fig. 6 presents magnetic lines whose starting points are near an equilibrium point of the differential system. We choose the point $(0, 4.472, 0)$, the initial conditions

(1) $x_0 = -0.1, \quad y_0 = 4.472, \quad z_0 = 0$

(2) $x_0 = 0.1, \quad y_0 = -4.40, \quad z_0 = 0$

(3) $x_0 = 0.1, \quad y_0 = 4.472, \quad z_0 = 0;$

maximum number of iterations = 900; integration step 0.2.

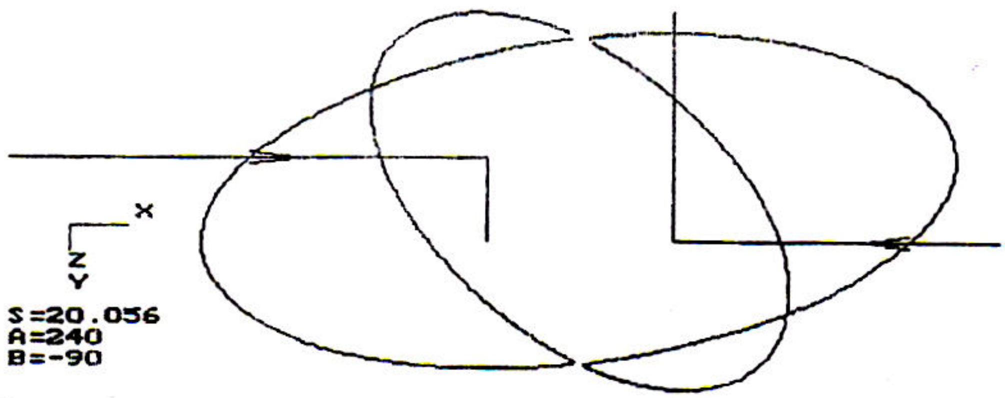

Fig. 6

For $a = -1$ and $b = 2$ (Figs. 7 and 8) we obtained magnetic lines (different projections) whose starting points are inside of the rectangle $[-1,1] \times [-2,2]$; initial conditions and maximum number of iterations

$$\begin{array}{lllll} (1) & x_0 = \pm 0.20, & y_0 = 0, & z_0 = 0, & \ldots\ 850 \\ (2) & x_0 = \pm 0.40, & y_0 = 0, & z_0 = 0, & \ldots\ 350 \\ (3) & x_0 = \pm 0.55, & y_0 = 0, & z_0 = 0; & \ldots\ 250 \end{array}$$

whereas the integration step is 0.1.

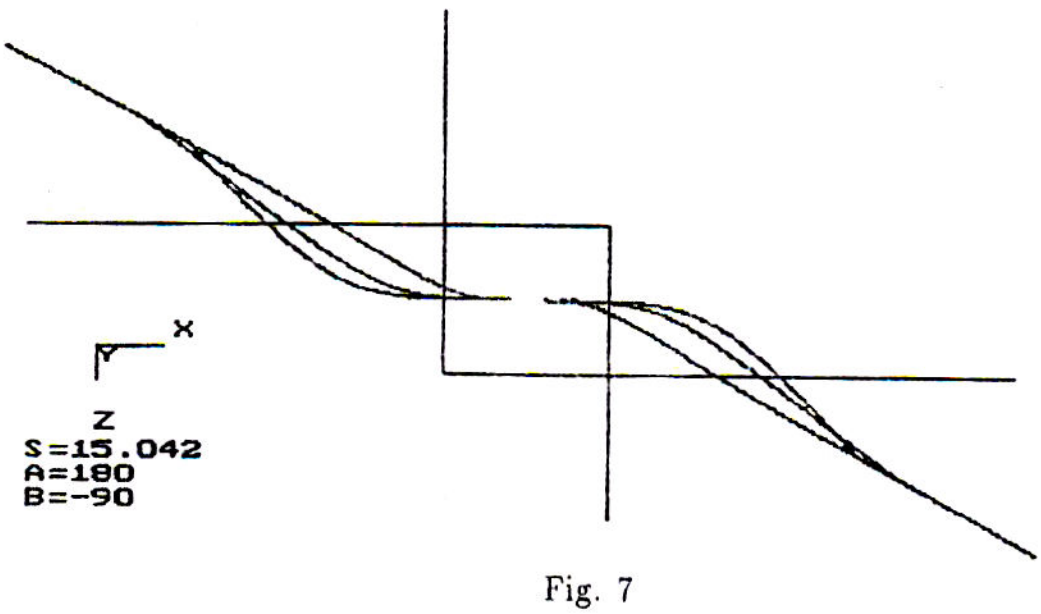

Fig. 7

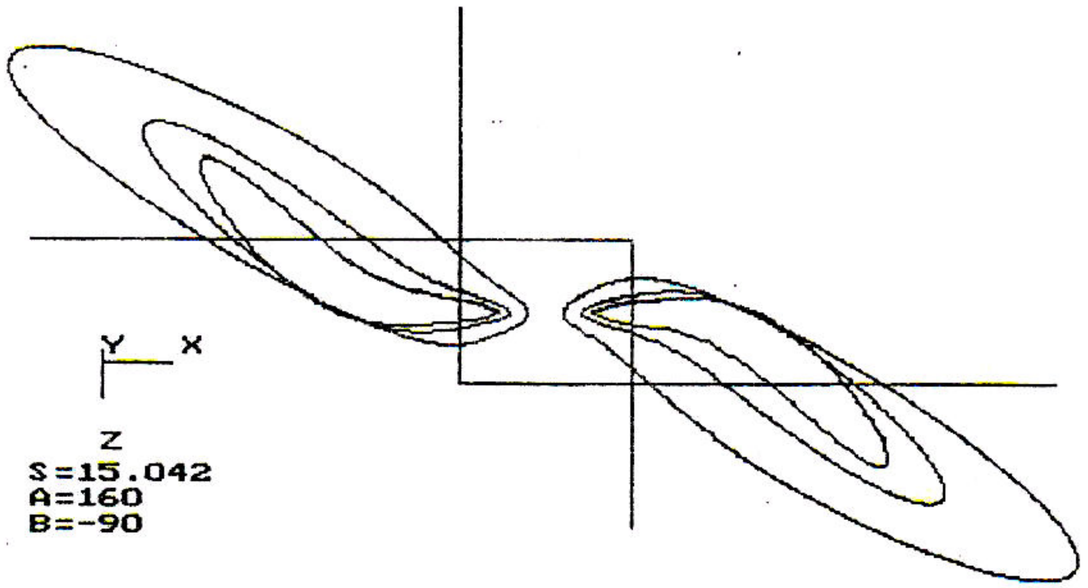

Fig. 8

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