DETERMINATION OF SPECIAL STRUCTURES ON A COMPACT RIEMANNIAN MANIFOLD BY DIFFERENT SPECTRA AND DETERMINATION OF (F_4, g_0) BY DIFFERENT SPECTRA

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Abstract

Let (M, g) be a compact Riemannian manifold of dimension n. Let V be a vector bundle over M. Let D be a second order elliptic differential operator on the cross sections $C^\infty(V)$ of V. This operator D gives a spectrum denoted by $Sp(V, M, D)$. The aim of the present paper is to study the influence of different spectra on special structure on a compact manifold and to prove that different spectra can determine completely the geometry on the exceptional Lie group (F_4, g_0).

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1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension n. We consider a vector bundle V over M. Let $C^\infty(V)$ be the cross sections of V. Let D be a second order elliptic differential operator with leading symbol given by the metric tensor g acting on $C^\infty(V)$, that

$$D : C^\infty(V) \to C^\infty(V),\ D : \theta \to D(\theta). \tag{1}$$

If we have the property $D(\theta) = \lambda\theta$, then θ is called eigensection and λ eigenvalue associated to θ. The set of all eigenvalues of D is called spectrum of D and denoted by $Sp(V, M, D)$ and has the form

$$Sp(V, M, D) = \{\lambda_0 = \cdots = \lambda_0 = \cdots < \lambda_1 = \lambda_1 \cdots < \cdots < \infty\}.$$

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This is discrete and each eigenvalue has finite multiplicity. One of the basic problems in the spectra theory is to study the influence of $Sp(V,M,D)$ on some structures on the compact Riemannian manifold (M,g), when V and D are given.

The whole paper contains six sections. Each of them is analyzed as follows.

The first section contains the introduction.

Some basic elements of fibre bundles and Riemannian geometry are included in the second section.

The third section has the different methods to compute the coefficients of this asymptotic expansion.

The calculations of these coefficients for special manifolds are included in the fourth section.

The fifth section contains some Riemannian manifolds, which can be determined by some spectra.

The determination of the geometry on the exceptional Lie group (F_4, g_0) by different spectra is given in the last section, where g_0 the Riemannian metric on F_4 coming from the Killing-Cartan form on the Lie algebra t_4 of F_4.

2 BASIC ELEMENTS OF AFFINE AND RIEMANNIAN GEOMETRY

Let M be a differential manifold of dimension n. In the local level we consider a chart (U,φ) on M, that is

$$\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$$

is a diffeomorphism of U onto $\varphi(U)$. Let $\{x_1, ..., x_n\}$ be a local coordinate system in U.

We denonte by $C^\infty(M)$ the set of all differentiable functions on M, that is

$$C^\infty(M) = \{f \ | \ f : M \to \mathbb{R}\}$$

which is an algebra over $\mathbb{R}$. Then we have the vector space $\Lambda^q(M,\mathbb{R})$, $q = 0, 1, ..., n$ of exterior q-forms on the manifold M. It is obvious that $\Lambda^0(M,\mathbb{R}) = C^\infty(M)$.

Let $B = (V, M, F)$ be a vector bundle over the compact manifold M, that means V is the total space, M the base manifold and F the fibre. It is obvious

$$F \xrightarrow{i} V \xrightarrow{\pi} M,$$

where i is the inclusion mapping and π the projection mapping and

$$F \approx V_x, \ \forall x \in M.$$

Let Γ be an affine connection on M. Let $C^\infty(V)$ be the set of cross sections of V. If X is a vector on M, then this connection Γ defines a linear mapping

$$\nabla_X : C^\infty(B) \to C^\infty(B), \ \nabla_X(\varphi + \psi) = \nabla_X\varphi + \nabla_X\psi,$$

$$\nabla_{\nu X}\varphi = \nu\nabla_X\varphi,\ \nabla_X(\nu\varphi) = \nu\nabla_X\varphi + (X\nu)\varphi,$$

where $\varphi, \psi \in C^\infty(V)$, $\nu \in C^\infty(M)$.

In the local coordinate system $(x_1, ..., x_n)$ in U and let $(y_1, ..., y_m)$ be a local coordinate system in W, such that $B_{|W} = (W, U, F)$ is local representation of $B = (V, M, F)$. Let $(\varphi_1, \ldots, \varphi_n)$ be a local base of the cross sections of $B_{|W}$. It is known that the local vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \ldots, X_n = \frac{\partial}{\partial x_n}$$

form a base of $D^1(U)$. Then the following formulas are valid:

$$\nabla_{X_i}X_j = \sum_{i=1}^{n}\Gamma_{ij}^l X_l.$$

Therefore we obtain Christoffel's functions

$$\Gamma_{jk}^l,$$

which are n^3.

These Christoffel's functions also determine the connection Γ on M. For every vector field X on M and the connection Γ on M we have the linear mapping

$$\nabla_X : C^\infty(V) \to C^\infty(V),\ \nabla_X : \varphi \to \nabla_X\varphi.$$

The torsion tensor field T of type $(1, 2)$ and the curvature tensor field R of type $(1, 3)$ in the local coordinate system $(x_1, \ldots, x_n)$ can be expressed as follows

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k,$$

$$R_{ijk}^l = \frac{\partial}{\partial x_i}\Gamma_{jk}^l - \frac{\partial}{\partial x_j}\Gamma_{ik}^l + \sum_{m=1}^{n}\left(\Gamma_{im}^l\Gamma_{jk}^m - \Gamma_{jm}^l\Gamma_{ik}^m\right).$$

The theory of connection can be faced by the means of exterior forms.

Let ω^i, ω_j^i , $(1 \le i, j \le n)$, be the 1-forms on U determined by

$$\omega^i(X_j) = \delta_j^i, \quad \omega_j^i = \sum_k \Gamma_{kj}^i\omega^k.$$

The structure equations of Cartan are given by

$$d\omega^i = -\sum_p \omega_p^i \wedge \omega^p + \frac{1}{2}\sum_{j,k} T_{jk}^i\omega^j \wedge \omega^k,$$

$$d\omega_j^i = -\sum_p \omega_p^i \wedge \omega^p + \frac{1}{2}\sum_{l,k} R_{jlk}^i\omega^l \wedge \omega^k.$$

These are related by previous theory using the tangent bundle over M.

Let (M,g) be a Riemannian manifold. Let (U,φ) be a chart on M with local coordinate system $(x_1,\ldots,x_n)$. The Riemannian metric g on U can be written

$$g_{|U} = g_{ij}dx^i dx^j.$$

This Riemannian metric g defines a connection Γ in the tangent bundle TM of M, whose components $\{\Gamma^i_{jk}\}$ with respect to the local coordinate system $(x_1,\ldots,x_n)$ are given by

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left\{\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l}\right\}.$$

This connection is free torsion and its curvature tensor field has components $\{R^h_{kji}\}$ with respect to $(x_1,...,x_n)$, which are given by

$$R^h_{kji} = \frac{\partial}{\partial x_k}\Gamma^h_{ji} - \frac{\partial}{\partial x_j}\Gamma^h_{ki} + \Gamma^h_{kt}\Gamma^t_{ji} - \Gamma^h_{jt}\Gamma^t_{kj}.$$

It is known that the tensor field R of type $(1,3)$ satisfies some conditions.

From this tensor field we can define the Ricci tensor field p of type $(0,2)$ and a scalar curvature T, which is a function on M. These in local coordinate system $(x_1,\ldots,x_n)$ are defined by

$$p_{ij} = \sum_{t=1}^{n} R^t_{tij},$$

$$T = g^{ij}p_{ij}.$$

Let λ be a plane in $T_P(M)$ which is spanned by two linearly independent vectors X and Y. Then the expression

$$\sigma(\lambda) = \frac{-\bar{R}_P(X,Y,X,Y)}{g_P(X,X)g_P(Y,Y) - g_P^2(X,Y)}$$

is called sectional curvature of λ, where $\bar{R}$ is the Riemannin curvature of (M,g), defined in local coordinates $(x_1,\ldots,x_n)$ as follows

$$\bar{R}_{ijkl} = R^t_{ijk}g_{it}$$

and $\bar{R}_P$ and g_P values of $\bar{R}$ and g respectively at the point P.

3.

Let (M,g) be a compact Riemannian manifold of dimension n. We denote by $\Lambda^q(M,\mathbb{R})$ the vector space of exterior q-forms on the manifold M, where $q=0,1,\ldots,n$ and $\Lambda^0(M,\mathbb{R}) = C^\infty(M)$. We consider the Laplace operator $\Delta = d\delta + \delta d$ acting on the exterior q-forms $\Lambda^q(M,\mathbb{R})$, that is

$$\Delta = d\delta + \delta d : \Lambda^q(M,\mathbb{R}) \to \Lambda^q(M,\mathbb{R}), \tag{2}$$

$$\Delta : \alpha \to \Delta\alpha. \tag{3}$$

If we have $\Delta\alpha = \lambda\alpha$, $\lambda \in I\!R$, then α is called eigen-q-form and λ eigenvalue associated to α. The set of all eigenvalues is called spectrum and denoted by $Sp^q(M, g)$. This has the form

$$Sp^q(M,g) = \{0 \leq \lambda_{1,q} \leq \lambda_{2,q} \leq \cdots \leq +\infty\}, \tag{4}$$

which is discrete and each eigenvalue has finite multiplicity. We also have the eigen-q-forms

$$\{0, \varphi_{1,q}, \varphi_{2,q}, \ldots, \}. \tag{5}$$

We form the sum

$$\sum_{m=0}^{\infty} e^{-\lambda_{m,q}t} \varphi_{m,q}(x)\varphi_{m,q}(y), \tag{6}$$

which converges uniformly on compact subsets of $(0, \infty) \times M$ to the fundamental solution $e_q(t, x, y)$ of the operator $\Delta - \partial/\partial t$ acting on q-forms, and the trace

$$Z_q = \sum_{m \geq 0} e^{-\lambda_m t} \tag{7}$$

can be expressed as the integral over the manifold of the pole

$$T_Q = \sum_{m \geq 0} e^{-\lambda_m t} \langle \varphi_m, \varphi_m \rangle, \tag{8}$$

where $\langle \varphi_m, \varphi_m \rangle$ is the Riemannian inner product of q-forms at a point of M, that is

$$Z_q = \int_M Tr\,(T_q)\, dM, \tag{9}$$

where dM is the volume of M. This technique is based the main idea in order to construct the fundamental solution of the operator $\Delta - \partial/\partial t$, which has the form

$$e_q\,(t,x,y) = G_N^q\,(t,x,y) + \sum_{m \geq 0} (-1)^{m+1} \int_0^t ds \int_M \left(K^m\,(s,x,z)\, G_N^q\,(t-s,z,y)\right) dM. \tag{10}$$

The quantities

$$K^m\,(s,x,z) \;\; and \; G_N^q\,(t-s,z,y) \tag{11}$$

depend on the Riemannian tensor, its tensor fields and their covariant derivatives. The asymptotic expansion of Z_q, which is given by (7), takes the form

$$Z_q = \sum_{m \geq 0} e^{-\lambda_m t} \approx (4\pi t)^{-n/2} \left\{a_{0,q} + a_{1,q}t + a_{2,q}t^2 + \cdots\right\}, \tag{12}$$

where

$$a_{i,q} = \int_M U_{i,q} dM \tag{13}$$

and $U_{i,q}$ a function on M, that is

$$U_{i,q} : M \to I\!R, \tag{14}$$

which depends on the curvature tensor field R, the Ricci tensor field p and the scalar curvature T and their covariant derivatives.

Some of these coefficients have the form

$$a_{0,q} = \binom{n}{q} Vol(M) \tag{15}$$

$$a_{1,q} = \left[\frac{-1}{6}\binom{n}{q} - \binom{n-2}{q-1}\right] \int_M T dM, \tag{16}$$

$$\begin{aligned} a_{2,q} = \int_M \Bigg\{ & \left[\frac{1}{72}\binom{n}{q} - \frac{1}{6}\binom{n-2}{q-1} + \frac{1}{2}\binom{n-4}{q-2}\right] T^2 + \\ & \left[\frac{-1}{180}\binom{n}{q} + \frac{1}{2}\binom{n-2}{q-1} - 2\binom{n-4}{q-2}\right] |p|^2 + \\ & \left[\frac{1}{180}\binom{n}{q} - \frac{1}{12}\binom{n-2}{q-1} + \frac{1}{2}\binom{n-4}{q-2}\right] |R|^2 \Bigg\} dM. \end{aligned} \tag{17}$$

The coefficients $a_{3,q}$ have been estimated only for the cases $q = 0, 1, 2$. Therefore we have

$$\begin{aligned} a_{3,0} = \frac{1}{7!9} \int & [-142 |\nabla T|^2 - 26 |\nabla \rho|^2 \, 7 |\nabla R|^2 - 35 T^3 \\ & + 42 T |\rho|^2 - 42 T |R|^2 + 35 |\rho|^3 - 20 L_1 + 8 L_2 - 8 L_3] dM, \end{aligned} \tag{18}$$

$$\begin{aligned} a_{3,1} = \frac{1}{9 \cdot 40 \cdot 7!} \int_M \Big[& - (980 + 5680n) |\nabla T|^2 - (1078 + 104n) |\nabla \rho|^2 + (49 + 280n) |\nabla R|^2 + \\ & + (1568 - 1680n) T |\rho|^2 + (343 - 1680n) T |R|^2 + (2548 + 1440n) |\rho|^3 + (215 - 1400n) T^3 + \\ & + (392 - 800n) L_1 + (-1392 + 320n) L_2 + (197 - 960n) L_3] dM, \end{aligned} \tag{19}$$

$a_{3,2} = \frac{1}{4 \cdot 7! \cdot 9 \cdot 10} \cdot$

$$\begin{aligned} \int \Big[& (-(2840n^2 - 3330n - 2438) |\nabla T|^2 + (-52n^2 - 1026n + 8036) |\nabla \rho|^2 + (-140n^2 + \\ & + 149n - 1568) |\nabla R|^2 - (-720n^2 + 265n - 1960) T^3 + (840n^2 - 2408n + 17836) T |\rho|^2 + \\ & + (-840n^2 + 1173n - 3626) T |R|^2 + (720n^2 + 1112n - 28616) |\rho|^3 + \\ & + (-800n^2 + 1192n - 18421) L_1 + (160n^2 - 1532n + 26246) L_2 - \\ & - (-480n^2 + 627n - 4708) L_3) \Big] dM, \end{aligned} \tag{20}$$

where

$$L_1 = \rho_{ij} \rho_{kl} R_{ikjl}, \; L_2 = \rho_{ij} R_{iklm} R_{jklm}, \; L_3 = R_{ijkl} R_{ijuv} R_{kluv} \,. \tag{21}$$

We can generalize the above results using theory of fibre bundles. Let V be a vector bundle over the compact Riemannian manifold (M, g). Let (U, φ) be a chart of M with local coordinate system $(x_1, ..., x_n)$. The restriction of g on U is given by

$$g_{/U} = ds^2 = g_{ij} dx_i dx_j \, .$$

Let (g^{ij}) be the inverse matrix of (g_{ij}) and let $|g| = (\det(g_{ij}))^{1/2}$. The Riemannian measure on M is given by

$$dM = g dx_1 \dots dx_m \, .$$

From this vector bundle V we obtain the vector space $C^\infty(V)$ of cross sections of the vector bundle V. Let $D : C^\infty(V) \to C^\infty(V)$ be a second order elliptic differential operator with leading symbol given by the metric tensor. If we choose a local frame for V, we can express D in the form

$$D = -(g^{ij} \partial^2 / \partial x_i \partial x_j + P^k \partial x_k + Q),$$

where P^k and Q are square matrices. These are not invariantly defined, but they depend on the choice of frame and coordinate system.

Let V_x be the fibre of V over a point x. We choose a smooth fibre metric on V. Let $L^2(V)$ be the completion of $C^\infty(V)$ with respect to the global integrated inner product. As a Banach space $L^2(V)$ is independent of the Riemannian and fibre metrics. For $t > 0$, we have

$$\exp(-tD) : L^2(V) \to C^\infty(V)$$

is an infinitely smoothing operator of trace class. The Kernel of $\exp(-tD)$ is defined by

$$K(t, x, y, D) : V_y \to V_x$$

and it is a smooth endomorphism valued function of (t, x, y). It can be proved ([4]) that $K(t, x, y, D)$ vanishes to infinite order for $x \neq y$. If $x = y$, then K has an asymptotic expansion as $t \to 0^+$ of the form

$$K(t, x, y, D) \sim (4\pi t)^{-n/2} \sum_{n=0}^{\infty} E_n \,(x, D) \, t^n,$$

where $E_0(x, D) = I$.

We must notice that $E_n(x, D)$ are local invariants of the differential operator D. If we use local frame for V and a local system of coordinates, then we can express $E_n(x, D)$ functorially as a non-commutative polynomial in the derivatives of the metric tensor and in the derivatives of the matrices P^k and Q with coefficients which are smooth functions of the metric. This polynomial is universal in the sense that the coefficients depend only on the dimension n and are independent of the vector bundle and the operator D.

If D is self-adjoint, let $\{\lambda_i, \theta_i\}$ be a spectral resolution of D into a complete orthonormal basis of eigensections θ_i and eigenvalues λ_i. For such an operator the Kernel function is given by

$$K\left(t, x, y, D\right) = \sum_{i=0} e^{-\lambda_i t} \theta_i\left(x\right) \otimes \theta_j\left(y\right).$$

Hence

$$Tr\left(\exp\left(-t\pi D\right)\right) = \sum_{i=0}^{\infty} e^{-t\lambda_i} = \int_M Tr\left(K\left(t, x, y, D\right)\right) dM.$$

This can be written as follows

$$Tr\left(\exp\left(-tD\right)\right) = \left(4\pi t\right)^{-n/2} \sum t^n \int_M Tr\left(E_n\left(x, D\right)\right) dM.$$

We set

$$a_n\left(x, D\right) = Tr\left(E_n\left(x, D\right)\right), \quad a_n(D) = \int_M a_n\left(x, D\right) dM.$$

The coefficients $a_n(D)$ are isospectral invariants or the operator D.

There are some techniques for estimates of $a_n\left(x, D\right)$. This can be done by tensorial expressions. This method is different than the first one [4].

4.

Let $T^n = I\!R^n / Z^n$ be the flat torus with metric $\bar{g}$ coming from the restriction of the Euclidean metric

$$g = dx_1^2 + ... + dx_n^2$$

on the n-dimensional Euclidean space $I\!R^n(x_1, ..., x_n)$ in T^n. The Laplace operator Δ on the vector space $C^\infty(T^n)$ of functions on T^n has the form

$$\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right).$$

The spectrum of Δ acting on $C^\infty(T^n)$ is given by the formula

$$Sp(T^n, \bar{g}) = \left\{0 < \lambda_1 = \cdots < \lambda_2 = \cdots < \cdots < \lambda_n = \cdots < \cdots < \infty\right\}.$$

The coefficients a_q, $q = 0, 1, \ldots$, are given by

$$a_0 = 2\pi n l, \quad a_q = 0, \quad q = 1, 2, \ldots, n,$$

where l is the radius of the circle.

Let (M, g) be a compact symmetric manifold of rank 1. These Riemannian manifolds are the following:

1. (S^n, g_0) n-dimensional sphere with constant sectional curvature 1.
2. $(\mathbb{P}^n(\mathbb{R}), g_0)$ real projective space of dimension n with constant sectional curvature 1.
3. $(\mathbb{P}^n(C), g_0)$ complex projective space of constant holomorphic sectional curvature 1.
4. $(\mathbb{P}^n(\mathbb{H}), g)$ quaternionic projective space.
5. $P^2(y) = F_4/Spin\, P$ Cayley projective plane.

The coefficients a_q $q = 0, 1, \ldots$, for these manifolds are given in ([1]).

It is still open to compute these coefficients for other manifolds.

Which we require all the operators should be with leading symbol given by the metric tensor.

In order to see this influence we give the following theorems.

Theorem 4.1 *([8]) Let (M, g) and (M', g') be two compact Riemannian manifolds of dimension n with the property $Sp^k(M, g) = Sp^k(M', g')$ for $k = 0$ and $k = 1$. If (M, g) is an Einstein so is (M', g').*

Theorem 4.2 *([7]) Let (M, g) and (M', g') be two compact simply connected Riemannian manifolds of dimension n. If n is given then there is at least one $q \in [0, n]$ such that if $Sp^q(M, g) = Sp^q(M', g')$ and (M, g) has constant sectional curvatnre, so does (M', g') and (M, g) is isometric onto (M', g').*

5.

Let (M, g) be a compact Riemannian manifold of dimension n. We consider the following second order elliptic differential operator:

$$D_r^\varepsilon = \varepsilon \Delta_r + (1 - \varepsilon)\bar{\Delta}_r \ , \tag{22}$$

where $0 \leq \varepsilon \leq 1$, $r = 1, 2, \ldots, n-1$ and $\bar{\Delta}_r$ the Beltrami-Laplace operator acting on r-forms on M. This operator has a spectrum

$$Sp^r\left(M, g, D_r^\varepsilon\right) = \left\{0 \leq \lambda_1\left(\varepsilon\right) \leq \lambda_2\left(\varepsilon\right) \leq \cdots < \infty\right\},$$

which is discrete and each eigenvalue has finite multiplicity. Some of the coefficients of $D_1(\varepsilon)$ are given by

$$a_{0,1}(\varepsilon) = nVol(M), \tag{23}$$

$$a_{1,1}(\varepsilon) = \frac{6\varepsilon - n}{n} \int_M T dM, \tag{24}$$

$$a_{2,1}(\varepsilon) = \frac{1}{360} \int_M [(5n - 60\varepsilon)T^2 + (180\varepsilon^2 - 2n)\left|\rho\right|^2 + (2n - 30)\left|R\right|^2]dM, \tag{25}$$

$$a_{3,1}(\varepsilon) = \frac{1}{360 \cdot 7!} \int_M \Big[(-98 + 588\varepsilon - 5680n) |\nabla T|^2 +$$

$$+(392 - 1470\varepsilon^2 - 1440n)) |\nabla \rho|^2 + (49 - 280n) |\nabla R|^2 +$$

$$+(245 - 1400n)T^3 + (-98\varepsilon - 1470\varepsilon^2 + 1680n)T |\rho|^2 +$$

$$+(245 + 98\varepsilon - 1680n)T |R|^2 + (245\varepsilon - 1400n) |\rho|^3$$

$$+(392 - 800n)L_1 + (98 - 1470\varepsilon + 320n)L_2 + (197 - 960n)L_3] \, dM. \tag{26}$$

The following theorem has been proved ([10]).

Theorem 5.1 *Let (M, g) and (N, h) be two compact Riemannian manifolds. We assume that $Sp^k(M, g, \Delta) = Sp^k(N, h, \Delta)$ for $k = 0, 1, 2$ and $Sp^1(M, g, D_1^\varepsilon) = Sp^1(N, h, D_1^\varepsilon)$ for three distinct values of $\varepsilon \neq 0$. If (M, g) is locally symmetric, so is (N, h).*

The main result of the present paper is to prove that the symmetric space (F_4, g_0), which is the exceptional Lie group with metric g_0 coming from the Killing-Cartan form on the Lie algebra t_4 of F_4.

Now, we prove the below theorem.

Theorem 5.2 *Let (M, g) be a compact Riemannian irreducible manifold of dimension 52. If we have the relations $Sp^k(M, g, \Delta) = Sp^k(F_4, g_0, \Delta)$ for $k = 0, 1, 2$ and $Sp^1(M, g, D_1^\varepsilon) = Sp^1(F_4, g_0, D_1^\varepsilon)$ for three distinct values of $\varepsilon \neq 0$, where (F_4, g_0) is the exceptional Lie group with Riemannian metric g_0 coming from the Killing-Cartan form on the Lie algebra t_4 of F_4, then (M, g) is isometric onto (F_4, g_0).*

Proof. From the assumption we conclude that the manifold (M, g) is symmetric. The only 52 dimensional irreducible symmetric manifold are the following:

$Sp(14)/Sp(13) \times Sp(1) = I\!P^{13}(I\!H)$;
$SU(27)/S(U(26) \times U(1)) = I\!P^{26}(C)$;
$SU(15)/S(U(13) \times U(2))$;
$SO(53)/SO(52) \times S0(1) = S^{52}$;
$SO(28)/SO(26) \times S0(2)$;
$SO(17)/SO(13) \times S0(4)$;

Since we have

$Sp(F_4, g_0, \Delta_0) \neq Sp(I\!P^{13}(I\!H), g_1, \Delta_0)$ ([1]), ([14]);
$Sp(F_4, g_0, \Delta_0) \neq Sp(I\!P^{26}(C), g_2, \Delta_0)$ ([1]), ([14]);
$Sp(F_4, g_0, \Delta_0) \neq Sp(SU(15)/S(U(13) \times U(2)), g_3, \Delta_0)$ ([12]), ([14]);
$Sp(F_4, g_0, \Delta_0) \neq Sp(SU(53)/SO(52) \times S0(1), g_4, \Delta_0)$ ([1]), ([14]);
$Sp(F_4, g_0, \Delta_0) \neq Sp(SO(26) \times S0(2), g_5, \Delta_0)$ ([15]), ([14]);
$Sp(F_4, g_0, \Delta_0) \neq Sp(SO(17)/SO(13) \times S0(4), g_6, \Delta_0)$ ([12]), ([14]),

where g_1, g_2, g_3, g_4, g_5 and g_6 are the Riemannian metrics on the manifolds $I\!P^{13}(I\!H)$, $I\!P^{26}(C)$, $SU(15)/S(U(13) \times U(2))$, $SU(53)/SO(52) \times S0(1)$, $SO(26) \times S0(2)$, $SO(17)/SO(13) \times S0(4)$ respectively coming from the Killing-Cartan form on the Lie algebras t_{14}, t_{27}, t_{15}, t_{53}, t_{28} and t_{17} respectively of the Lie groups $Sp(14)$, $SU(27)$, $SU(5)$, $SU(53)$, $SO(28)$ and $SO(12)$ respectively.

Finally we conclude that the irreducible Riemannian symmetric manifold coincides with the exceptional Lie groups (F_4, g_0).

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