SOME REMARKS ON I-DERIVABLE CURVES

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Abstract

This paper studies a special class of curves, called by the author i-derivable curves. The special circles define an interesting Lagrangean structure permitting to characterize the special circles. The existence of the circular points on the special circles leads to a central symmetry for the inverse of the given i-derivable curve. An interesting metric spaces class is highlighted, so that the distance between two close points has the same Lagrangean form as those described by special circles.

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We shall consider in the two-dimensional Euclidean plane known the concepts: curve, closed curve, convex set, tangent in a point to a curve as it appears in [1], [4]. Let P_0 be a fixed point belonging to a given curve $c : I \subset I\!R \to I\!R^2$ and let P be a variable point on c in the neighborhood $U(P_0) \cap c$ of P_0.

Definition 1 The curve c is called *dual derivable* if the limit of the intersection of the tangents in P_0 and P, when P is moving on curve to P_0, is the fixed point P_0.

Observation 1 The dual derivability excludes the existence of rectilinear components of the curve.

Observation 2 Obviously, a simple closed curve having its interior as convex set is not dual derivable.

Definition 2 We shall call *parallel derivable curve* any simple closed curve belonging to the two-dimensional Euclidean plane which satisfies the conditions:

i) for any direction it allows just two tangents parallel with a given direction;

ii) the tangents described above do not intersect again the interior of the curve.

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Observation 3 The parallel derivability does not imply necessary a dual derivability for a curve, such that the following definition makes sense.

Definition 3 We shall call that the curve K is *i-derivable* if K is a simple closed curve of the two-dimensional Euclidean plane which proceeds from a parallel and dual derivable curve K^* by geometric inversion of an arbitrary power with respect to an arbitrary point as pole, pole which is contained in the interior of K^*.

Lemma 1 *A parallel and dual derivable curve K^* has as interior a convex set.*

Proof. If not, there exist $M, N \in K^*$ such that the segment line MN intersects K^*. That means K^* has points in the both sides of MN . In each side there exists, using Lagrange's theorem, tangent lines parallel with MN. It is obvious that one of this tangent will intersect the interior of K^*, in collision with the parallel derivability of K^*. $\square$

Lemma 2 *An i-derivable curve is dual and parallel derivable and its interior is a convex set.*

Proof. Consider a point A contained in the interior of the given i-derivable curve denoted by K.

Taking into account that the inverse K^* is dual and parallel derivable, the geometric inversion $I(A, \mu)$ of arbitrary power will conserve both the angles between curves and the tangence, that means that K will be a dual and parallel derivable curve.

The convex interior of K^* in Lemma 1 will be transformed into a convex set bounded by the initial curve K,with respect to $I(A, \mu)$.$\square$

Lemma 3 *In any point A situated in its interior, an i-derivable curve K permits a pair of circles both mutually tangent in A and being each one also tangent in a unique point at K. The common tangent line in A of the two circles may have any direction.*

Proof. The i-derivable curve K proceeds from the inversion of K^* with respect to A, an interior point of K^*. The parallel lines having a given direction are transformed in tangent circles passing by A with the tangent line in A parallel with Δ. Taking into account Lemma 2, the circles tangent in A will intersect each one K in only one point.$\square$

Denote by s, S the tangent points at K of the circles described by Lemma 3 and by r, R the length of the radii of the same circles. We can observe that r, R depend on the point A and by the direction Δ.

Definition 4 We shall call *special circle,* the circle determined by the points *s, A and S.*

Lemma 4 *An i-derivable curve allows a Lagrangean structure in its interior.*

Proof. Using x_1, x_2 as usual coordinates in the plane of the curve, we shall consider the arclength element described by

$$ds = M\left(x_1, x_2, \dot{x}_1, \dot{x}_2\right)\sqrt{dx_1^2 + dx_2^2},$$

where we denote by $M\left(x_1, x_2, \dot{x}_1, \dot{x}_2\right)$ the expression $\frac{1}{2}(\frac{1}{r} + \frac{1}{R})$, according to the special circle determined by A and $\Delta = \frac{\dot{x}_2}{\dot{x}_1}$.

Suppose that M is a differentiable function and let us denote by $L\left(x_1, x_2, \dot{x}_1, \dot{x}_2\right) := M\left(x_1, x_2, \dot{x}_1, \dot{x}_2\right)\sqrt{\dot{x}_1^2 + \dot{x}_2^2}$. Taking into account that the zeros of the first differential form $\frac{\partial L}{\partial \dot{x}_1}dx_1 + \frac{\partial L}{\partial \dot{x}_2}dx_2$ over $I\!R^2$ are straightlines having like equation $\frac{\partial L}{\partial \dot{x}_1}x_1 + \frac{\partial L}{\partial \dot{x}_2}x_2$ with a precise slope, it is justified the following definition by the colligation slope-transversal direction.

Definition 5 We shall call *transversal direction* the expression $\frac{dx_2}{dx_1}$ defined by $\frac{\partial L}{\partial \dot{x}_1}dx_1 + \frac{\partial L}{\partial \dot{x}_2}dx_2 = 0$.

Theorem 1 *If $L(x_1, x_2, \dot{x}_1, \dot{x}_2)$ is a differentiable function then the transversal direction in the point A is coincident with the orthogonal direction to the tangent in A at the special circle determined by the point A and the direction $\Delta = \frac{\dot{x}_2}{\dot{x}_1}$.*

Proof. It is obvious that L may be thought as $M(x_1, x_2, \frac{\dot{x}_2}{\dot{x}_1})\sqrt{\dot{x}_1^2 + \dot{x}_2^2}$ or as $M(x_1, x_2, \theta)\sqrt{\dot{x}_1^2 + \dot{x}_2^2}$, where $\theta = \arctan \frac{\dot{x}_2}{\dot{x}_1}$.

We have successively:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial \left(M\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right)}{\partial \dot{x}_1} = \frac{\partial M}{\partial \dot{x}_1}\sqrt{\dot{x}_1^2 + \dot{x}_2^2} + M\frac{\partial \left(\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\right)}{\partial \dot{x}_1} =$$

$$\frac{dM}{d\theta} \cdot \frac{d\theta}{d\dot{x}_1}\sqrt{\dot{x}_1^2 + \dot{x}_2^2} + M\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\frac{dM}{d\theta} \cdot \frac{-\frac{\dot{x}_2}{\dot{x}_1^2}}{1 + \left(\frac{\dot{x}_2}{\dot{x}_1}\right)^2} + M \cdot \frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}.$$

Analogously, we have:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2} \cdot \frac{dM}{d\theta} \cdot \frac{\frac{1}{\dot{x}_1}}{1 + \left(\frac{\dot{x}_2}{\dot{x}_1}\right)^2} + M \cdot \frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}.$$

Then the transversal direction is defined by:

$$\left(\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\frac{dM}{d\theta}\frac{-\frac{\dot{x}_2}{\dot{x}_1^2}}{1 + \left(\frac{\dot{x}_2}{\dot{x}_1}\right)^2} + M\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}\right)dx_1 +$$

$$\left(\sqrt{\dot{x}_1^2+\dot{x}_2^2}\frac{dM}{d\theta}\frac{\frac{1}{\dot{x}_1}}{1+\left(\frac{\dot{x}_2}{\dot{x}_1}\right)^2}+M\frac{\dot{x}_1}{\sqrt{\dot{x}_1^2+\dot{x}_2^2}}\right)dx_2=0$$

By calculating we obtain:

$$\frac{dM}{d\theta}:M=\frac{\dot{x}_1dx_1+\dot{x}_2dx_2}{\dot{x}_2dx_1-\dot{x}_1dx_2}. \tag{1}$$

Consider the geometric inversion having A as pole and 1 as power. The two tangent circles described in Lemma 3 become two parallel tangent lines at K^* orthogonal to the direction $\frac{\dot{x}_2}{\dot{x}_1}$. These parallel lines have the point A between them, spaced at $\frac{1}{2r}$, $\frac{1}{2R}$ respectively. Obviously, $M=\frac{1}{2}\left(\frac{1}{r}+\frac{1}{R}\right)$ is the distance between the parallel lines.

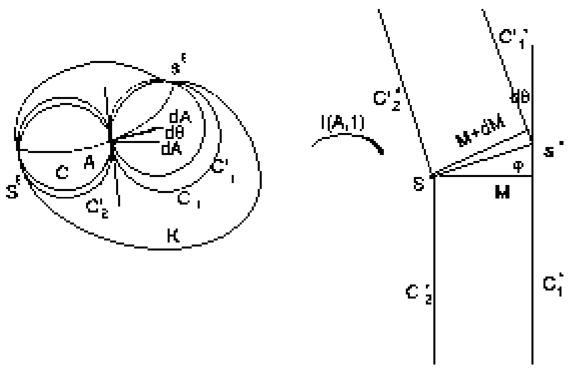

Figure 1.

Consider two pairs of orthogonal circles corresponding both to the point A and to the sufficiently close directions Δ and Δ', so that $d\theta$ is the angle between Δ and Δ'. Denoting by dA, dA' the orthogonal directions corresponding to Δ and Δ' respectively, and by S^ε, s^ε the contacts with the curve of the second pair of circles, we will obtain after the inversion

$$S^*s^*=\frac{M}{\sin\varphi}=\frac{M+dM}{\sin(\pi-\varphi-d\theta)},$$

where S^*, s^* are the inverses of S^ε, s^ε and φ is the angle between the transverse circle and K. Therefore

$$\frac{M+dM}{M}=1+d\theta\cdot\cot\varphi,$$

or, equivalent,

$$\frac{dM}{d\theta} : M = \cot\varphi. \tag{2}$$

Thus, (1) and (2) imply

$$\frac{\dot{x}_1 dx_1 + \dot{x}_2 dx_2}{\dot{x}_2 dx_1 - \dot{x}_1 dx_2} = \cot\varphi. \tag{3}$$

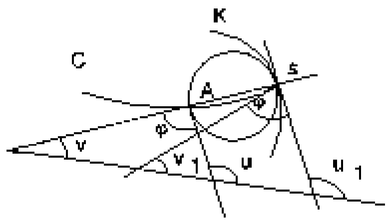

Figure 2.

Let us use Figure 2 and let us denote by:

i) α the slope of the tangent line of the first pair of circles (therefore $\tan u = \alpha$);

ii) τ_1 the slope of the tangent line in A at the first special circle (therefore $\tan v = \tau_1$ and also $\dfrac{\dot{x}_2}{\dot{x}_1} = \tau_1$).

It results

$$\cot\varphi = \cot(u - v) = \frac{1 + \alpha\tau_1}{\alpha - \tau_1} = \frac{\dot{x}_1 dx_1 + \dot{x}_2 dx_2}{\dot{x}_2 dx_1 - \dot{x}_1 dx_2}. \tag{4}$$

Using (3) and (4) we obtain

$$\left(1 + \frac{\dot{x}_2}{\dot{x}_1}\frac{dx_2}{dx_1}\right)(\alpha - \tau_1) = \left(\frac{\dot{x}_2}{\dot{x}_1} - \frac{dx_2}{dx_1}\right)(1 + \alpha\tau_1),$$

or, in the final form,

$$\left(\frac{\dot{x}_2}{\dot{x}_1} - \alpha\right)\left(1 + \frac{dx_2}{dx_1}\cdot\tau_1\right) = \left(1 + \frac{\dot{x}_2}{\dot{x}_1}\alpha\right)\left(\frac{dx_2}{dx_1} - \tau_1\right).$$

Since $1+\dfrac{\dot{x}_2}{\dot{x}_1}\alpha = 0$, it results $1+\dfrac{dx_2}{dx_1}\cdot\tau_1 = 0$, or, in the older form, $\dfrac{-\frac{\partial L}{\partial \dot{x}_1}}{\frac{\partial L}{\partial \dot{x}_2}}\cdot\tau_1 = -1$. This means that the transverse direction is orthogonal in A to the special circle corresponding to the direction Δ. $\square$

We shall show that a particular distance that we shall introduce in the interior of an i-derivable curve leads to the same finslerian metric as the one introduced by the simple circles of i-derivable curves.

Consider A and B as fixed points in the interior of the i-derivable curve denoted by K and P an arbitrary point on K.

The Euclidean distances $\|PA\|$, $\|PB\|$ determine a function $f(P) := \dfrac{\|PA\|}{\|PB\|}$, $f : K \to \mathbb{R}_+^*$, which has a maximum M_{AB} and a minimum m_{AB}, when P is moving on K.

Theorem 2 *$d(A,B) := \ln M_{AB} \cdot m_{AB}^{-1}$ is a distance between A and B.*

Proof. If $A = B$ then $f(P) = \dfrac{\|PA\|}{\|PB\|}$ for any $P \in K$ and that means $\ln \dfrac{M_{AB}}{m_{AB}} = \ln 1 = 0$. If $\ln \dfrac{M_{AB}}{m_{AB}} = 0$ for a pair A, B then $M_{AB} = m_{AB}$ and that means that the function is constant. Or, if $A \neq B$, it results that P which belongs to K also belongs to the Apolloniu's circle of the pair A, B. But A and B are separated by the Apolloniu's circle which coincides with K, in collision with A, $B \in int\,K$.

For $d(A,B) = d(B,A)$, it is enough to observe that

$$\min_{P\in K} \frac{\|PA\|}{\|PB\|} = \frac{1}{\max\limits_{P\in K} \frac{\|PB\|}{\|PA\|}}.$$

We wish to prove that for any three points A, B, C in $int\,K$ we have

$$d(A,B) + d(B,C) \geq d(A,C). \tag{5}$$

Let S_1, S_2, S_3; s_1, s_2, s_3 be the points for which the maximum and the minimum of the three ratios is reached:

$$\frac{\frac{\|S_1A\|}{\|S_1B\|}}{\frac{\|s_1A\|}{\|s_1B\|}} = \frac{M_{AB}}{m_{AB}};\ \frac{\frac{\|S_2B\|}{\|S_2C\|}}{\frac{\|s_2B\|}{\|s_2C\|}} = \frac{M_{BC}}{m_{BC}};\ \frac{\frac{\|S_3A\|}{\|S_3C\|}}{\frac{\|s_3A\|}{\|s_3C\|}} = \frac{M_{AC}}{m_{AC}}.$$

Therefore, for the substitutions with minorant role S_1, $S_2 \to S_3$; s_1, $s_2 \to s_3$, we obtain

$$\frac{M_{AB}}{m_{AB}} \cdot \frac{M_{BC}}{m_{BC}} \geq \frac{\|S_3A\|}{\|S_3C\|} : \frac{\|s_3A\|}{\|s_3C\|} = \frac{M_A}{m_A},$$

equivalently with (5). See also [2] , [3]. $\square$

Let A be a point belonging to the interior of the i-derivable curve K, Δ be a given direction and $A + dA$ be another point in a small neighborhood of A such that dA is orthogonal to Δ. In accordance with Theorem 1 the special circle determined by A and Δ has dA as tangent; let us denote by R, r the radii of the circles which appear in Lemma 3, and by ds the infinitesimal distance established by Theorem 2, between the points A and $A+dA$,i.e.

$$ds = d\,(A, A+dA) = \ln \frac{\max\limits_{P\in K} \frac{\|PA\|}{\|P(A+dA)\|}}{\min\limits_{P\in K} \frac{\|PA\|}{\|P(A+dA)\|}}.$$

Let us denote by d the Euclidean distance between the points A, $A+dA$.

Theorem 3 *The distance between two close points A, $A+dA$ has the same form as the Lagrangean arclength determined by the special circle corresponding to the point A and to the direction Δ.*

Proof. We have to prove that $ds = \frac{1}{2}(\frac{1}{R}+\frac{1}{r})d\sigma$.
In the given conditions $ds = \dfrac{M_{A(A+dA)} - m_{A(A+dA)}}{m_{A(A+dA)}}$. For A, $A+dA$, P with the coordinates (x_1, x_2), (x_1^1, x_2^1), (x^1, x^2) the Apolloniu's circle determined by A, $A+dA$ and the constant $\sqrt{\lambda}$ has the equation

$$\sum_{i=1}^{2} \left(\left(x^i - x_i\right)^2 - \lambda \left(x^i - x_i^1\right)^2 \right) = 0.$$

Its radius will be

$$\rho^2 = \frac{\lambda}{(1-\lambda)^2} \sum_{1}^{2} \left(x^i - x_i^1\right)^2.$$

For the maximum $M_{A(A+dA)}$ and the minimum $m_{A(A+dA)}$ of the expression $\dfrac{\|PA\|}{\|P\,(A+dA)\|}$, it appears

$$R^2 = \frac{M_{A(A+dA)}}{\left(1 - M_{A(A+dA)}\right)^2} d\sigma^2,\ r^2 = \frac{m_{A(A+dA)}}{\left(1 - m_{A(A+dA)}\right)^2} d\sigma^2,$$

so it results:

$$\frac{M_{A(A+dA)} - m_{A(A+dA)}}{m_{A(A+dA)}} = \frac{2\left(\sqrt{d\sigma^2 + 4r^2} + \sqrt{d\sigma^2 + 4R^2}\right) d\sigma}{\left(-d\sigma + \sqrt{d\sigma^2 + 4R^2}\right)\left(d\sigma + \sqrt{d\sigma^2 + 4r^2}\right)}.$$

Taking into account that we can neglect small infinities of second order, we obtain $\dfrac{2\,d\sigma}{d\sigma + \sqrt{d\sigma^2 + 4a^2}} = \dfrac{d\sigma}{a}$ and also $ds = \frac{1}{2}(\frac{1}{R}+\frac{1}{r})d\sigma$.

Definition 6 The point Q belonging to the interior of an i-derivable curve denoted by K is called a *circular point* if:

i) after the revolution with π as angle of the direction Δ the points s_Δ,S_Δ describe completely the curve K and, *ii)* on each special circle which passes through Q there exists a point O_Δ such that after the previous revolution the set of points O_Δ determines a simple closed curve.

Theorem 4 *If an i-derivable curve allows a circular point, then its inverse with respect to the circular point allows a central symmetry.*

Proof. Consider a geometric inversion with an arbitrary power having as pole the circular point. We shall show that the set of points O_Δ from the previous definition is formed by an element only. We introduce a system of cartesian coordinates and let Δ be an arbitrary direction having as slope $\tan\varphi$. Let S^*, s^* be the contacts of tangent lines parallel with Δ. We shall give up Δ in our notations.

We have that the inverse of the point O, denoted by O^*, belongs to the segment line s^*S^*.

Let (x^1, x^2) , (x_1^*, x_2^*) , (X_1^*, X_2^*) be the coordinates of the points O^*, s^*, S^* and $\tan\Psi$ be the slope of the straightline s^*S^*. Then we have:

$$\begin{aligned} x_1^* &= x_1 + \lambda\cos\Psi, \\ x_2^* &= x_2 + \lambda\sin\Psi, \\ X_1^* &= x_1 + \Lambda\cos\Psi, \\ X_2^* &= x_2 + \Lambda\sin\Psi, \end{aligned} \tag{6}$$

where $\lambda := \overline{o^*s^*}$, $\Lambda := \overline{O^*S^*}$ are oriented segments and

$$\|s^*S^*\| = \Lambda - \lambda.$$

Therefore O^* describes a continuous bounded curve Ω^* which is contained in the interior of K^*, the inverse of K. Both the coordinates functions (x^1, x^2) for Ω^* and $\tan\Psi$ depend continuously by $\tan\varphi$. We observe that $\tan\Psi$ is strictly increasing and that means $d\Psi > 0$. We have

$$\begin{cases} dx_1 = \cos\Psi\ d\omega \\ dx_2 = \sin\Psi\ d\omega \end{cases},$$

where $d\omega^2 = dx_1^2 + dx_2^2$. The condition of parallelism of the two tangent lines at K^* can be written:

$$\begin{vmatrix} dx_1^* & dx_2^* \\ dX_1^* & dX_2^* \end{vmatrix} = 0$$

so, in accordance with (6), we obtain

$$\begin{vmatrix} d(\omega+\Lambda)\cos\Psi - \Lambda d\Psi\sin\Psi & d(\omega+\Lambda)\sin\Psi + \Lambda d\Psi\cos\Psi \\ d(\omega+\lambda)\cos\Psi - \lambda d\Psi\sin\Psi & d(\omega+\lambda)\sin\Psi + \lambda d\Psi\cos\Psi \end{vmatrix} = 0$$

or, equivalently,

$$\begin{vmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{vmatrix} \cdot \begin{vmatrix} d(\omega+\Lambda) & d(\omega+\lambda) \\ \Lambda & \lambda \end{vmatrix} \cdot d\Psi = 0$$

i.e.

$$\begin{vmatrix} d(\omega+\Lambda) & d(\omega+\lambda) \\ \Lambda & \lambda \end{vmatrix} = 0. \tag{7}$$

If we denote by $\rho = \dfrac{\lambda}{\Lambda - \lambda}$, we have

$$d\rho = \frac{\Delta d\lambda - \lambda d\Lambda}{(\Lambda - \lambda)^2}$$

and using (7) we obtain

$$\frac{d\omega}{\Lambda - \lambda} + \frac{\Delta d\lambda - \lambda d\Lambda}{(\Lambda - \lambda)^2} = 0$$

or, equivalently, $d\rho = -\dfrac{d\omega}{\Lambda - \lambda}$.

This means that

$$\rho = -\int_0^{\varphi} \frac{d\omega}{\Lambda - \lambda},\ \rho_0 = -\int_0^{\varphi_0} \frac{d\omega}{\Lambda - \lambda}\ ,$$

where $\lambda_0 = \lambda(\varphi_0)$, $\Lambda_0 = \Lambda(\varphi_0)$. Taking into account the geometric signification of the revolution with 2π we obtain

$$\lambda_0 = -(\Lambda_0 - \lambda_0)\int_0^{\varphi_0} \frac{d\omega}{\Lambda - \lambda}$$

and

$$\lambda_0 = -(\Lambda_0 - \lambda_0)\int_0^{\varphi_0+2\pi} \frac{d\omega}{\Lambda - \lambda}.$$

The last two relations lead to

$$\int_0^{\varphi_0+2\pi} \frac{d\omega}{\Lambda - \lambda} = 0. \tag{8}$$

But the geometric meaning of the ratio $\dfrac{d\omega}{\Lambda - \lambda}$ leads to a constant positive sign during the previous revolution. This and (8) assert that $d\omega = 0$ for any $\varphi \in [0, 2\pi)$, that means $dx_1 = 0$, $dx_2 = 0$. It results that the coordinates of the point O^* are constants. Since after a revolution of the segment line S^*s^* having π as angle we have $s^* \to S^*$,$s^* \to S^*$, it results that O^* is the midpoint of the segment line S^*s^*. Therefore it appears the symmetry for the curve K^*.

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