

# NONPERTURBATIVE METHODS FOR CALCULATING THE HEAT KERNEL

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## Abstract

We study the low-energy approximation for calculation of the heat kernel which is determined by the strong slowly varying background fields in strongly curved quasi-homogeneous manifolds. A new covariant algebraic approach, based on taking into account a finite number of low-order covariant derivatives of the background fields and neglecting all covariant derivatives of higher orders, is proposed. It is shown that a set of covariant differential operators together with the background fields and their low-order derivatives generates a finite dimensional Lie algebra. This algebraic structure can be used to present the heat semigroup operator in the form of an average over the corresponding Lie group. Closed covariant formulas for the heat kernel diagonal are obtained. These formulas serve, in particular, as the generating functions for the whole sequence of the Hadamard-Minakshisundaram-De Witt-Seeley coefficients in all symmetric spaces.

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## 1 Introduction

The heat kernel is an important tool in quantum field theory and mathematical physics [1-13]. In particular, the one-loop contribution of quantized bosonic fields on a Riemannian manifold  $(M, g)$  to the effective action is given by the functional determinant of some elliptic differential operator  $\Delta$  acting on the smooth sections  $\varphi \in C^\infty(V)$  of a vector bundle  $V$  over the manifold  $M$  [1]

$$\Gamma_{(1)} = \frac{1}{2} \log \text{Det } \Delta. \quad (1.1)$$

Using the standard spectral  $\zeta$ -function [14]

$$\zeta(p) = \text{Tr } \Delta^{-p}, \quad (1.2)$$

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where "Tr" means the functional trace, one can present the effective action in terms of the derivative of the zeta-function at  $p = 0$ :

$$\Gamma_{(1)} = -\frac{1}{2}\zeta'(0). \quad (1.3)$$

The most convenient way to evaluate the zeta-function in the general case is to express it in terms of the corresponding heat kernel  $\exp(-t\Delta)$  [15]

$$\zeta(p) = \frac{1}{\Gamma(p)} \int_0^\infty dt t^{p-1} \int_M d\text{vol} \text{tr} U(t), \quad (1.4)$$

where "tr" means the usual bundle (matrix) trace and  $U(t)$  is the heat kernel diagonal, i.e. the restriction to the diagonal  $M \times M$  of the heat kernel:

$$U(t) = \exp(-t\Delta)\delta(x, x')|_{x=x'}, \quad (1.5)$$

where  $\delta(x, x')$  is the covariant distribution on  $M \times M$ .

By choosing the appropriate gauge and parametrization one can almost always reduce the problem to the Laplace type operators:

$$\Delta = -\square + Q + m^2, \quad (1.6)$$

where  $\square = g^{\mu\nu}\nabla_\mu\nabla_\nu$  is the generalized Laplacian,  $\nabla$  is the covariant derivative on  $C^\infty(V)$ ,  $Q$  is an arbitrary endomorphism of the vector bundle  $V$  and  $m$  is a mass parameter.

The effective action and the zeta function are determined by the spectrum of the operator  $\Delta$  and are very complicated functionals of the background fields, i.e. the metric  $g$ , the bundle connection  $\nabla$  and the endomorphism  $Q$ . Obviously, the effective action can be calculated exactly only for some very specific simple backgrounds. In quantum field theory, however, one needs the effective action for the *generic* background. Therefore, one has to develop consistent *approximate* methods for its calculation. Moreover, these approximations should be manifestly *covariant*, i.e. they have to preserve the gauge invariance at each order.

For the essentially *local* analysis that is carried out in this paper it is sufficient to characterize the background fields only by the local covariant objects, i.e. the curvatures, the Riemann curvature of  $(M, g)$  and the curvature of the bundle connection  $\nabla$ , and their covariant derivatives. We denote the components of the Riemann curvature and the curvature of the bundle connection by  $R_{\mu\nu\alpha\beta}$  and  $\mathcal{R}_{\mu\nu}$  and call below all the quantities  $\mathfrak{R} = \{R_{\mu\nu\alpha\beta}, \mathcal{R}_{\mu\nu}, Q\}$  just the *background curvatures*.

Further, following [16] we introduce the infinite set of all covariant derivatives of the curvatures:

$$\mathcal{J} = \{\mathfrak{R}_{(i)}; (i = 1, 2, \dots)\}, \quad \mathfrak{R}_{(i)} = \{\underbrace{\nabla \cdots \nabla}_{i} \mathfrak{R}\} \quad (1.7)$$

and call them the *background jets*. The whole set of jets,  $\mathcal{J}$  completely describes the background locally.

Since it is not possible to calculate the heat kernel exactly, one is forced to consider different asymptotic expansions. A consistent way to construct the asymptotic expansions was developed in [16]. The idea is the following. One makes a deformation of the background fields with two deformation parameters,  $\alpha$  and  $\epsilon$ ,

$$g \rightarrow g(\alpha, \epsilon), \quad \nabla \rightarrow \nabla(\alpha, \epsilon), \quad Q \rightarrow Q(\alpha, \epsilon), \quad (1.8)$$

in such a way that the jets transform uniformly

$$\mathfrak{R}_{(i)} \rightarrow \alpha \epsilon^i \mathfrak{R}_{(i)}. \quad (1.9)$$

This deformation changes the operator  $\Delta$  and, of course, the heat kernel  $U(t)$ :

$$U(t) \rightarrow U(t; \alpha, \epsilon) \quad (1.10)$$

and is manifestly covariant because of the transformation law (1.9).

Thus it gives a natural framework to develop various asymptotic expansions with respect to the parameter  $t$  and the deformation parameters  $\alpha$  and  $\epsilon$ . The limit  $t \rightarrow 0$  corresponds to small background jets,  $t^{1+i/2} \mathfrak{R}_{(i)} \ll 1$ , the limit  $\alpha \rightarrow 0$  corresponds to the situation when the powers of curvatures are much smaller than the derivatives of them, so called short-wave, or *high-energy approximation*,  $\nabla \nabla \mathfrak{R} \gg \mathfrak{R} \mathfrak{R}$ , and the limit  $\epsilon \rightarrow 0$  corresponds to the case when the derivatives of the curvatures are much smaller than the products of the curvatures of corresponding dimension, so called long-wave, or *low-energy approximation*,  $\nabla \nabla \mathfrak{R} \ll \mathfrak{R} \mathfrak{R}$ . For a more detailed discussion see [16-22].

As  $t \rightarrow 0$ , one has the well known asymptotic expansion [1-9]

$$U(t) \sim (4\pi t)^{-d/2} \exp(-tm^2) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} b_k, \quad (1.11)$$

where  $d$  is the dimension of the manifold. The coefficients  $b_k$  are the famous Hadamard-Minakshisundaram-De Witt-Seeley (HMDS) coefficients [1,4-9,23-28]. They are purely local universal invariants built from the background curvatures and their covariant derivatives that do not depend on the global structure of the manifold and the boundary conditions. They play a very important role both in physics and mathematics [6,27,28]. The HMDS-coefficients are known now in general case up to  $b_4$  [4,5,23-26]. The  $b_4$  coefficient in general case was calculated for the first time in our PhD thesis [4] and has been published then in [24,25,5].

However, the asymptotic expansion (1.11) is of very limited applicability. It is absolutely inadequate for large  $t$  ( $t\mathfrak{R} \gg 1$ ) in strongly curved manifolds and strong background fields. Therefore, this approximation cannot describe essentially nonperturbative *nonlocal* and *nonanalytical* effects. The investigation of such effects requires consideration of other approximation schemes.

In the high-energy limit,  $\alpha \rightarrow 0$ , there is an expansion [16]

$$U(t; \alpha, \epsilon) \sim (4\pi t)^{-d/2} \exp(-tm^2) \sum_{n=0}^{\infty} (\alpha t)^n h_n(t; \epsilon), \quad (1.12)$$

where  $h_n(t; \epsilon)$  are some *nonlocal* functionals. They are *resummed perturbative* objects. This approximation was studied in details in our PhD thesis [4] and in the papers [5,29,30], where the explicit form of the functionals  $h_1$  and  $h_2$  was obtained and analyzed. The third coefficient  $h_3$  has been investigated in [31].

The long-wave (or low-energy) approximation is determined by strong slowly varying background fields. It corresponds to the asymptotic expansion of the deformed heat kernel as  $\epsilon \rightarrow 0$  [16]

$$U(t; \alpha, \epsilon) \sim (4\pi t)^{-d/2} \exp(-tm^2) \sum_{l=0}^{\infty} (\epsilon^2 t)^l u_l(t; \alpha). \quad (1.13)$$

The coefficients  $u_l$  are essentially *non-perturbative* functionals. They cannot be obtained in any perturbation theory and are much complicated than the HMDS-coefficients  $b_k$  and the high-energy functionals  $h_n$ .

We consider in this paper mostly the zeroth order of this approximation, i.e. the coefficient  $u_0$ , which corresponds simply to covariantly constant background curvatures

$$\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \quad \nabla_\mu R_{\alpha\beta} = 0, \quad \nabla_\mu Q = 0. \quad (1.14)$$

The coefficient  $u_0$  depends, of course, on the global structure of the manifold. However, the asymptotic expansion of  $u_0$  as  $t \rightarrow 0$  is purely local and determines all the terms without covariant derivatives in all HMDS-coefficients  $b_k$ . Therefore, it can be viewed on as the *generating function* for all HMDS-coefficients in covariantly constant background.

The conditions (1.14) determine the geometry of *locally* symmetric spaces [32,33]. The *globally* symmetric manifold satisfies additionally some topological restrictions and the condition (1.14) is valid everywhere in the manifold. However, in typical *physical* problems, the situation is rather different. One has usually a complete noncompact asymptotically flat space-time manifold without boundary that is homeomorphic to  $\mathbb{R}^d$ . In the low-energy approximation a finite (not small, in general) region of the manifold exists that is locally strongly curved and quasi-homogeneous, i.e., the local invariants of the curvature in this region vary very slowly. Then the geometry of this region is *locally* very similar to that of a symmetric space. However, globally, the manifold can be completely different from the symmetric space and one should keep in mind that there are *always* regions in the manifold where the condition (1.14) is not fulfilled. See the discussion in [17-22].

Thus the problem is to calculate the low-energy heat kernel diagonal (1.5) for covariantly constant background (1.14). In other words one has to construct a local covariant function of the invariants of the curvatures that would describe adequately the low-energy limit of the heat kernel diagonal and that would, when expanded in curvatures, reproduce *all terms without covariant derivatives* in the asymptotic expansion of the heat kernel. If one finds such an expression, then one can simply determine the  $\zeta$ -function, (1.4), and, therefore, the effective action, (1.3).

## 2 Algebraic approach

There exist a very elegant indirect way to construct the heat kernel without solving the heat equation but using only the commutation relations of some covariant first order differential operators [16-22]. The main idea is in a generalization of the usual Fourier transform to the case of operators and consists in the following.

Let us consider for a moment a trivial case, where the curvatures vanish but not the potential term:

$$R_{\alpha\beta\gamma\delta} = 0, \quad \mathcal{R}_{\alpha\beta} = 0, \quad \nabla Q = 0. \quad (2.1)$$

In this case the operators of covariant derivatives obviously commute and form together with the potential term an Abelian Lie algebra

$$[\nabla_\mu, \nabla_\nu] = 0, \quad [\nabla_\mu, Q] = 0. \quad (2.2)$$

It is easy to show that the *heat semigroup operator* can be presented in the form

$$\begin{aligned} \exp(-t\Delta) &= (4\pi t)^{-d/2} \exp[-t(m^2 + Q)] \\ &\times \int_{\mathbb{R}^d} dk g^{1/2} \exp\left(-\frac{1}{4t} \langle k, gk \rangle + k \cdot \nabla\right), \end{aligned} \quad (2.3)$$

where  $\langle k, gk \rangle = k^\mu g_{\mu\nu} k^\nu$ ,  $k \cdot \nabla = k^\mu \nabla_\mu$  and  $g^{1/2} = \sqrt{\det g_{\mu\nu}}$ . Here, of course, it is assumed that the covariant derivatives commute also with the metric

$$[\nabla_\mu, g_{\alpha\beta}] = 0. \quad (2.4)$$

Acting with this operator on the  $\delta$ -function and using the obvious relation

$$\exp(k \cdot \nabla) \delta(x, x') \Big|_{x=x'} = g^{-1/2} \delta(k), \quad (2.5)$$

one integrates easily over  $k$  and obtains the heat kernel diagonal

$$U(t) = (4\pi t)^{-d/2} \exp[-t(m^2 + Q)]. \quad (2.6)$$

In fact, the commutators of the covariant differential operators  $\nabla$  do not vanish but are proportional to the curvatures  $\mathfrak{R}$ . The commutators of covariant derivatives  $\nabla$  with the curvatures  $\mathfrak{R}$  give the first derivatives of the curvatures, i.e. the jets  $\mathfrak{R}_{(1)}$ , the commutators of covariant derivatives with  $\mathfrak{R}_{(1)}$  give the second jets  $\mathfrak{R}_{(2)}$ , etc. Thus the operators  $\nabla$  together with the whole set of the jets  $\mathcal{J}$  form an *infinite* dimensional Lie algebra  $\mathcal{G} = \{\nabla, \mathfrak{R}_{(i)}; (i = 1, 2, \dots)\}$  [16]. To evaluate the low-energy heat kernel one can take into account a *finite* number of low-order jets, i.e. the low-order covariant derivatives of the background fields,  $\{\mathfrak{R}_{(i)}; (i \leq N)\}$ , and neglect all the higher order jets, i.e. the covariant derivatives of higher orders, i.e. put  $\mathfrak{R}_{(i)} = 0$  for  $i > N$ . Then one can show that there exist a set of covariant differential operators that together with the background fields and their low-order derivatives generate a *finite* dimensional Lie algebra  $\mathcal{G}' = \{\nabla, \mathfrak{R}_{(i)}; (i = 1, 2, \dots, N)\}$  [16].

Thus one can try to generalize the above idea in such a way that (2.3) would be the zeroth approximation in the commutators of the covariant derivatives, i.e. in the curvatures. Roughly speaking, we are going to find a representation of the heat semigroup *operator* in the form

$$\exp(-t\Delta) = \int_{\mathbb{R}^D} dk \Phi(t, k) \exp\left(-\frac{1}{4t} \langle k, \Psi(t)k \rangle + k \cdot T\right), \quad (2.7)$$

where  $\langle k, \Psi(t)k \rangle = k^A \Psi_{AB}(t) k^B$ ,  $k \cdot T = k^A T_A$ , ( $A = 1, 2, \dots, D$ ),  $T_A = X_A^\mu \nabla_\mu + Y_A$  are some first order differential operators and the functions  $\Psi(t)$  and  $\Phi(t, k)$  are expressed in terms of commutators of these operators, i.e., in terms of the curvatures.

In general, the operators  $T_A$  do not form a closed finite dimensional algebra because at each stage taking more commutators there appear more and more derivatives of the curvatures. It is the *low-energy reduction*  $\mathcal{G} \rightarrow \mathcal{G}'$ , i.e. the restriction to the low-order jets, that actually closes the algebra  $\mathcal{G}$  of the operators  $T_A$  and the background jets, i.e. makes it finite dimensional.

Using this representation one could, as above, act with  $\exp(k \cdot T)$  on the  $\delta$ -function on  $M$  to get the heat kernel. The main point of this idea is that it is much easier to calculate the action of the exponential of the *first* order operator  $k \cdot T$  on the  $\delta$ -function than that of the exponential of the second order operator  $\square$ .

### 3 Heat kernel in flat space

#### 3.1 Covariantly constant potential term

Let us consider now the more complicated case of nontrivial covariantly constant curvature of background connection in flat space:

$$R_{\alpha\beta\gamma\delta} = 0, \quad \nabla_\mu \mathcal{R}_{\alpha\beta} = 0, \quad \nabla_\mu Q = 0. \quad (3.1)$$

Using the condition of covariant constancy of the curvatures (1.14), one can show that in this case the covariant derivatives form a *nilpotent* Lie algebra

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] &= \mathcal{R}_{\mu\nu}, \\ [\nabla_\mu, \mathcal{R}_{\alpha\beta}] &= [\nabla_\mu, Q] = 0 \\ [\mathcal{R}_{\mu\nu}, \mathcal{R}_{\alpha\beta}] &= [\mathcal{R}_{\mu\nu}, Q] = 0. \end{aligned} \quad (3.2)$$

For this algebra one can prove a theorem expressing the heat semigroup operator in terms of an average over the corresponding Lie group [17,18]

$$\begin{aligned} \exp(-t\Delta) &= (4\pi t)^{-d/2} \exp[-t(m^2 + Q)] \det\left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})}\right)^{1/2} \\ &\times \int_{\mathbb{R}^d} dk g^{1/2} \exp\left(-\frac{1}{4t} \langle k, gt\mathcal{R} \coth(t\mathcal{R})k \rangle + k \cdot \nabla\right), \end{aligned} \quad (3.3)$$

where  $k \cdot \nabla = k^\mu \nabla_\mu$ ,  $\mathcal{R}$  means the matrix with coordinate indices  $\mathcal{R} = \{\mathcal{R}^\mu_\nu\}$ ,  $\mathcal{R}^\mu_\nu = g^{\mu\lambda} \mathcal{R}_{\lambda\nu}$ , and the determinant is taken with respect to these indices, other (bundle) indices being intact.

It is not difficult to show that also in this case we have:

$$\exp(k \cdot \nabla) \delta(x, x') \big|_{x=x'} = g^{-1/2} \delta(k). \quad (3.4)$$

Subsequently, the integral over  $k^\mu$  becomes trivial and we obtain immediately the heat kernel diagonal

$$U(t) = (4\pi t)^{-d/2} \exp[-t(m^2 + Q)] \det \left( \frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right)^{1/2}. \quad (3.5)$$

Expanding it in a power series in  $t$  one can find *all* covariantly constant terms in *all* HMDS-coefficients  $b_k$ .

As we have seen the contribution of the bundle curvature  $\mathcal{R}_{\mu\nu}$  is not as trivial as that of the potential term. However, the algebraic approach does work in this case too. It is a good example how can one get the heat kernel without solving any differential equations but using only the algebraic properties of the covariant derivatives.

### 3.2 Contribution of two first derivatives of the potential term

In fact, in flat space it is possible to do a bit more, i.e. to calculate the contribution of the first and the second derivatives of the potential term  $Q$  [22]. That is we consider the case when the derivatives of the potential term vanish only starting from the *third* derivative, i.e.

$$R_{\alpha\beta\gamma\delta} = 0, \quad \nabla_\mu \mathcal{R}_{\alpha\beta} = 0, \quad \nabla_\mu \nabla_\nu \nabla_\lambda Q = 0. \quad (3.6)$$

Besides we assume the background to be *Abelian*, i.e. all the nonvanishing background quantities,  $\mathcal{R}_{\alpha\beta}$ ,  $Q$ ,  $Q_{;\mu} \equiv \nabla_\mu Q$  and  $Q_{;\nu\mu} \equiv \nabla_\nu \nabla_\mu Q$ , commute with each other. Thus we have again a nilpotent Lie algebra

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] &= \mathcal{R}_{\mu\nu}, \\ [\nabla_m, Q] &= Q_{;\mu}, \\ [\nabla_m, Q_{;\nu}] &= Q_{;\nu\mu}, \end{aligned} \quad (3.7)$$

all other commutators being zero.

By parametrizing the potential term according to

$$Q = \Omega - \alpha^{ik} N_i N_k, \quad (3.8)$$

where  $(i = 1, \dots, q; q \leq d)$ ,  $\alpha^{ik}$  is some constant symmetric nondegenerate  $q \times q$  matrix,  $\Omega$  is a covariantly constant matrix and  $N_i$  are some matrices with vanishing second covariant derivative:

$$\nabla_\mu \Omega = 0, \quad \nabla_\mu \nabla_\nu N_i = 0, \quad (3.9)$$

and introducing the operators  $X_A = (\nabla_\mu, N_i)$ ,  $(A = 1, \dots, d+q)$ , one can rewrite the commutation relations (3.7) in a more compact form [22]

$$\begin{aligned} [X_A, X_B] &= \mathcal{F}_{AB}, \\ [X_A, \mathcal{F}_{CD}] &= [X_A, \Omega] = 0, \\ [\mathcal{F}_{AB}, \mathcal{F}_{CD}] &= [\mathcal{F}_{AB}, \Omega] = 0, \end{aligned} \quad (3.10)$$

where  $\mathcal{F}_{AB}$  is a matrix

$$(\mathcal{F}_{AB}) = \begin{pmatrix} \mathcal{R}_{\mu\nu} & N_{i;\mu} \\ -N_{k;\nu} & 0 \end{pmatrix}, \quad (3.11)$$

with  $N_{i;\mu} \equiv \nabla_\mu N_i$ .

The operator (1.6) can now be written in the form

$$\Delta = -\lambda^{AB} X_A X_B + \Omega + m^2, \quad (3.12)$$

where

$$(\lambda^{AB}) = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \alpha^{ik} \end{pmatrix}. \quad (3.13)$$

The algebra (3.10) is a nilpotent Lie algebra of the type (3.2). Thus one can apply the theorem (3.3) in this case too to get [22]

$$\begin{aligned} \exp(-t\Delta) &= (4\pi t)^{-D'/2} \exp[-t(\Omega + m^2)] \det\left(\frac{\sinh(t\mathcal{F})}{t\mathcal{F}}\right)^{-1/2} \\ &\times \int_{\mathbb{R}^{d+q}} dk \lambda^{1/2} \exp\left(-\frac{1}{4t} \langle k, \lambda t \mathcal{F} \coth(t\mathcal{F}) k \rangle + k \cdot X\right), \end{aligned} \quad (3.14)$$

where  $\lambda = \det \lambda_{AB}$ ,  $\langle k, \lambda t \mathcal{F} \coth(t\mathcal{F}) k \rangle = k^A \lambda_{AB} (t\mathcal{F} \coth(t\mathcal{F}))^A_C k^C$ ,  $k \cdot X = k^A X_A$ .

Thus we have expressed the heat semigroup operator in terms of the operator  $\exp(k \cdot X)$ . The integration over  $k$  in (4.16) is Gaussian except for the noncommutative part. Splitting the integration variables  $(k^A) = (q^\mu, \omega^i)$  and using the Campbell-Hausdorff formula we obtain [22]

$$\exp(k \cdot X) \delta(x, x') \Big|_{x=x'} = g^{-1/2} \exp(\omega \cdot N) \delta(q), \quad (3.15)$$

where  $\omega \cdot N = \omega^i N_i$ .

Further, after taking off the trivial integration over  $q$  and a Gaussian integral over  $\omega$ , we obtain the heat kernel diagonal in a very simple form [22]

$$U(t) = (4\pi t)^{-d/2} \Phi(t) \exp\left[-t(m^2 + Q) + \frac{1}{4} t^3 \langle \nabla Q, \Psi(t) g^{-1} \nabla Q \rangle\right], \quad (3.16)$$

where  $\langle \nabla Q, \Psi(t) g^{-1} \nabla Q \rangle = \nabla_\mu Q \Psi^\mu_\nu(t) g^{\nu\lambda} \nabla_\lambda Q$ ,

$$\Phi(t) = \det\left(\frac{\sinh(t\mathcal{F})}{t\mathcal{F}}\right)^{-1/2} \det(1 + t^2 C(t) P)^{-1/2}, \quad (3.17)$$



$$\Psi(t) = \{\Psi_\nu^\mu(t)\} = (1 + t^2 C(t)P)^{-1} C(t), \quad (3.18)$$

$P$  is the matrix determined by second derivatives of the potential term,

$$P = \{P_\nu^\mu\}, \quad P_\nu^\mu = \frac{1}{2} g^{\mu\lambda} \nabla_\nu \nabla_\lambda Q, \quad (3.19)$$

and the matrix  $C(t) = \{C_\nu^\mu(t)\}$  is defined by

$$C(t) = \oint_C \frac{dz}{2\pi i} t \coth(tz^{-1}) (1 - z\mathcal{R} - z^2 P)^{-1}. \quad (3.20)$$

The formula (3.16) exhibits the general structure of the heat kernel diagonal. Namely, one sees immediately how the potential term and its first derivatives enter the result. The complete nontrivial information is contained only in a scalar,  $\Phi(t)$ , and a tensor,  $\Psi_{\mu\nu}(t)$ , functions which are constructed purely from the curvature  $\mathcal{R}_{\mu\nu}$  and the *second* derivatives of the potential term,  $\nabla_\mu \nabla_\nu Q$ . So we conclude that the coefficients  $b_k$  of the heat kernel asymptotic expansion (1.11) are constructed from three different types of scalar (connected) blocks,  $Q$ ,  $\Phi_{(n)}(\mathcal{R}, \nabla \nabla Q)$  and  $\nabla_\mu Q \Psi_{(n)}^{\mu\nu}(\mathcal{R}, \nabla \nabla Q) \nabla_\nu Q$ .

## 4 Heat kernel in symmetric spaces

Let us now generalize the algebraic approach to the case of the *curved* manifolds with covariantly constant Riemann curvature and the trivial bundle connection [19,20]:

$$\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \quad \mathcal{R}_{\alpha\beta} = 0, \quad \nabla_\mu Q = 0. \quad (4.1)$$

First of all, we give some definitions [32,33]. The condition (4.1) defines, as we already said above, the geometry of locally symmetric spaces. A Riemannian locally symmetric space which is simply connected and complete is globally symmetric space (or, simply, symmetric space). A symmetric space is said to be of *compact*, *noncompact* or *Euclidean type* if *all* sectional curvatures  $K(u, v) = R_{abcd}u^a v^b u^c v^d$  are positive, negative or zero. A direct product of symmetric spaces of compact and noncompact types is called *semisimple* symmetric space. A generic complete simply connected Riemannian symmetric space is a direct product of a flat space and a semisimple symmetric space.

It should be noted that our analysis in this paper is purely *local*. We are looking for a *universal* local function of the curvature invariants,  $u_0$  (introduced in Sect.1) that describes adequately the low-energy limit of the heat kernel diagonal  $U(t)$ . Our minimal requirement is that this function should reproduce *all* the terms without covariant derivatives of the curvature in the local asymptotic expansion of the heat kernel (1.11), i.e. it should give *all* the HMDS-coefficients  $b_k$  for *any* symmetric space.

It is well known that the HMDS-coefficients have a *universal* structure, i.e. they are polynomials in the background jets (just in curvatures in case of symmetric spaces) with the numerical coefficients that do not depend on the global properties of the manifold, on the dimension, on the signature of the metric etc. [6].

It is obvious that any flat subspaces do not contribute to the HMDS-coefficients  $b_k$ . Therefore, to find this universal structure it is sufficient to consider only semisimple symmetric spaces. Moreover, since HMDS-coefficients are analytic in the curvatures, one can restrict oneself only to symmetric spaces of *compact* type. Using the factorization property of the heat kernel and the duality between compact and noncompact symmetric spaces, one can obtain then the results for the general case by analytical continuation. That is why in this paper we consider only the case of *compact* symmetric spaces when the sectional curvatures and the metric are *positive* definite.

Let  $e_a^\mu$  be a *covariantly constant (parallel)* frame along the geodesic. The frame components of the curvature tensor of a symmetric space are, obviously, constant and can be presented in the form [19,20]

$$R_{abcd} = \beta_{ik} E_{ab}^i E_{cd}^k, \quad (4.2)$$

where  $E_{ab}^i$ , ( $i = 1, \dots, p$ ;  $p \leq d(d-1)/2$ ), is some set of antisymmetric matrices and  $\beta_{ik}$  is some symmetric nondegenerate  $p \times p$  matrix. The traceless matrices  $D_i = \{D_{ib}^a\}$  defined by

$$D_{ib}^a = -\beta_{ik} E_{cb}^k g^{ca} = -D_{bi}^a \quad (4.3)$$

are known to be the generators of the *holonomy algebra*  $\mathcal{H}$

$$[D_i, D_k] = F_{ik}^j D_j, \quad (4.4)$$

where  $F_{ik}^j$  are the structure constants.

In symmetric spaces a much richer algebraic structure exists [19,20]. Indeed, let us define the quantities  $C_{BC}^A = -C_{CB}^A$ , ( $A = 1, \dots, D$ ;  $D = d + p$ ):

$$C_{ab}^i = E_{ab}^i, \quad C_{ib}^a = D_{ib}^a, \quad C_{kl}^i = F_{kl}^i, \quad (4.5)$$

$$C_{bc}^a = C_{ka}^i = C_{ik}^a = 0,$$

and the matrices  $C_A = \{C_{AC}^B\} = (C_a, C_i)$ :

$$C_a = \begin{pmatrix} 0 & D_{ai}^b \\ E_{ac}^j & 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} D_{ia}^b & 0 \\ 0 & F_{ik}^j \end{pmatrix}. \quad (4.6)$$

One can show that they satisfy the Jacobi identities

$$[C_A, C_B] = C_{AB}^C C_C \quad (4.7)$$

and, hence, define a Lie algebra  $\mathcal{G}$  of dimension  $D$  with the structure constants  $C_{BC}^A$ , the matrices  $C_A$  being the generators of the adjoint representation.

In symmetric spaces one can find explicitly the generators of the infinitesimal isometries, i.e. the Killing vector fields  $\xi_A$ , and show that they form a Lie algebra of isometries that is (in case of semisimple symmetric space) isomorphic to the Lie algebra  $\mathcal{G}$  (4.7), [20]

$$[\xi_A, \xi_B] = C_{AB}^C \xi_C. \quad (4.8)$$

Moreover, introducing a symmetric nondegenerate  $D \times D$  matrix

$$\gamma_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix}, \quad (4.9)$$

that plays the role of the metric on the algebra  $\mathcal{G}$ , one can express the operator (1.6) in semisimple symmetric spaces in terms of the generators of isometries [20]

$$\Delta = -\gamma^{AB} \xi_A \xi_B + Q + m^2, \quad (4.10)$$

where  $\gamma^{AB} = (\gamma_{AB})^{-1}$ .

Using this representation one can prove a theorem that presents the heat semi-group operator in terms of some average over the group of isometries  $G$  [19,20]:

$$\begin{aligned} \exp(-t\Delta) &= (4\pi t)^{-D/2} \exp \left[ -t(Q + m^2 - \frac{1}{6}R_G) \right] \\ &\times \int_{\mathbb{R}^D} dk \gamma^{1/2} \det \left( \frac{\sinh(k \cdot C/2)}{k \cdot C/2} \right)^{1/2} \exp \left( -\frac{1}{4t} \langle k, \gamma k \rangle + k \cdot \xi \right), \end{aligned} \quad (4.11)$$

where  $\gamma = \det \gamma_{AB}$ ,  $k \cdot C = k^A C_A$ ,  $k \cdot \xi = k^A \xi_A$  and  $R_G$  is the scalar curvature of the group of isometries  $G$

$$R_G = -\frac{1}{4} \gamma^{AB} C_{AD}^C C_{BC}^D. \quad (4.12)$$

Acting with this operator on the delta-function  $\delta(x, x')$  one can, in principle, evaluate the off-diagonal heat kernel  $\exp(-t\Delta)\delta(x, x')$ , i.e. for non-coinciding points  $x \neq x'$  [20]. Since in this paper we are going to calculate only the heat kernel diagonal (1.5), it is sufficient to compute only the coincidence limit  $x = x'$ . Splitting the integration variables  $k^A = (q^a, \omega^i)$  and solving the equations of characteristics one can obtain the action of the isometries on the  $\delta$ -function (Lemma 2 in [20])

$$\exp(k \cdot \xi) \delta(x, x') \Big|_{x=x'} = \det \left( \frac{\sinh(\omega \cdot D/2)}{\omega \cdot D/2} \right)^{-1} \eta^{-1/2} \delta(q), \quad (4.13)$$

where  $\omega \cdot D = \omega^i D_i$  and  $\eta = \det g_{ab}$ . Using this result, one can easily integrate over  $q$  in (4.11) to get the heat kernel diagonal. After changing the integration variables  $\omega \rightarrow \sqrt{t}\omega$  it takes the form

$$\begin{aligned} U(t) &= (4\pi t)^{-d/2} \exp \left[ -t \left( m^2 + Q - \frac{1}{8}R - \frac{1}{6}R_H \right) \right] \\ &\times (4\pi)^{-p/2} \int_{\mathbb{R}^p} d\omega \beta^{1/2} \exp \left( -\frac{1}{4} \langle \omega, \beta \omega \rangle \right) \\ &\times \det \left( \frac{\sinh(\sqrt{t}\omega \cdot F/2)}{\sqrt{t}\omega \cdot F/2} \right)^{1/2} \det \left( \frac{\sinh(\sqrt{t}\omega \cdot D/2)}{\sqrt{t}\omega \cdot D/2} \right)^{-1/2}, \end{aligned} \quad (4.14)$$

where  $\omega \cdot F = \omega^i F_i$ ,  $F_i = \{F_{ik}^j\}$  are the generators of the holonomy algebra  $\mathcal{H}$ , (4.4), in adjoint representation, and

$$R_H = -\frac{1}{4} \beta^{ik} F_{il}^m F_{km}^l \quad (4.15)$$

is the scalar curvature of the holonomy group.

The remaining integration over  $\omega$  in (4.14) can be done in a rather *formal* way. Namely, one can prove that for any analytic function  $f(\omega)$  that falls off at the infinity there holds [20]

$$\begin{aligned} (4\pi)^{-p/2} \int_{\mathbb{R}^p} d\omega \beta^{1/2} \exp\left(-\frac{1}{4} \langle \omega, \beta \omega \rangle\right) f(\omega) \\ = f\left(i \frac{\partial}{\partial q}\right) \exp\left(-\langle q, \beta^{-1} q \rangle\right) \Big|_{q=0}, \end{aligned} \quad (4.16)$$

where  $\langle q, \beta^{-1} q \rangle = q_j \beta^{jk} q_k$ .

Introducing an abstract dynamical system with a normalized "vacuum state"  $|0\rangle$ ,

$$\langle 0|0\rangle = 1, \quad (4.17)$$

and the "coordinate" and "momentum" operators  $\hat{q}_i$  and  $\hat{p}^k$  satisfying the commutation relations

$$\begin{aligned} [\hat{p}^j, \hat{q}_k] &= i\delta^j_k, \\ [\hat{p}^i, \hat{p}^k] &= [\hat{q}_i, \hat{q}_k] = 0, \end{aligned} \quad (4.18)$$

and the rules

$$\hat{p}^i |0\rangle = 0, \quad \langle 0|\hat{q}_k = 0, \quad (4.19)$$

one can present the equation (4.16) in the form

$$\begin{aligned} (4\pi)^{-p/2} \int_{\mathbb{R}^p} d\omega \beta^{1/2} \exp\left(-\frac{1}{4} \langle \omega, \beta \omega \rangle\right) f(\omega) \\ = \langle 0|f(\hat{p}) \exp\left(-\langle \hat{q}, \beta^{-1} \hat{q} \rangle\right) |0\rangle. \end{aligned} \quad (4.20)$$

Using this equation we have finally from (4.14) the heat kernel diagonal in an formal algebraic form, without any integration

$$\begin{aligned} U(t) &= (4\pi t)^{-d/2} \exp\left[-t\left(m^2 + Q - \frac{1}{8}R - \frac{1}{6}R_H\right)\right] \\ &\times \left\langle 0 \left| \det\left(\frac{\sinh(\sqrt{t}\hat{p}\cdot F/2)}{\sqrt{t}\hat{p}\cdot F/2}\right)^{1/2} \det\left(\frac{\sinh(\sqrt{t}\hat{p}\cdot D/2)}{\sqrt{t}\hat{p}\cdot D/2}\right)^{-1/2} \right. \right. \\ &\times \exp\left(-\langle \hat{q}, \beta^{-1} \hat{q} \rangle\right) \Big| 0 \rangle, \end{aligned} \quad (4.21)$$

where  $\hat{p}\cdot F = \hat{p}^k F_k$  and  $\hat{p}\cdot D = \hat{p}^k D_k$ . This formal solution should be understood as a power series in the operators  $\hat{p}^k$  and  $\hat{q}_k$  and determines a well defined asymptotic expansion in  $t \rightarrow 0$ .

Let us stress that the formulae (4.14) and (4.21) are exact (up to topological contributions) and *manifestly covariant* because they are expressed in terms of the invariants of the holonomy group  $H$ , i.e. the invariants of the Riemann curvature tensor. They can be used now to generate *all* HMDS-coefficients  $b_k$  for *any* symmetric space, i.e. for any manifold with covariantly constant curvature, simply by expanding it in an asymptotic power series as  $t \rightarrow 0$ . Thereby one finds *all* covariantly constant terms in *all* HMDS-coefficients in a manifestly covariant way. This gives a very nontrivial example how the heat kernel can be constructed using only the Lie algebra of isometries of the symmetric space.

## 5 Conclusion

In present paper we have presented our recent results in studying the heat kernel obtained in the papers [16-22]. We discussed some ideas connected with the problem of developing consistent covariant approximation schemes for calculating the heat kernel. Especial attention is paid to the low-energy limit of quantum field theory. It is shown that in the local analysis there exists an algebraic structure (the Lie algebra of background jets) that turns out to be extremely useful for the study of the low-energy approximation. Based on the background jets algebra we have proposed a new promising approach for calculating the low-energy heat kernel.

Within this framework we have obtained closed formulas for the heat kernel diagonal in the zeroth order, i.e. in case of covariantly constant background curvatures. Besides, we were able to take into account the first and second derivatives of the potential term in flat space (Sect. 3.1).

The obtained formulas are exact, covariant and general, i.e. they are applicable for *any* covariantly constant background fields. This enables to treat the results of this paper as the *generating functions* for the whole set of the Hadamard-Minakshisundaram-De Witt-Seeley-coefficients. In other words, we have calculated *all* covariantly constant terms in *all* HMDS-coefficients. This is the opposite case to the leading derivatives terms which were calculated in [4,29,30,5] and [34].

Needless to say that the investigation of the low-energy effective action is of great importance in quantum gravity and gauge theories because it describes the dynamics of the vacuum state of the theory. The algebraic approach described in this paper was applied to calculate explicitly the effective potential in Yang-Mills theory and to study the structure of the vacuum of this model [35].

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