COMPUTER EXPERIMENTS FOR CONFORMAL DYNAMICAL SYSTEMS

Mihai Postolache and Sorin Udrişte

Abstract

The object of this paper is to present a link between the conformal differential geometry, dynamical systems theory and the numerical analysis. More precisely, our work describes the behaviour of some conformal dynamical systems of order one and their second order prolongations.

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1 Introduction

Let $D \subset R^n$ be an open and connected set and $X = (X_1, \ldots, X_n)$ a vector field of class C^1 on D. A curve $\alpha : I \to D$ of class C^1, $\alpha(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, solution of the Cauchy problem

$$(C) \quad \begin{cases} \dot{x}_1 = X_1(x_1, \ldots, x_n), \ldots, \dot{x}_n = X_n(x_1, \ldots, x_n) \\ x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0 \end{cases}$$

is called *field line*. The image $\alpha(I) \subset D$ of a field line is called *orbit* of X.

Sometimes we are interested to know geometrical and/or topological properties of the dynamical systems of type (C). A first idea would be to obtain portraits of the orbits of the field and the second to get a prolongation of the dynamical system of order one to a dynamical system of order two.

The conformal vector field generates a local group of geometric conformal transformations which depends on a parameter. The portraits of the orbits suggest the kind of the conformal transformation. This is the reason for our study.

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2 Conformal Dynamical Systems

Let $X = (X_1, X_2, \ldots, X_n)$ be a vector field of class C^∞ on R^n. This vector field is called *conformal* if the following equations hold good

$$\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i} = \psi\delta_{ij}, \quad i,j = 1,2,\ldots,n, \tag{1}$$

where δ_{ij} is the Kronecker delta symbol and $\psi : R^n$ are real functions. If ψ is a constant, then the conformal field is called *omothetic* while if $\psi = 0$ the conformal field is called *Killing*. We remark that (1) implies

$$\frac{\partial X_1}{\partial x_1} = \frac{\partial X_2}{\partial x_2} = \cdots = \frac{\partial X_n}{\partial x_n} = \frac{\psi}{2},$$

hence $\psi = \frac{2}{n}\mathrm{div}X$. We remark that in the 1-dimensional case the conformal fields become functions of class C^∞, and in the 2-dimensional case they become complex monogene functions.

It can be shown that the most general conformal vector field on R^n, $n \geq 3$, has the components [5]

$$X_i(x) = \frac{1}{2}x_i\sum_{j=1}^{n} c_j x_j - \frac{1}{4}c_i\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} a_{ij}x_j + \frac{c}{2}x_i + d_i, \quad i = 1,2,\ldots,n, \tag{2}$$

where a_{ij}, c_i, d_i, c are constants and (a_{ij}) is a skew-symmetric matrix.

Some remarks are in order.

1. There are $(n+1)(n+2)/2$ conformal vector fields on R^n;

2. Define the Lie bracket as follows

$$[X,Y] = X(Y_i) - Y(X_i),$$

where $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$. If both X and Y are solutions of (1), then $[X,Y]$ is solution of (1) namely the bracket of two conformal vector fields is a conformal vector field too. Consequently

The set of conformal vector fields on R^n, $n \geq 3$ is a Lie algebra of dimension $(n+1)(n+2)/2$.

Let $X = (X_1, \ldots, X_n)$ be a conformal vector field. The dynamical system of order one, $\frac{dx}{dt} = X(x)$ induces a new dynamics since it can be prolonged to a conservative second order differential system, according to a recent idea of Prof. Dr. C. Udriste ([6], [7]).

Denote $F_{ij} = \frac{\partial X_j}{\partial x_i} - \frac{\partial X_i}{\partial x_j} = c_i x_j - c_j x_i + c_{ji} - c_{ij}$. Then, we have the following result.

Theorem 2.1 *The conformal dynamical system of order one can be prolonged to the nonpotential dynamical system with three degress of freedom*

$$\frac{d^2x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_j F_{ij}\frac{dx_j}{dt}, \tag{3}$$

where $f = \frac{1}{2}||X||^2$ is the energy of the conformal vector field X.

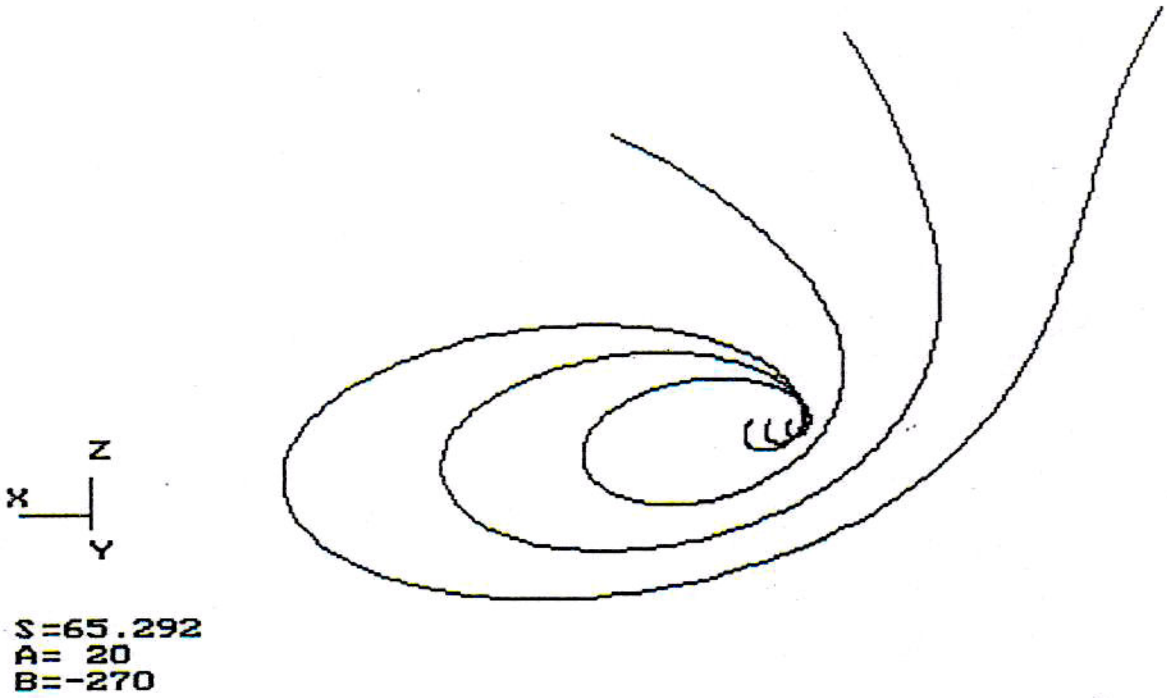

Figure 1

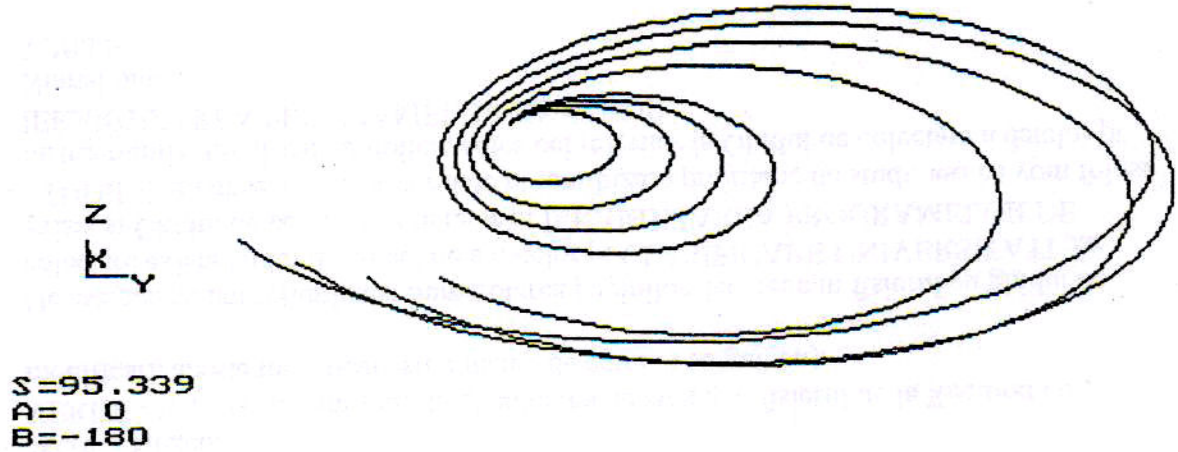

Figure 2

and the trajectories classify as follows:
- the set of original conformal lines with energy $\mathcal{H} = 0$,
- the set of trajectories with the energy $\mathcal{H} = \text{const} < 0$,
- the set of trajectories with the energy $\mathcal{H} = \text{const} > 0$.

The preceding dynamical system describes in fact a new Lorentz world-force law since we have

Theorem 2.2 *Every trajectory of the dynamical system (3) which has the total constant energy $\mathcal{H} > -f$, is a reparametrized horizontal geodesic of the Riemann-Jacobi metric $g_{ij} = (\mathcal{H} + f)\delta_{ij}$ and the nonlinear connection $N^i{}_j = \Gamma^i{}_{jk}y^k + F^i{}_j$, $F^i{}_j = g^{ih}F_{hj}$, where $\Gamma^i{}_{jk}$ is the Riemannian connection induced by the metric g_{ij}.*

3. In the three dimensional case, and $d_i = 0$ the matrix of linear approximation is

$$(a'_{ij}) = \begin{bmatrix} \frac{c}{2} & \alpha & \beta \\ -\alpha & \frac{c}{2} & \gamma \\ \beta & -\gamma & \frac{c}{2} \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = \frac{c}{2}$ and $\lambda_{2,3} = \frac{c}{2} \pm i\sqrt{\alpha^2 + \beta^2 + \gamma^2}$. Consequently, if $c > 0$, then the equilibrium point $(0,0,0)$ will not be stable, while if $c < 0$, this point is stable.

In the three dimensional case and for given constants we obtained interested conclusions concerning the behaviour of the orbits of conformal dynamical systems of order one. These results are presented in the sequel.

3 Computational Results

For the case $n = 3$, in (2) we shall consider $d_i = 0$, $c_1 = i$, $i = 1,2,3$. Also consider the matrix (a_{ij}) defined as follows

$$(a_{ij}) = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}.$$

The case $c = 2$. We obtain the conformal vector field of components

$$\begin{cases} F(x,y,z) = \frac{x}{2}(x + 2y + 3z) - \frac{1}{4}(x^2 + y^2 + z^2) + x + y + 2z \\ G(x,y,z) = \frac{y}{2}(x + 2y + 3z) - \frac{1}{2}(x^2 + y^2 + z^2) - x + y + 3z \\ H(x,y,z) = \frac{z}{2}(x + 2y + 3z) - \frac{3}{4}(x^2 + y^2 + z^2) - 2x - 3y + z. \end{cases}$$

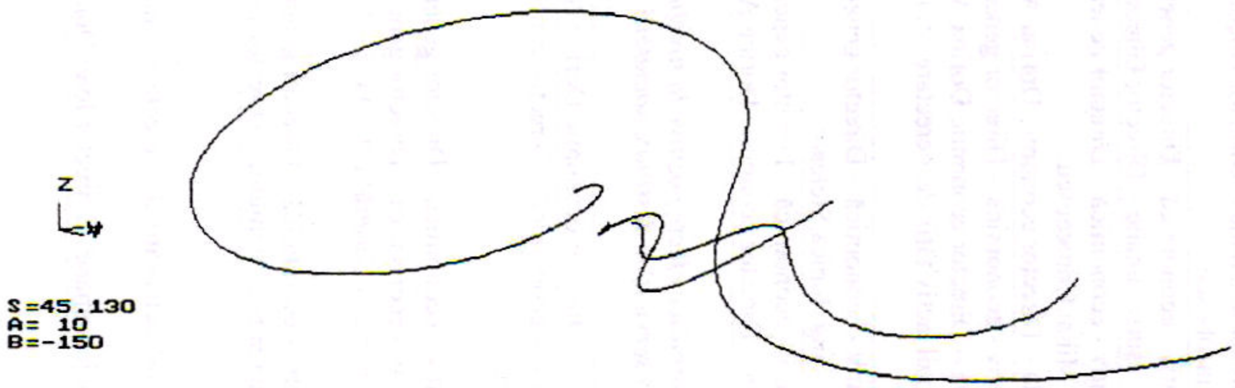

Figure 3

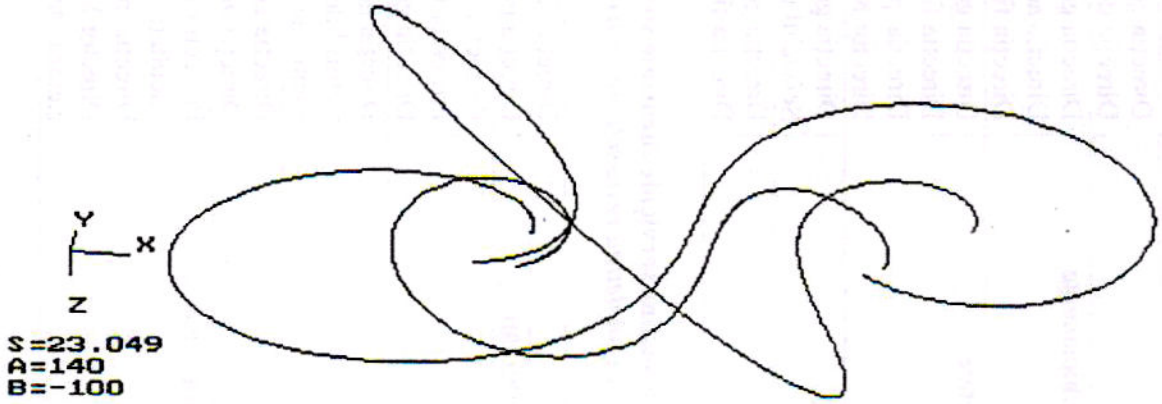

Figure 4

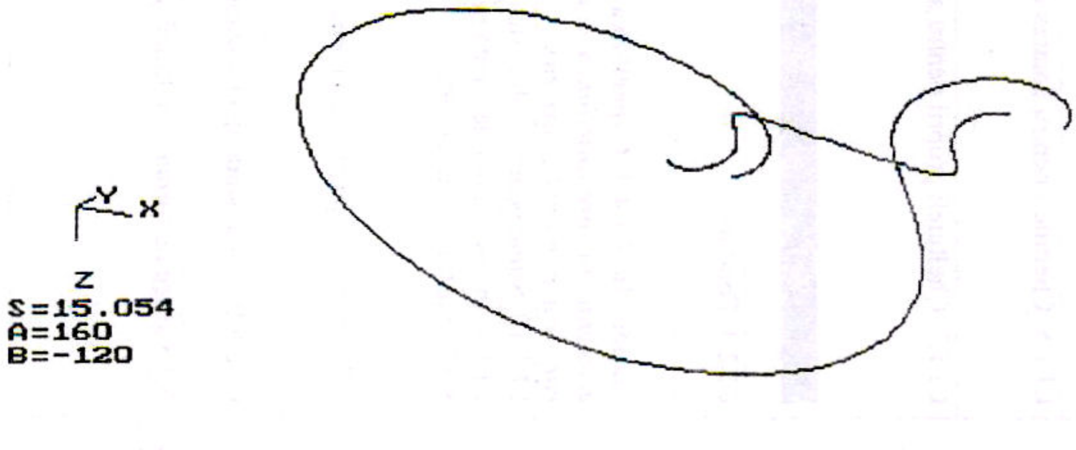

Figure 5

tem and the matrix of the linear approximation

$$\begin{cases} \dot{x} = x + y + 2z \\ \dot{y} = -x + y + 3z \\ \dot{z} = -2x - 3y + z \end{cases}$$

has the eigenvalue $\lambda_1 = 1$. Consequently, the equilibrium point $(0,0,0)$ is not a stable point.

The Figure 1 presents numerical simulated portrait of the orbits fixed by the initial conditions $(x_0, y_0, z_0) \in \{(0.1,0,0); (0.15,0,0); (0.2,0,0); (0.2,0,0)\}$. The Figure 2 exhibits a simulated portrait fixed by the initial conditions

$$(x_0, y_0, z_0) \in \{(-0.25,0,0); (-0.2,0,0); (-0.15,0,0); (-0.1,0,0)\}.$$

For initial conditions $(x_0, y_0, z_0) \in \{(-2,1,0); (-3,0,1); (3,0,-1)\}$ located on to the plane $x + 2y + 3z = 0$ we found the behaviour from the Figure 3.

Now, we shall choose initial conditions located on to the sphere described by the equation $x^2 + y^2 + z^2 = 1$ namely $\{(x_0, y_0, z_0) \in (1,0,0); (0,1,0); (0,0,1)\}$ and $\{(x_0, y_0, z_0) \in (0.709, 0.709, 0); (0.709, -0,709, 0)\}$ respectively. In this case, the behaviour of the orbits are presented in the Figures 4 and 5.

If $\{(x_0, y_0, z_0) \in (0,0,1); (0,0,-1); (0.709, 0.709, 0)\}$ the orbits are presented in the Figure 6.

The case $c = -2$. We obtain the conformal vector field of components

$$\begin{cases} F(x,y,z) = \frac{x}{2}(x + 2y + 3z) - \frac{1}{4}(x^2 + y^2 + z^2) - x + y + z \\ G(x,y,z) = \frac{y}{2}(x + 2y + 3z) - \frac{1}{2}(x^2 + y^2 + z^2) - x - y + z \\ H(x,y,z) = \frac{z}{2}(x + 2y + 3z) - \frac{3}{4}(x^2 + y^2 + z^2) - x - y - z. \end{cases}$$

We remark that $(0,0,0)$ is also an equilibrium point for the associated dynamical system and the matrix of the linear approximation

$$\begin{cases} \dot{x} = -x + y + z \\ \dot{y} = -x - y + z \\ \dot{z} = -x - y - z \end{cases}$$

has the eigenvalue $\lambda_1 = -1$ and $\lambda_{2,3} = -1 \pm i$. Consequently, the equilibrium point $(0,0,0)$ is a stable point.

The Figures 7 and 8 present numerical simulated portraits of the orbits fixed by the conditions $(x_0, y_0, z_0) \in \{(-1,0,0); (0,1,0); (0,-1,0); (1,0,0); (0,0,1)\}$ and by the conditions

$$(x_0, y_0, z_0) \in \{(-5,1,1); (-3,0,1); (3,0,-1); (1,1,-1); (-1,-1,1); (-1,2,-1)\}$$

respectively.

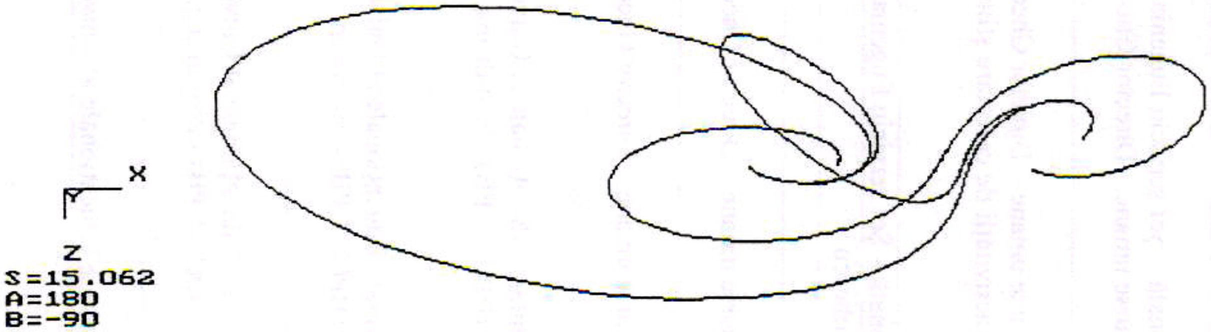

Figure 6

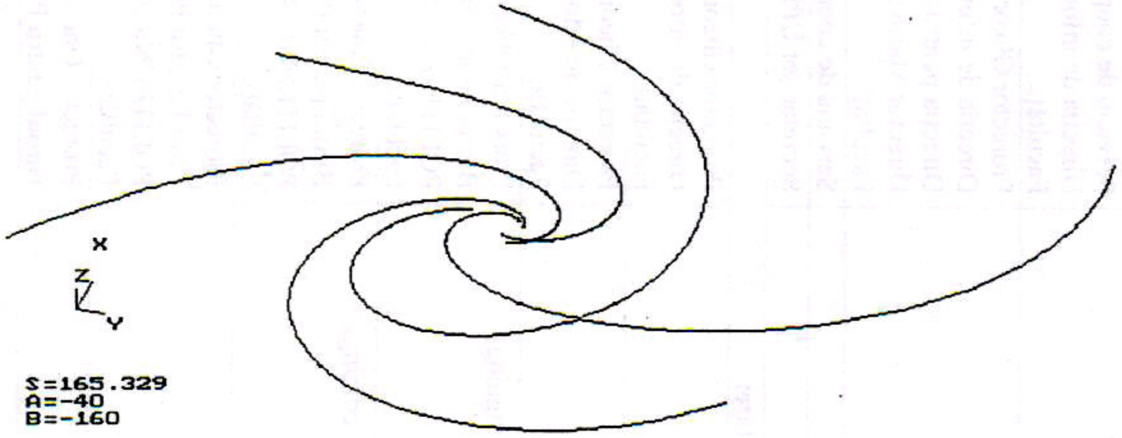

Figure 7

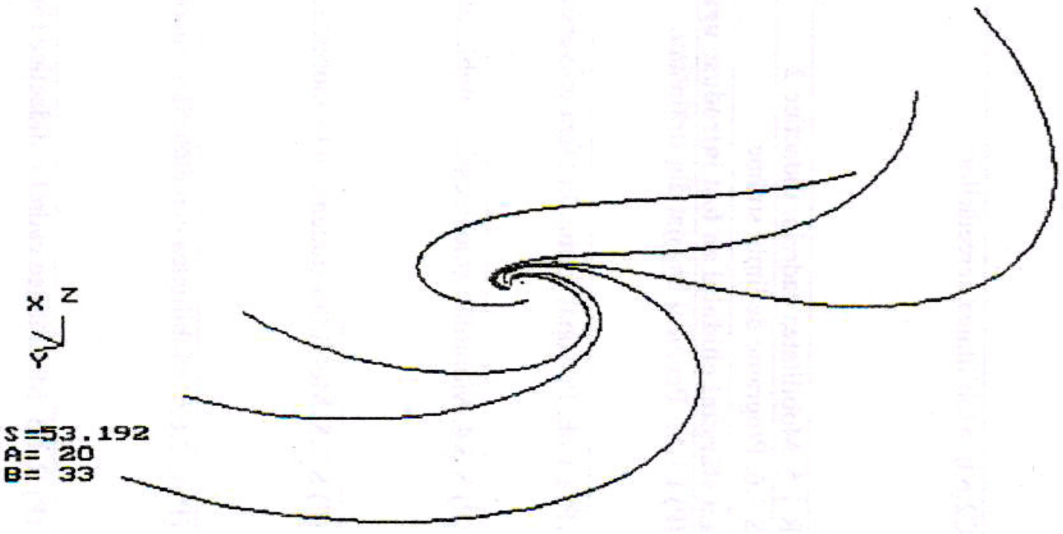

Figure 8

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Author's address:

Mihai Postolache and Sorin Udriste
"Politehnica" University of Bucharest,
Splaiul Independenţei 313, 2206 Bucharest (Ro),
e-mail: postol@math.pub.ro