HOMOMORPHISMS IN THE THEORY OF THE M-POLYSYMMETRICAL HYPERGROUPS AND MONOGENE M-POLYSYMMETRICAL HYPERGROUPS

Constantinos N. Yatras

Abstract

J. Mittas in his paper [6], which has been announced in the French Academy of Sciences, has introduced a special type of hypergroup that he has named polysymmetrical. Also, in the same paper J. Mittas has given certain fundamental properties of this hyperstructure. Starting from the above paper and having called Mittas’ structure M-polysymmetrical hypergroup (in order to distinguish this polysymmetrical hypergroup from other types of polysymmetrical hypergroups [2], [8], [9] we have proceeded to a profound analysis of this hypergroup [11] and its subhypergroups [12]. In this paper appear the initial results that have been reached during the study of the homomorphisms of the M-polysymmetrical hypergroups and also of the monogene M-polysymmetrical hypergroups.

AMS Subject Classification: 20N20

Key words: hypergroup, subhypergroup, hyperoperation, homomorphism of the hypergroups, monogene

1 Introduction

Let’s see, as introductory elements, two definitions and some important properties, which derive from [6], [11], [12].

A set $H$ is called a \textbf{M-polysymmetrical hypergroup} (M-P-H) if is is endowed with a hyperoperation $x + y$ that satisfies the following axioms:

1. $(x + y + +z = x + (y + z)$ for every $x, y, z \in H$
2. $x + y = y + x$ for every $x, y \in H$
3. $(\exists 0 \in H) (\forall x \in H) [x \in 0 + x]$
4. $(\forall x \in H) (\exists x' \in H) [x + x' = 0]$


©Balkan Society of Geometers, Geometry Balkan Press
(x′ is an opposite or symmetrical of x, with regard to the considered 0, and the set of
the opposites $S(x) = \{x′ ∈ H : x + x′ = 0\}$ is the symmetrical set of x).

5. For every $x, y, z ∈ H$, $x′ ∈ S(x)$, $y′ ∈ S(y)$, $z′ ∈ S(z)$ we have
$$z ∈ x + y ⇒ z′ ∈ x′ + y′.$$ 

We remind that on such a hypergroup, whe $x$ runs in $H$, the sets $C_0(x) = 0 + x$
form a partition of $H$, which is denoted by $mod(0)$, or simply (0) and for which we
have $C_0(x) = C_0(y) ⇔ 0 + x = 0 + y$. Also, for every $x ∈ H$, $x′ ∈ S(x)$ we have
$S(x) = C(x′)$ and the set of classes, $H/0 = G(H)$ is an abelian group which is
defined group of reduction of $H$. A subhypergroup of a M-P.H. $(H, +)$ which is also
M-P.H. with regard to the hyperoperation of $H$ and which has the same zero with $H$,
is called $M$-polysymmetrical subhypergroup $(M-P. SH)$ of $H$.

Every subhypergroup of a M-P.H. $H$ is M-P. SH of $H$. Also, a non void subset $h$
of $H$ is a subhypergroup of $H$ only if, for all $x, y ∈ h$ we have $x + S(y) ⊆ h$. If
$h$ is a subhypergroup of $H$ then $y ≡ x mod h ⇔ y + x′ ⊆ h$ holds for every $x, y ∈ H$.
It must be noted that for every normal equivalence relation $R$, the class $C_R(0)$ is a
subhypergroup of $H$ and moreover $R ≡ mod(C_R(0))$.

2 Homomorphisms of M-P.Hs

Having as a starting point the general definition of the homomorphism [1], [3] and
especially the definition of the homomorphism in the theory of the canonical hyper-
groups [5], [7] we proceed to the study of the homomorphisms of M-P.Hs.

Let’s suppose that $H$ and $H_1$ are two M-P.Hs and let $ϕ$ be a normal homomor-
phism$^1$ from $H$ to $H_1$. Also let 0, 0$_1$ be the zeros of $H$, $H_1$ respectively.

Regarding the above, we have the propositions

**Proposition 2.1**  

i) $ϕ(0) = 0_1$,  

ii) $ϕ((C_0(x)) = C_0(ϕ(x))$ thus $ϕ(S(x)) = S(ϕ(x))$ for every $x ∈ H$ (where ($C_0(x)$,
$C_0(ϕ(x))$ are the classes $mod(0)$ of x and $mod(0_1)$ of $ϕ(x)$, respectively).

iii) The homomorphic image $ϕ(H)$ of $H$ is a subhypergroup of $H_1$, (thus $M$-P.SH of
$H_1$).

iv The Kernel $Kϕ = ϕ^{-1}(ϕ(0)) = ϕ^{-1}(0_1)$ of the homomorphism $ϕ$, is a subhy-
pergroup of $H$.

---

$^1$Let $H$ and $H_1$ be hypergroups. A mapping $ϕ : H → P(H_1)$ is named homomorphism from $H$
to $P(H_1)$, if the relation $ϕ(x - y) ⊆ ϕ(x) - ϕ(y)$ is valid for every $x, y ∈ H$.

ϕ is named strong homomorphism if the above relation hold as an equality, i.e. $ϕ(x - y) = ϕ(x) - ϕ(y)$.

A homomorphism $ϕ : H → H_1$, from $H$ to $H_1$ is named strict.

A strong and strict homomorphism is named normal.

A homomorphism $ϕ : H → H_1$, which is one to one mapping from $H$ onto $H_1$ is named isomor-
phism. Generally, in the case of $H$, $H_1$ being hypergroupoids, we define, in the same way as above,
the homomorphism from $H$ to $H_1$ [1], [3], [5].
Proof. i) Let $x \in H$, so $\varphi(x) \in \varphi(H)$. Then $\varphi(x) \in \varphi(x + 0) = \varphi(x) + \varphi(0)$ thus $\varphi(0) = 0_1$ (it has been proved that if $y \in x + y \Rightarrow y = 0 [11]$).

Also:

\[ \varphi(C_0(x)) = \varphi(0 + x) = \varphi(0) + \varphi(x) = a_1 + \varphi(x) = C_0_1(\varphi(x)) \]

ii) $\varphi(C_0(x)) = \varphi(0 + x) = \varphi(0) + \varphi(x) = a_1 + \varphi(x) = C_0_1(\varphi(x))$

Indeed $\varphi(S(x)) = \varphi(C_0(x')) = C_0_1(\varphi(x')) = 0_1 + \varphi(x') = S(\varphi(x))$, since $\varphi(0) = 0_1 \Rightarrow \varphi(x + x') = 0_1 \Rightarrow \varphi(x) + \varphi(x') = 0_1 \Rightarrow \varphi(x') \in S(\varphi(x))$

and thus $0_1 + \varphi(x') = S(\varphi(x))$.

iii) There exists $x \in H$ such that $\varphi(x) = a$ for every $a \in \varphi(H)$ and also $\varphi(H) \subseteq H_1$. Moreover it holds that $a + \varphi(H) = \varphi(H)$.

Indeed $a + \varphi(H) = \varphi(x) + \varphi(H) = \cup_{y \in H} \{\varphi(x) + \varphi(y)\} = \cup_{y \in H} \varphi(x + y) = \varphi(x + H) = \varphi(H)$.

iv) For every $x, y \in N(\varphi)$ we have:

\[ \varphi(x + y) = \varphi(x) + \varphi(y) = 0_1 + 0_1 \]

thus $x + y \subseteq N(\varphi)$. Also, for every $x \in N(\varphi)$, we have:

\[ 0_1 = \varphi(0) = \varphi(x + x') = \varphi(x) + \varphi(x') = 0_1 + \varphi(x') \Rightarrow \varphi(x') = 0 \Rightarrow x' \in N(\varphi) \]  

for every $x' \in N(\varphi)$, $\Rightarrow S(x) \subseteq N(\varphi)$. Thus $N(\varphi)$ is M-P.SH of $H$. Q.e.d.

Proposition 2.2  

i) The homomorphic image $\varphi(h)$ of every subhypergroup $h$ of $H$ is a subhypergroup of $\varphi(H)$ (thus a subhypergroup of $H_1$).

ii) The inverse image $\varphi^{-1}(h)$ of every subhypergroup $h$ of $\varphi(H)$ is a subhypergroup of $H$ and $\varphi^{-1}(h) \subseteq \varphi^{-1}(h)$.

Proof. i) Because of proposition 2.1 (iii).

ii) Let $x, y \in \varphi^{-1}(h)$. So $\varphi(x), \varphi(y) \in h$ and since $\varphi(S(y)) = \varphi(S(y)) \subseteq h$ we have that $\varphi(x + S(y)) = \varphi(x) + \varphi(S(y)) = \varphi(x) + S(\varphi(y)) \subseteq h$ thus $x + S(y) \subseteq \varphi^{-1}(h)$, hence $\varphi^{-1}(h)$ is a subhypergroup of $H$.

Finally, it is obvious that $N(\varphi) \subseteq \varphi^{-1}(h)$, since $N(\varphi) = \varphi^{-1}(0_1)$ and $0_1 \in h$.Q.e.d.

Proposition 2.3 The mapping $x + N(\varphi) \rightarrow \varphi(x) + 0_1$ is an isomorphism from the group $H/N(\varphi)$ onto $\varphi(H)/(0_1)$ [where $\varphi(H)/(0_1)$ is the group of reduction of $\varphi(H)$].

Proof. Let $\psi$ be the mapping

\[ H/N(\varphi) \rightarrow \varphi(H)/(0_1) \]

such that $\psi(x + N(\varphi)) = f(x) + 0_1$ for every $x + N(\varphi) = C_N(\varphi)(x) \in H/N(\varphi)$.

For the $\psi$ we have:

i) $\psi(x + N(\varphi)) + (y + N(\varphi)) = \psi(z + y) = \{\varphi(z) + 0_1 : z = x + y\} = \varphi(x + y) + 0_1 = \varphi(\varphi(x) + \varphi(y) + 0_1) = [\varphi(x) + 0_1] + [\varphi(y) + 0_1] = \psi(x + N(\varphi)) + \psi(y + N(\varphi))$.

Thus $\psi$ is a homomorphism between groups.

ii) It is obvious that $\psi$ is a mapping onto $\varphi(H)/(0_1)$. 

HOMOMORPHISMS IN THE THEORY OF THE HYPERGROUPS
Corollary 2.1 For every normal monomorphism\(^1\) \(\varphi : H \to H_1\), the groups \(H/\mathcal{N}(\varphi)\) and \(H_1/(0_1)\) are isomorphic.

Corollary 2.2 If \(\mathcal{N}(\varphi) = \{0\}\) then the groups \(H/(0)\) and \(\varphi(H)/(0_1)\) are isomorphic.

Now, having as a starting point a normal homomorphism \(\varphi : H \to H_1\) we consider the mapping:

\[
\bar{\varphi} : H \to \varphi(H)/(0_1)
\]

such that \(\bar{\varphi}(x) = \varphi(x) + 0_1\) for every \(x \in H\).

For every \(x, y \in H\) we have for the \(\bar{\varphi}\):

\[
\varphi(x + y) = \{\varphi(z) : z \in x + y\} = \{\varphi(z) + 0_1 : z \in x + y\} = \\
\varphi(x) + 0_1 = [\varphi(x) + 0_1] + [\varphi(y) + 0_1] = \bar{\varphi}(x) + \bar{\varphi}(y).
\]

Thus \(\bar{\varphi}\) is a normal homomorphism from \(H\) onto the group of reduction of image \(\varphi(H)\) of \(H\).

So we have the proposition:

Proposition 2.4 To every normal homomorphism \(\varphi : H \to H_1\) corresponds the mapping \(\bar{\varphi} : H \to \varphi(H)/(0_1)\), from \(H\) onto \(\varphi(H)/(0_1)\) such that \(\bar{\varphi}(x) = \varphi(x) + 0_1\) for every \(x \in H\) which is also a normal homomorphism.

We have the mapping \(\bar{\varphi}\) homomorphism of reduction of the homomorphism \(\varphi\) and because of proposition 2.3 we have:

Proposition 2.5 For every normal homomorphism \(\varphi : H \to H_1\) the homomorphism of reduction \(\bar{\varphi}\) is being factorized as follows: \(\bar{\varphi} = \psi n\), where \(n\) is the canonical mapping \(x \to x + \mathcal{N}(\varphi)\) from \(H\) onto \(H/\mathcal{N}(\varphi)\), and \(\psi\) is the isomorphism from \(H/\mathcal{N}(\varphi)\) onto \(\varphi(H)/(0_1)\).

Proposition 2.6 For every subhypergroup \(h\) of \(H\), the canonical homomorphism \(\varphi : x \to x + h\) is a normal homomorphism from \(H\) onto \(H/h\) and obviously \(\mathcal{N}(\varphi) = h\).

\(^1\)I.e. a homomorphism from \(H\) onto \(H_1\).
HOMOMORPHISMS IN THE THEORY OF THE HYPERGROUPS

Proof. It is obvious that \( \varphi \) is strict. Also we have:

\[
\varphi(x + y) = x + y + h = (x + h) + (y + h) = \varphi(x) + \varphi(y)
\]

thus \( \varphi \) is also strong. Therefore \( \varphi \) is normal. Q.e.d.

Corollary 2.3 For every normal equivalence relation \( R \) in \( H \), the mapping \( \varphi : H \rightarrow H/R \) such that \( \varphi(x) = C_R(x) \) for every \( x \in H \), is a normal homomorphism.

Proposition 2.7 For every subhypergroup \( h \) of \( H \) holds:

\[
h = N^h(\varphi) \text{ where } N^h(\varphi) \text{ is the kernel of the canonical mapping } \varphi : H \rightarrow H/h.
\]

Proof. Indeed the kernel of \( \varphi \) is exactly the \( h \) because \( x \in N^h(\varphi) \iff f(x) = h \), also \( \varphi(x) = x + h \) and so \( x \in N^h(\varphi) \iff x + h = h \). Thus \( N^h(\varphi) = h \). Q.e.d.

In the theory of homomorphisms between M-P.Hs, except the equivalence relation of the homomorphism, we also have the following equivalence relation, which appears due to the feature or the M-P.H:

Definition 2.1 Let \( \varphi : H \rightarrow H_1 \) be a normal homomorphism. We name equivalence relation (in \( H \)) of the normal homomorphism, related with the \( 0_1 \in H_1 \), the equivalence relation \( R \) which is defined as follows:

\[
xRy \iff \varphi(x) \equiv \varphi(y) \mod (0_1).
\]

Now, let \( R' \) be the equivalence relation of the homomorphism, that is:

\[
xR'y \iff \varphi(x) = \varphi(y).
\]

We can see that, for every \( x, y \in H \), we have:

\[
(x, y) \in R' \Rightarrow \varphi(x) + 0_1 = \varphi(y) + 0_1 \Rightarrow \varphi(x) \equiv \varphi(y) \mod (0_1) \Rightarrow (x, y) \in R.
\]

[The converse generally is not valid because the cancellation law does not hold for the M-P.Hs, i.e. \( \varphi(x) + 0_1 = \varphi(y) + 0_1 \not\Rightarrow \varphi(x) \equiv \varphi(y) \) [11]]. Thus \( R' \subseteq R \) and so the proposition:

Proposition 2.8 Every class \( \mod(R) \) of \( H \) is saturated with respect to the equivalence \( R' \).

Proposition 2.9 For every \( x, y \in H \) we have:

\[
xRy \iff x \equiv y \mod(N(\varphi))
\]

or, with other words, starting with the kernel \( N(\varphi) \) of \( \varphi \), we define the relation \( R \) as follows:

\[
xRy \iff x + S(y) \subseteq N(\varphi).
\]
Proof. Let \( x \) \( R \) \( y \). Then we have:
\[
\varphi(x) \equiv \varphi(y) \mod(01) \iff \varphi(x) + 0_1 = \varphi(y) + 0_1 \iff \varphi(x) + \varphi(y') + 0_1 = \varphi(y) + \varphi(y') + 0_1 \iff \varphi(x + y') + 0_1 = \varphi(y + y') + 0_1 \iff \varphi(x + y') + \varphi(0) = \varphi(0) + 0_1 = 0_1 = 0_1 \iff \varphi(x + y' + 0) = 0_1 \iff \varphi(x + y') = 0_1 \iff x + y' \subseteq \mathcal{N}(\varphi) \iff x + S(y) \subseteq \mathcal{N}(\varphi) \text{ and thus } x \equiv y \mod(\mathcal{N}(\varphi)) \ [12]. \ Q.e.d.
\]

**Corollary 2.4** The equivalence relation \( R \) is normal and the class \( \mathcal{C}_R(0) \) is the kernel \( \mathcal{N}(\varphi) \) of the homomorphism \( \varphi \) and for every \( x \in H \) holds:
\[
\mathcal{C}_R(x) = x + \mathcal{N}(\varphi).
\]

In the end of the paragraph of the homomorphisms we give the following example:

**Example 2.1:** Let’s consider the M-P.H (\( H, + \)) of the example 2.1 of [12], where \( H_1 = \bigcup_{X \in G_1} X^j \), \( G = \{X^0, X^1, X^2, X^3\} \) and the M-P.H (\( H_1, + \)) of the example 3.1 of [11] by taking \( X = Y' \), so \( \tilde{G}_1 = \{O^1, Y^1\}, H_1 = \bigcup_{Y \in G_1} Y^i \). Both M-P.Hs are equipped with the hyperoperations of the above mentioned examples. We also consider that the sets \( X^j, Y^j \) have the same cardinality. Let, for instance be \( X^j = \{x^j_1, x^j_2, x^j_3\} \), \( j \in \{1, 2, 3\} \), \( Y^i = \{y^i_1, y^i_2, y^i_3\} \) and \( X^0 = O = \{0\}, Y^0 = O^1 = \{0\} \). Then, the mapping \( \varphi : H \to H_1 \) such that: \( \varphi(0) = 0_1, \varphi(x^j_k) = 0_1 \) and \( \varphi(1) = \varphi(x^0_1) = y^1_1 \) for every \( k \in \{1, 2, 3\} \) is a normal homomorphism with kernel \( \mathcal{N}(\varphi) = X^0 \cup X^2 \) subhypergroup of \( H \). According to corollary 2.1 and proposition 2.3 the groups \( H/\mathcal{N}(\varphi) \) and \( H_1/(0_1) \) are isomorphic. Indeed
\[
\begin{align*}
\bar{x}^1 + \mathcal{N}(\varphi) &= x^1_1 + (X^0 \cup X^2) = (x^1_1 + X^0) \cup (x^1_1 + X^2) = X^1 \cup X^3, \\
x^1_2 + (X^0 \cup X^2) &= (x^1_2 + X^0) \cup (x^1_2 + X^2) = X^1 \cup X^3, \\
x^1_3 + (X^0 \cup X^2) &= (x^1_3 + X^0) \cup (x^1_3 + X^2) = X^1 \cup X^3, \\
x^1_0 + (X^0 \cup X^2) &= (x^1_0 + X^0) \cup (x^1_0 + X^2) = X^1 \cup X^3, \\
x^1_0 + 0_1 + \mathcal{N}(\varphi) &= X^0 \cup X^2 \cup X^0 = X^0 \cup X^2,
\end{align*}
\]
\( H/\mathcal{N}(\varphi) = \{X^0 \cup X^2, X^1 \cup X^3\} \) and \( H_1/(0_1) = \{Y^0, Y^1\} \).

**Remark 2.1:** If we reconsider the above example, taking as M-P.H (\( H, + \)) the one which appears on the example 3.2 of [11] and if we take \( X = Y^1, Y = Y^2, \) so \( G = \{O, Y^1, Y^2\} \), then the proposition 2.3 does not hold for none of the candidate homomorphisms, with \( \mathcal{N}(\varphi) = \{0\} = X^0 \) or \( \mathcal{N}(\varphi) = X^0 \cup X^2 \) \( [X^0, X^0 \cup X^2 \] are the only proper subhypergroups of \( H \) because \( \text{card } H/\mathcal{N}(\varphi) = 4, H/\mathcal{N}(\varphi_1) = 2, \) while \( H/(t_1) = 3. \) Thus the hypergroups \( H \) and \( H_1 \) are not isomorphic.

### 3 Monogene M-P.Hs

Let \( H \) be a M.P.H and \( x \in H \). The subhypergroup \( \overline{\{x\}} \) of \( H \) (thus M-P.SH of \( H \)) which is generated by \( x \) is named, (in an analogous way to the classical case of the groups and the case of the canonical hypergroups [5], [7]), *monogene* subhypergroup of \( H \) generates by \( x \). A subhypergroup \( h \) of \( H \) is named *monogene*, if there is \( x \in H \) such that \( h = \overline{\{x\}} \). A M-P.H is named *monogene* M-P.H if there is \( x \in H \) such that \( h = \overline{\{x\}} \).
First of all, for every \( x_1, \ldots, x_n \in C_0(x), \ x_1', \ldots, x_n' \in S(x) \) for \( n > 1 \), we have (see [11]):

\[
x_1 + x_2 + \cdots + x_n = x + x + \cdots + x = C_0(x) + \cdots + C_0(x)
\]

\[
x_1' + x_2' + \cdots + x_n' = x' + x' + \cdots + x' = S(x) + \cdots + S(x) = C_0(x') + \cdots + C_0(x')
\]

Also for \( n = 1 \) we have:

\[
0 + x_1 = 0 + x = C_0(x), \ 0 + x_1' = 0 + x' = S(x) = C_0(x')
\]

So, we introduce the definition of a multiplication of an integer \( n \) with an element \( x \in H \) as follows:

\[
n \cdot x = \begin{cases} x + x + \cdots + x & \text{n times for } n > 0, \ n \neq 1 \\
0 & \text{for } n = 0 \\
x' + x' + \cdots + x' & \text{n times for } n < 0, \ n \neq -1
\end{cases}
\]

Also for \( n = 1 \) we define \( 1 \cdot x = 0 + x \)

and for \( n = -1 \) we define \( (-1) \cdot x = 0 + x', \ x' \in S(x), \) arbitrary.

We easy conclude that the following relations are valid:

\[
n \cdot (x + y) = n \cdot x + n \cdot y
\]

\[
m \cdot x + n \cdot x = (m + n) \cdot x
\]

Hence, based on the proposition 2.6 of [12], we have for the subhypergroup \( \overline{\{x\}} \) which is generated by \( x \):

\[
\overline{\{x\}} = \bigcup_{k, \ell \in \mathbb{Z}^+} (k \cdot x - \ell \cdot x) = \bigcup_{n \in \mathbb{Z}} n \cdot x = \bigg( \bigcup_{m \in \mathbb{Z} - \{1, -1\}} n \cdot x \bigg) \cup C_0(x) \cup S(x)
\]

And so the proposition:

**Proposition 3.1** For every \( x \in H \) we have:

\[
\overline{\{x\}} = \bigcup_{n \in \mathbb{Z}} n \cdot x = \bigg( \bigcup_{m \in \mathbb{Z} - \{1, -1\}} n \cdot x \bigg) \cup C_0(x) \cup S(x)
\]

**Remarks 3.1**:

a) If \( 0 \in k \cdot x \), then \( k \cdot x = k \cdot x' = 0 \) for every \( k \in \mathbb{Z}, \ x \in H \). Indeed, \( 0 \in k \cdot x \Rightarrow 0 + k \cdot x' \subseteq k \cdot x + k \cdot x' = k(x + x') = 0 \Rightarrow 0 + k \cdot x' = 0 \Rightarrow k \cdot x' + k \cdot x = 0 + k \cdot x \Rightarrow 0 + k \cdot x = k \cdot (x + x') = 0 \Rightarrow k \cdot x = 0 \).

b) \( m \cdot x \cap n \cdot x \neq \emptyset \) for every \( x \in H \) and \( m, n \in \mathbb{Z} \). Because, \( m \cdot x \cap n \cdot x \neq \emptyset \Leftrightarrow 0 \in m \cdot x - n \cdot x = (m - n) \cdot x \).

Now, we distinguish between two cases which are contradictory to each other, that is:
I. either, for every \( m, n \in \mathbb{Z} \), \( m \neq n \) is \( m \cdot x \cap n \cdot x = \emptyset \), thus \( 0 \in m \cdot x - n \cdot x = (m-n) \cdot x \) and therefore \( 0 \not\in h \cdot x \) for none of the \( h \in \mathbb{Z} - \{0\} \).

In this case the element \( x \) and the monogene subhypergroup \( \langle x \rangle \), is said to have infinite \( (+\infty) \) order.

II. or, there exist, \( m, n \in \mathbb{Z} \), \( m \neq n \) such that: \( m \cdot x \cap n \cdot x \neq \emptyset \) and thus there exists \( h \in \mathbb{Z} - \{0\} \) such that \( 0 \in n \cdot x \) (so \( h \cdot x = 0 \)).

Because of the previous remark 3.1, a when \( h < 0 \) then we have for \( -h > 0 \), that \( 0 \in -h \cdot x \). Therefore, in this case, there exists a minimum positive integer \( h \), hence the element \( x \) and the subhypergroup \( \langle x \rangle \) is said to have order \( h \). In this case the subhypergroup \( \langle x \rangle \) is named cyclic.

Denoting the order of an element \( x \) and also the order of the subhypergroup \( \langle x \rangle \) by \( \omega(x) \), we give in briefly the above cases in the following proposition:

**Proposition 3.2** We say that:

i) \( \omega(x) = +\infty \), when for every \( m, n \in \mathbb{Z} \), \( m \neq n \) is valid that \( m \cdot x \cap n \cdot x = \emptyset \) or, equivalently, when for every \( h \in \mathbb{Z} - \{0\} \) we have \( h \cdot x \neq 0 \).

ii) \( \omega(x) = l \in \mathbb{N} - \{0\} \), when there exist \( m, n \in \mathbb{Z} \), \( m \neq n \) such that \( m \cdot x \cap n \cdot x \neq \emptyset \) or, equivalently, when there exist \( h \in \mathbb{Z} - \{0\} \) such that \( h \cdot x = 0 \). \( l \) is the minimum positive integer which has the previous mentioned property.

**Remarks 3.2:**

a) The zero of \( H \) has order 1 and it is the only element oh \( H \) which has order 1 (Indeed, let \( x \in H \), \( x \neq 0 \) and \( \omega(x) = 1 \), then \( 1 \cdot x = 0 \) but \( 1 \cdot x = x \cdot 0 \) and \( 0 \not\in 0 + x \)). b) Obviously, the elements \( x, x' \in H \), \( x' \in S(x) \) generate the same monogene subhypergroup.

Now let’s consider a monogene M-P.SH, with \( \omega(x) = \lambda \), so \( \lambda \cdot x = 0 \). Also let \( m \in \mathbb{Z} - \{0\} \) such that \( m \cdot x = 0 \). Then \( m = k \cdot \lambda + u \), \( k \in \mathbb{Z} \), \( 0 \leq u < \lambda \).

So: \( (k \cdot \lambda + u) \cdot x = 0 \Leftrightarrow k \cdot \lambda \cdot x + u \cdot x = 0 \Leftrightarrow k \cdot 0 + u \cdot x = 0 \Leftrightarrow 0 + u \cdot x = 0 \Leftrightarrow u \cdot x = 0 \). But \( u < \lambda \) and \( \lambda \) is the minimum non zero positive integer which has this property, consequently \( u = 0 \).

And so the proposition:

**Proposition 3.3** If \( \omega(x) = \lambda \) then \( m \cdot x = 0 \), \( m \in \mathbb{Z} - \{0\} \) if and only if \( m = \lambda \cdot x \), \( x \in Z \).

Further on, we easily observe that \( \Omega(x) = \Omega \left( \frac{x}{\Omega(x)} \right) = 0^1 \), that is the quotient set \( \left\{ \frac{x}{\Omega(x)} = \frac{x}{(0)} \right\} \) is the group of reduction of \( \langle x \rangle \) [11]. Obviously \( \left\{ \frac{x}{(0)} \right\} \subseteq H/(0) \) is a subgroup of \( H/(0) \).

\(^1\)In the theory of M-P.Hs holds that \( \Omega(x) = \{0\} \), \( X \subseteq H \), \( X \neq \emptyset \) where \( H \) is a M-P.H. See also [4], [7], [8].
Taking into consideration the above, we come to the conclusion that for every $x \in H$ the sets $m \cdot x$, when $m$ runz into $\mathbb{Z}$ are either disjoint sets or coincide to each other. Specifically, when $\omega(x) = +\infty$, for every $m_1, m_2 \in \mathbb{Z}$, $m_1 \neq m_2$ then the sets $m_1 \cdot x$, $m_2 \cdot x$ are disjoint sets, but when $\omega(x) = \lambda$, $\lambda \in \mathbb{N} - \{0\}$, then every set $m \cdot x$ coincides with one of the sets $m \cdot x$, $m \in \{0, 1, \ldots, \lambda - 1\}$. (Because $m = k \cdot \lambda + u \Rightarrow m \cdot x = k \cdot \lambda \cdot x + u \cdot x = k \cdot 0 + u \cdot x$, $0 \leq u < \lambda$). So the sets $m \cdot x$ form a partition in $\{x\}$, denoted by $\text{mod}(\omega(x))$. For the partition $\text{mod}(\omega(x))$ we have:

$$z_1 \equiv z_2 \mod(\omega(x)) \iff (\exists k \in \mathbb{Z})[(z_1 \in k \cdot x) \wedge (z_2 \in k \cdot x)] \iff$$

$$\Leftrightarrow z_1 + z_2' \leq k \cdot x + k \cdot x' = k \cdot (x + x') = k \cdot 0 = 0 \Rightarrow$$

$$z_1 + z_2' = 0 \iff z_1 + z_2' + z_2 = 0 + z_1 = 0 + z_2 \Leftrightarrow z_1 \equiv z_2 \mod(0).$$

Thus we deduce to the proposition:

**Proposition 3.4** For every $x \in H$, the sets $m \cdot x$, $m \in \mathbb{Z}$ when $m$ runs into $\mathbb{Z}$, form a partition of the monogene M-P.H $\{x\}$, which coincides with the partition $\text{mod}(0)$ of $\{x\}$.

**Corollary 3.1** When $z \in m \cdot x$ then it is valid that $m \cdot x = 0 + z = C_0(z)$.

**Proposition 3.5** If $\omega(x) = +\infty$ then the group of reduction $\{x\}/(0)$ of $\{x\}$ is isomorphic to the additive group $\mathbb{Z}$ of integers. If $\omega(x) = \lambda$, $\lambda \in \mathbb{N} - \{0\}$ then the group $\{x\}/(0)$ is isomorphic to the additive group $\mathbb{Z}/(\lambda)$ of the classes $\text{mod}(\lambda)$ of $\mathbb{Z}$.

As it is in the theory of the groups, a M-P.H is said to be without torsion if everyone of its elements, except zero, has infinite order and a M-P.H is said to be with torsion or periodic if everyone of its elements has finite order.

The above can be seen in two examples of monogene M-P.Hs (one without torsion and one with torsion):

**Example 3.1**: 1. Taking into consideration the proposition 2.3 of [11] and starting with the example 2.1 of [12] we consider as the group $G$, the group $\mathbb{Z}$ of integers and an union $H = \bigcup_{x \in \mathbb{Z}} X^t$ of disjoint sets, with $X^0 = O = \{X_0^t\} = \{0\}$. The set $H$, endowed with the hyperoperation

$$x_k^k + x_m^m = \varphi^{-1} \left[ \varphi \left( x_k^k \right) + \varphi \left( x_m^m \right) \right],$$

where $\varphi : H \to \mathbb{Z}$ from $H$ onto $\mathbb{Z}$, such that $\varphi(x_k^k) = k$ for every $x_k^k \in X^k$, $k \in \mathbb{Z}$ (thus $\varphi(X^k) = k$), is a monogene M-P.H without torsion, according to proposition 3.4, having as generator an arbitrary $e \in X^t$ (thus $\varphi(e) = 1$). Indeed, for every $k \in \mathbb{Z}^+, k \neq 1$, we have:

$$k \cdot e = e \underbrace{\ldots + e}_{k \text{ times}} = \varphi^{-1} \left[ \varphi(e) + \cdots + \varphi(e) \right] = \varphi^{-1} \left( 1 + \cdots + 1 \right) \varphi^{-1}(k) =$$

$$= X^k \neq \emptyset.$$
Analogous are the cases, for $k \in \mathbb{Z}$, $k \neq -1$ and for $k = 0, 1, -1$.

2. Based again on the proposition 2.3 of [11] and on example 2.1 of [12] we consider as the group $G$, the additive group $\mathbb{Z}/(n)$ of the classes mod$(n)$ of $\mathbb{Z}$, $n \in \mathbb{N} - \{0\}$ and an union $H = \bigcup_{i \in E} X^i$ of disjoint sets, where $E = \{0, 1, \ldots, (n-1)\}$, with $X^0 = O = \{X^0\} = \{0\}$. Here the set of $H$ which is endowed with the same hyperoperation $x^k + x^n$ as above, but with $\varphi : H \to \mathbb{Z}/(n)$ from $H$ onto $\mathbb{Z}/(n)$ such that $\varphi(x^k) = k$ for every $x^k \in X^k$, $k \in E$ (thus $\varphi(X^k) = k$), is a monogene M-P.H with torsion.

According to proposition 3.4, $H$ has order $n$ and the generator of $H$ is an arbitrary element $q \in X^0$, $\delta \in E - \{0\}$ with $n$ being the order of $\delta$, thus $n \cdot \delta = 0$ and $\varphi(q) = \delta$. Indeed $n \cdot q = q + \cdots + q = \varphi^{-1}([\varphi(q) + \cdots + \varphi(q)]) = \varphi^{-1}(\delta + \cdots + \delta) = \varphi^{-1}(n\delta) = \varphi^{-1}(0) = \{0\}$ thus $\omega(q) = n$. Moreover, for every $\tau \in X^0$ (thus $\varphi(\tau) = \bar{\rho}$) and for every $k \in E$ we have:

$$k \cdot \tau = \underbrace{\tau + \cdots + \tau}_{k \text{ times}} = \underbrace{\varphi(\tau) + \cdots + \varphi(\tau)}_{k \text{ times}} = \varphi^{-1}(\bar{\rho} + \cdots + \bar{\rho}) = \varphi^{-1}(k\bar{\rho}) = X^{k\rho}$$

and also for some $k_1 \in E$ it will be $k_1 \cdot \bar{\rho} = 0$. Therefore $X^{k_1\rho} = X^0 = \{0\}$. Hence $\omega(\tau) = k_1$, that is $\tau$ has finite order.

**Remark 3.3**: Resulting from the above examples there derives the way of construction of all the monogene M-P.Hs with or without torsion.

**References**


[12] YATRAS, C.N.: Subhypergroups of M-polysymmetrical hypergroups. Proceeed-
ings of the Fifth International Congress on Algebraic Hyperstructures and Ap-

Author’s address:
Constantinos N. Yatras
8, Papatsoni Str., 11636 Athens, Greece