SOME REMARKS ON 3-DIMENSIONAL CONTACT RIEMANNIAN MANIFOLDS

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Abstract

In this communication we prove that a conformally flat 3 - manifold is either flat or a Sasakian manifold

AMS Subject Classification: 53C15, 53C25
Key words: flat and conformally flat Riemannian metrics, Sasakian structure

1 Introduction.

D. E. Blair in [3] showed that there are no flat Riemannian metrics associated to a contact structure on a contact manifold of dimension > 3. In [7] M. Okumura proved that every Sasakian conformally flat manifold of dimension is of constant curvature 1. In [9] S. Tanno showed that a 3 - dimensional conformally flat K - contact Riemannian manifold has constant curvature 1. In [4] D. E. Blair and T. Koufogiorgos proved that a conformally flat contact metric manifold M^{2n+4} with $Q\varphi = \varphi Q$ is of constant curvature 1 if $n > 1$, and 0 or 1 if $n = 1$. In [1] K. Bang proved that in dimension greater than 3, there are no conformally flat contact metric manifolds with . In [8] Z. Olszak proved that on a conformally flat contact metric manifold of dimension $2n + 1 \geq 5$, the scalar curvature S satisfies $S \leq 2n()2n + 1)$.

In the present communication we shall give a result of [6]. This is an extension of Babg's result in [1].

2 Preliminaries

By a contact manifold we mean a C^∞ manifold M^{2n+1} , endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. Then , it has an underlyng contact metric structure (η, g, φ, ξ) where g is a Rimannian metric (called assosiated metric) ,φ a global tensor of type $(1, 1)$ and ξ a global vector field (called characteristic vector field). These structure tensors satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta = g(\xi, \cdot), \quad \eta(\xi) = 1,$$

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1995,* 151-154

$$d\eta(Y,Z) = g(Y,\varphi Z), \quad g(\varphi Y,\varphi Z) = g(Y,Z) - \eta(Y)\eta(Z).$$

The associated metrics can be constructed by the polarization $d\eta$ of evaluated on a local orthonormal basis of an arbitrary meric on the contact distribution B defined by $\ker\eta$.

Denoting by L and R, Lie differentiation and curvature tensor respectively we define the tensors τ, h and l by

$$\tau = L_\xi g, \quad h = \frac{1}{2}L_\xi\varphi, \quad l = R(\cdot,\xi)\xi.$$

Letting ∇ denote the Riemannian connection of g, we have the following formulas (see e. g. [2])

$$\begin{array}{ll}
\varphi\xi = h\xi = l\xi = 0 & \eta\circ\varphi = \eta\circ h = 0 \\
d\eta\,(\xi,Z) = 0 & Tr\,h = Tr\,h\varphi = 0 \\
h\varphi = -\varphi h & \nabla_Z\xi = -\varphi Z - \varphi hZ \\
\nabla_\xi h = \varphi - \varphi l - \varphi h^2 & \varphi l\varphi - l = 2\left(\varphi^2 + h^2\right) \\
\nabla_\xi\varphi = 0 & Tr\,l = g\,(Q\xi,\xi) = 2 - Tr\,h^2
\end{array}$$

A contact meric structure is K - contact if ξ is a Killing field, this is the case if and only if $h = 0$. If the structure is normal it is Sasakian . A. Sasakian structure is K-contact only for dimension 3.

The sectional curvature $K(X,\xi)$ $(= g(R(X,\xi)\xi,X))$of a plane section spanned by ξ and a vector X orthogonal to ξ is called ξ-sectional curvature. The sectional curvature $K(X,\varphi X)$ $(= g(R(X,\varphi X)\varphi X,X))$ of a plane section spanned by vectors X and φX with X orthogonal to ξ is called φ-sectional curvature.

On a 3-dimensional Riemannian manifald M, we denote by Q the Ricci operator, by $S = Tr\,Q$ the scalar curvature and by P the tensor field $-Q + \frac{3}{4}l$. The curvature tensor $R(Y,Z)W$ is given by

$$R(Y,Z)W = g\,(Z,W)\,QY - g\,(Y,W)\,QZ + g\,(QZ,W)\,Y - g\,(QY,W)\,Z -$$
$$\frac{S}{2}\,[g(Z,W) - g(Y,W)Z] \tag{1}$$

A. Riemannian manifold is said to be conformally flat if it is conformally equivalent to a Euclidean space. A 3-dimensional Riemannian manifold M is conformally flat if and only if

$$(\nabla_Y P)\,Z = (\nabla_Z P)\,Y \tag{2}$$

for all vector fields Y and Z on M.

In what follows sayng 3-τ-manifold we mean a 3-dimensional contact metric manifold satisfyng $\nabla_\xi\tau$, (see e.g. [5]) . Now we shall give some results of [5] which we will use in the present communication.

Proposition 1 *Let M^3 be a non - Sasakian 3-τ- manifold. If X is a unit-eigenvector of h with eigenvalue λ and orthogonal to ξ, the*

$$\nabla_\xi X = \nabla_\xi(\varphi X) = 0$$

$$\nabla_X\xi = -(\lambda+1)\varphi X, \quad \nabla_{fX}\xi = (1-\lambda)X$$

$$\nabla_X X = \frac{1}{2\lambda}\left[\varphi X \cdot \lambda + \eta(QX)\right]\varphi X$$

$$\nabla_{\varphi X}\varphi X = \frac{1}{2\lambda}\left[X \cdot \lambda + \eta(Q\varphi X)\right]X$$

$$\nabla_X \varphi X = -\frac{1}{2\lambda}\left[\varphi X \cdot \lambda + \eta(QX)\right]X + (\lambda+1)\xi$$

$$\nabla_{\varphi X} X = -\frac{1}{2\lambda}\left[X \cdot \lambda + \eta(Q\varphi X)\right]\varphi X + (\lambda-1)\xi$$

Corollary 2 *On a* 3-τ*-manifold holds* $\xi \cdot Tr\, l = 0$.

Proposition 3 *Let* M^3 *be a* 3*-dimensional contact metric manifold. If for every* Z *of the contact distribution* B*, holds* $QZ \in B$ *then the conditions* $Q\varphi = \varphi Q$ *and* $\nabla_\xi \tau = 0$ *are equivalent.*

If $\lambda(\neq 0)$ is the eigenvalue of h with unit-eigevector X orthogonal to ξ then

$$Tr\, l = 2(1-\lambda^2) \leq 2 \tag{3}$$

For every vector field Z on a 3-τ-manifold M^3 we have

$$QZ = aZ + b\eta(Z)\xi + \eta(Z)Q\xi + \eta(QZ)\xi \tag{4}$$

where $a = \frac{1}{2}\left(S - Tr\, l\right)$ and $b = -\frac{1}{2}\left(S + Tr\, l\right)$.

3 Conformally flat 3-τ-manifolds

In the paper [6] we proved the following

Proposition 4 *A conformally flat* 3-τ *manifold with* $Tr\, l =$ *constant is either flat or Sasakian with constant curvature* 1.

The prove the above proposition we use the relations (1), 2, 3 and 4 and the propositions 1, 3 and corollary 2. Using the propositions 4. and 3 , [4] we obtain the main result:

Theorem 1 *A conformally flat* 3-τ*-manifold is either flat or a Sasakian manifold*

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