SPECTRA GEOMETRY FOR RICCI MANIFOLDS

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Abstract

The aim of the present paper is to study the influence of the spectra of different second order elliptic differential operators, on the cross sections of different vector bundles over a compact Riemannian manifold (M,g), on the Ricci structure of (M,g).

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1 Introduction

Let (M,g) be a Riemannian manifold of dimension n. If the Ricci tensor field ρ on (M,g) is parallel (resp. zero), that means $\nabla\rho = 0$ (resp. $\rho = 0$), the (M,g) is called Ricci manifold (resp. flat Ricci manifold). One of the problems of Differential Geometry is to study Ricci manifolds as well as flat Ricci manifolds.

Let Δ_k, $k = 0,1,\ldots,n$, be the Laplacian acting on the vector space $\Lambda^k(M,IR)$ of differential exterior k-forms. The spectrum of Δ_k is denoted by $Sp(M,g,\Delta_k)$ or briefly $Sp(M,\Delta_k)$. We also consider the Bochner-Laplace operator B_k, $k = 0,1,\ldots,n-1$, acting on $\Lambda^k(M,IR)$ whose spectrum is denoted by $Sp(M,B_k)$. Let $D_k^\varepsilon = \varepsilon\Delta_k + (1-\varepsilon)B_k$, $k = 0,1,\ldots,n-1$, be one parameter family of second order elliptic differential operators on $\Lambda^k(M,IR)$, whose spectrum is denoted by $Sp(M,D_k^\varepsilon) = \{\lambda_{m,k}(\varepsilon)\}$. This spectrum is distinct and each eigenvalue has finite multiplicity. The following theorem has been proved ([6]).

Theorem 1.1 *Let (M,g) and (M',g') be two compact Riemannian manifolds. Let $Sp(M,\Delta_k) = Sp(M',\Delta_k)$, $k = 0,2$ and $Sp(M,D_1^\varepsilon) = Sp(M',D_1^\varepsilon)$ for $n+1$ distinct values of ε. If $\mid \nabla\rho \mid^2 (M) = 0$, then $\mid \nabla\rho' \mid^2 (M) = 0$ and the eigenvalues of the Ricci tensor fields ρ and ρ' on M and M' respectively are the same.*

The aim of the present paper is to improve this theorem. We also atudy some other properties of Ricci manifold and flat Ricci manifold.

The improvement of the theorem 1.1 can be stated as follows

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Theorem 1.2 *Let (M,g) and (M',g') be two compact Riemannian manifolds. If $Sp(M,\Delta_k) = Sp(M',\Delta_k)$, $k = 0,1,2$ and $Sp(M,D_1^\varepsilon) = Sp(M',D_1^\varepsilon)$ only for three distinct values of $\varepsilon \neq 0$, then if (M,g) is locally Ricci manifold, the, so is (M',g').*

This paper contains five paragraphs. Each of them is analised as follows.

The second paragraph includes a general theory for Ricci manifolds and flat ricci manifolds. It also contains the relation between Ricci and Einstein manifolds.

The general theory of second order eliptic differential operator is studiedin the third paragraph.

The fourth paregraph has the proof of theorem 1.2, another basic theorem and some other isospectral properties of the Ricci tensor field.

Exemples of manifolds, which carry or not parallel Ricci tensor fields, are given in the last paragraph. Some spectra of these manifolds are studied in this paragraph.

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Let (M,g) be a compact Riemannian manifold of dimension n. Let (U,φ) be a chart of M with local coordinate system $(x_1,\dots x_n)$. If the restriction of the Ricci tensor field ρ on U satisfies the relation

$$\nabla\rho = 0 \tag{1}$$

then (M,g) is called locally Ricci manifold. If this property (1) is valid for the whole manifold, then M is called Ricci manifold. Similarly we can define locally flat Ricci manifold and as well as flat Ricci manifold.

Proposition 2.1 *Let (M,g) be a Ricci manifold.Then the scalar curvature of (M,g) is constant.*

Proof. It is known that the curvature tensor field R, with components (R_{hijk}) on the chart (U,φ) with local coordinate system $(x_1,\dots,x_n)$, satisfies the relations :

$$\nabla_l R^h_{ijk} + \nabla_j R^h_{ikl} + \nabla_k R^h_{ilj} = 0$$

which can be written

$$\nabla_l R^h_{ijk} - \nabla_j R^h_{ilk} + g^{hm}\nabla_k R_{milj} = 0 \tag{2}$$

Contracting (2) for h and k we obtain

$$\nabla_l \rho_{ij} - \nabla_j \rho_{il} + g^{hm}\nabla_k R_{milj} = 0 \tag{3}$$

Multiplying (3) by g^{il} we get

$$\nabla_l \rho^l_j - \nabla_j T + \nabla_l \rho^l_j = 0 \tag{4}$$

which implies

$$\nabla_l \rho^l_j = \frac{1}{2}\nabla_l T = \frac{1}{2}\frac{\partial T}{\partial x^j}, \tag{5}$$

where

$$\rho^l_j = g^{lm}\rho_{mj} \tag{6}$$

T the scalar curvature on M and (ρ_{ij}) the components of ρ on U with respect to $(x_1, \ldots, x_n)$.

From 6 we obtain

$$\begin{aligned}\nabla_l \rho^l_j = \nabla_l g^{lm}\rho_{mj} &= g^{lm}\nabla_l \rho_{mj} + \rho_{mj}\nabla_l g^{lm} \\ &= g^{lm}\nabla_l \rho_{mj}.\end{aligned} \tag{7}$$

If the Riemannian manifold (M, g) is Ricci manifold, then from (7) we have

$$\nabla_l \rho^l_j = 0 \tag{8}$$

From (5) and (8) we obtain
$\frac{\partial T}{\partial x^i} = 0 \Rightarrow T = const.$ □

Proposition 2.2 *Let (M, g) be a compact Riemannian manifold of dfimension n. (M, g) is a Ricci manifold if and only if the Einstein tensor field G is parallel.*

Proof. Let (U, φ) be a chart of (M, g) with local coordinate system $(x_1, \ldots, x_n)$. If G is the Einstein tensor field on M, then we have

$$G_{ij} = \rho_{ij} - \frac{T}{n} g_{ij}, \tag{9}$$

where $\{G_{ij}\}$ are the components of G with respect to $(x_1, \ldots, x_n)$.

From (9) we obtain

$$\nabla_l G_{ij} = \nabla_l \rho_{ij} - \nabla_l \frac{T}{n} g_{ij} = \nabla_l \rho_{ij} - \frac{1}{n} g_{ij} \nabla_l T. \tag{10}$$

If (M, g) is Ricci, then $\nabla_l \rho_{ij} = 0$, $\nabla_l T = 0$ and (10) implies
$\nabla_l G_{ij} = 0 \Rightarrow G$ is parallel.
Conversely if G is parallel then from (10) we have

$$\nabla_l \rho_{ij} = \frac{1}{n} g_{ij} \nabla_l T$$

which by means of (5) becomes

$$2n\nabla_l \rho_{ij} = g_{ij}\nabla_l \rho^l_j. \tag{11}$$

If $(x_1, \ldots, x_n)$ is a normal coordinate system with center the point $P \in U$ such that

$$g_{ij}(P) = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j. \end{cases}$$

Then the relation (10) implies

$$2n\nabla_l\rho_{ij} = \nabla_l\rho_{ij} \Rightarrow \nabla_l\rho_{ij} = 0.$$

From the continuity of ρ we conclude

$$\nabla_l\rho = 0$$

which means the Riemannian manifold (M, g) is Ricci. □

Let ρ be the Ricci tensor field on the Riemannian manifold (M, g). Then for each point $P \in M$ $\rho(P)$ is a symmetric contravariant tensor field of order two obtained by the tangent space $T_P(M)$ of M at P. If $\{e_1, \ldots, e_n\}$ is an orthonormal base of $T_P(M)$, then $\rho(P)$ can be represented by the symmetric matrix

$$\begin{pmatrix} \rho_{11}(P) & \rho_{12}(P) & \ldots & \rho_{1n}(P) \\ \rho_{21}(P) & \rho_{22}(P) & \ldots & \rho_{2n}(P) \\ \ldots & \ldots & \ldots & \ldots \\ \rho_{n1}(P) & \rho_{n2}(P) & \ldots & \rho_{nn}(P) \end{pmatrix} \tag{12}$$

with respect to $\{e_1, \ldots, e_n\}$. The eigenvalues of (12) are real numbers

$$\lambda_1(P), \lambda_2(P), \ldots, \lambda_n(P).$$

If $\lambda_i(P) > 0$ (resp. $\lambda_i(P) < 0$) $i = 1, \ldots, n$ for every $P \in M$, then the Ricci tensor field ρ is called positive definite (resp. negative definite). If $\lambda_1(P) \geq 0$ (resp. $\lambda_i(P) \leq 0$) $i = 1, \ldots, n$ for every $P \in M$, then the Ricci tensor field ρ is called semi-positive (resp. semi-negative).

If $\lambda_i(P) = 0$ $i = 1, \ldots, n$ for every $P \in M$, then $\rho = 0$ and (M, g) is Ricci flat manifold.

3

Let (M, g) be a compact Riemannian manifold of dimension n. Let (U, φ) be a chart of M with local coordinate system $(x_1, \ldots, x_n)$. The Riemannian metric ρ on U takes the form

$$ds^2 = g^{ij}dx_i dx_j.$$

Let $\left[g^{ij}\right]$ be the metric on the cotangent bundle T^*M over M and let dM be the Riemannian measure of M.

Let V be a smooth vector bundle over M. We consider

$$D : C^\infty(V) \to C^\infty(V)$$

a second order elliptic differential operator with leading symbol given by the metric tensor g. We choose a local orthonormal frame

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

for V which corresponds to the chart (U, φ) with local coordinate system $(x_1, \ldots, x_n)$. Hence D in a local level can be expressed by

$$D = -(g^{ij}\partial^2/\partial x_i \partial x_j + P_k \partial/\partial x_k + S)$$

where P_k and S are square matrices which are not invariantly defined but depend on the choise of frame and local coordinates.

Let V_x be the fibre of V over x. We choose a smooth fibre metric on V. Let $L^2(V)$ be the completion of $C^\infty(V)$ with respect to global integtaded inner product, that is

$$L^2(V) = \left\{ S \in C^\infty(V) / \int_M \| S \| \, dM < \infty \right\}.$$

As a banach space $L^2(V)$ is independent on the Riemannian and fibre metric and for $t > 0$

$$\exp(-tD) : L^2(V) \to C^\infty(V)$$

is an infinitely smoothing operator of trace class. Let $K(t, x, y) : V_y \to V_x$ be the Kernell of $\exp(-tD)$. K is a smooth endomorphism valued function of (t, x, y).

We define

$$f(t, D, x) = Trace_{V_x}(K(t, x, x))$$

and

$$f(t, D) = \int_M K(t, x, x) dM.$$

It is known that $f(t, D, x)$ has an asymptotic expansion, that is ([2])

$$f(t, D, x) \underset{t \to 0^+}{\cong} (4nt)^{-n/2} \sum_{m=0}^{\infty} \alpha_m(D, x) t^m$$

The coefficients $\alpha_m(D, x)$ are smooth functions of x, which can be estimated functionally of the derivatives of the total symbols of the differential operator D. If we integrate the function

$$\alpha_m(D, x) : M \to IR, \; m = 0, 1, 2, \ldots$$

on the manifold M we obtain the numbers

$$\alpha_m(D) = \int_M \alpha_m(D, x) dM$$

It is known that the numbers $\alpha_m(D)$, $m = 0, 1, 2, \ldots$, are isospectral invariants.

Let $D = \Delta_q$, $q = 0, 1, \ldots, n$, be the Laplacian which is a second order elliptic differential operator with leading symbol defined by the metric tensor on the cross sections of the vector bundle of exterior q-form $\Lambda^q(M)$ over the manifold M, that is

$$\Delta_q = d\delta + \delta d : C^\infty(\Lambda^q(M)) \to C^\infty(\Lambda^q(M)),$$

where d and δ are the exterior differentiation and codifferentiation respectively.

The coefficients $\alpha_m(\Delta_q)$ for $m = 0, 1, 2, 3$ and for $q = 0, 1, 2$ are given by ([1],[4],[6])

$$\alpha_0(\Delta_0) = Vol(M),\ \alpha_1(\Delta_0) = -\frac{1}{6}\int\limits_M T\, dM \tag{13}$$

$$\alpha_2(\Delta_0) = \frac{1}{360}\int\limits_M \left(5T^2 - 2 \mid \rho \mid^2 + 2 \mid R \mid^2\right) dM \tag{14}$$

$$\begin{aligned}\alpha_3(\Delta_0) = \tfrac{1}{9\cdot 7!}\int\limits_M [-142 \mid \nabla T \mid^2 -26 \mid \nabla\rho \mid^2 -7 \mid \nabla R \mid^2 -35T^3 + \\ 42T \mid \nabla\rho \mid^2 -42T \mid R \mid^2 +35 \mid \rho \mid^3 -20L_1 + 8L_2 - 24L_3]dM\end{aligned} \tag{15}$$

$$\alpha_0(\Delta_1) = \binom{n}{1} Vol(M),\ \alpha_1(\Delta_1) = \frac{6-n}{6}\int T\, dM \tag{16}$$

$$\begin{aligned}\alpha_2(\Delta_1) = \tfrac{1}{360}\int\limits_M [(5n-60)T^2 - (2n-180) \mid \rho \mid^2 + \\ (2n-30) \mid R \mid^2]\, dM\end{aligned} \tag{17}$$

$$\begin{aligned}\alpha_3(\Delta_1) = \tfrac{1}{4\cdot 7!\cdot 9\cdot 10}\int\limits_M [-(5680n+980) \mid \nabla T \mid^2 -(104n+1078) \mid \nabla\rho \mid^2 + \\ +(280n+49) \mid \nabla R \mid^2 -(1400n-215)T^3 + (1680n-1568)T \mid \rho \mid^2 + \\ +(343-1680n)T \mid R \mid^2 +(1440+2548) \mid \rho \mid^3 -(800n-392)L_1 + \\ +(320n-1392)L_2 - (960n-147)L_3]dM\end{aligned} \tag{18}$$

$$\alpha_0(\Delta_2) = \binom{n}{2} Vol(M),\ \alpha_1(\Delta_2) = \frac{(n-1)(n-12)}{12}\int T\, dM \tag{19}$$

$$\begin{aligned}\alpha_2(\Delta_2) = \tfrac{1}{720}\int\limits_M [(5n^2-125n+600)T^2 + (2n^2+362n-2 \mid \rho \mid^2 + \\ +(2n^2-62n+480) \mid R \mid^2)]\, dM\end{aligned} \tag{20}$$

$$\begin{aligned}\alpha_2(\Delta_2) = \tfrac{1}{4\cdot 7!\cdot 9\cdot 10}\int\limits_M [-(2840n^2-3330n-2438) \mid \nabla T \mid^2 + \\ +(-52n^2-1026n+8036) \mid \nabla\rho \mid^2 +(-140n^2+149n-1568) \mid \nabla R \mid^2 - \\ -(-720n^2+265n-1960)T^3 + (840n^2-2408n+17836)T \mid \rho \mid^2 + \\ (-800n^2+1192n-18421)L_1 + (160n^2-1532n+26246)L_2 - \\ -(-480n^2+627n-4708)L_3]dM\end{aligned} \tag{21}$$

where

$$L_1 = \rho_{ij}\rho_{km}R_{ijkm}, \quad L_2 = \rho_{ij}R_{iklm}R_{jklm} \tag{22}$$

$$L_3 = R_{ijkm}R_{ljuv}R_{knmv} \tag{23}$$

and R, ρ and T the curvature tensor field, Ricci tensor field, Ricci tensor field and the scalar curvature respectively, $\mid R \mid$ and $\mid \rho \mid$ the norm of R and ρ respectively, (ρ_{ij}) and (R_{ijkl}) are the components of ρ and R, respectively with respect to the local coordinate system $(x_1, \ldots, x_n)$ on the chart (U, φ) of the manifold M and ∇T, $\nabla\rho$, ∇R are the covariant derivatives of T, ρ, R respectively.

Now, we can definethe reduced or Bochner Laplacian operator B_k^{∇} by the following diagram

$$B_k^{\nabla} : C^{\infty}(M) \to C^{\infty}(T^*M \otimes V) \overset{\nabla_g \otimes 1 + 1 \otimes \nabla}{\longrightarrow} C^{\infty}(T^*M \otimes T^*M \otimes V) \to \overset{-g \otimes 1}{\longrightarrow} C^{\infty}(V) \tag{24}$$

where g is the Riemannian metric on M, ∇_g the Levi-Civita connection on TM, extend ∇_g on the tensor fields of all type and ∇ any connection on V. The Bochner Laplacian B_k^{∇} defined by Levi-Civita connection in local coordinate system has the form

$$B_k^{\nabla} = -g_{ij}\nabla_i\nabla_j \tag{25}$$

Now, we form one parameter family of second order elliptic differential operators

$$D_k^{\varepsilon} = \varepsilon\Delta_k + (1-\varepsilon)B_k \tag{26}$$

The coefficients $\alpha_m(D_k^{\varepsilon})$ for $m = 0, 1, 2, 3$ are given by ([4])

$$\alpha_0(D_1^{\varepsilon}) = nVol(M), \quad \alpha_1(D_1^{\varepsilon}) = \frac{6\varepsilon - 1}{6} \int_M T\, dM \tag{27}$$

$$\alpha_2(D_1^{\varepsilon}) = \frac{1}{360} \int_M [(5n - 6\varepsilon)T^2 - (180\varepsilon^2 - 2n) \mid \rho \mid^2 + (2n - 30) \mid R \mid^2] dM \tag{28}$$

$$\begin{gathered} \alpha_3(D_1^{\varepsilon}) = \tfrac{1}{360 \cdot 7!} \int_M [(-98 + 588\varepsilon - 5680n) \mid \nabla T \mid^2 + \\ +(392 - 1470\varepsilon^2 - 2480n) \mid \nabla\rho \mid^2 + (49 - 280n) \mid \nabla R \mid^2 + (245 - 1400n)T^3 + \\ +(-980 - 1470\varepsilon^2 + 1680n)T \mid \rho \mid^2 + (245 + 98\varepsilon - 1680n)T \mid R \mid^2 + \\ +(245 + 245\varepsilon - 1400n) \mid \rho \mid^3 + (392 + 800n)L_1 + \\ (98 - 1470\varepsilon^2 + 320n)L_2 + (147 - 960)L_3] dM \end{gathered} \tag{29}$$

4

Now we prove the theorem

Theorem 4.1 *Let (M, g) and (N, h) be two compact Riemannian manifolds with the properties $S_\rho(M, \Delta_k) = S_\rho(N, \Delta_k)$, $k = 0, 1, 2$ and $Sp(M, D_k^{\varepsilon}) = Sp(N, D_k^{\varepsilon})$ for three distinct values of $\varepsilon \neq 0$. If (M, g) is Ricci, so is (N, h).*

Proof. From the assumptions of the theorem we obtain

$$\alpha_k(M, \Delta_\nu) = \alpha_k(N, \Delta_\nu),\ \nu = 0, 1, 2 \quad k = 0, 1, 2, 3 \tag{30}$$

$$\alpha_k(D_1^{\varepsilon}, M) = \alpha_k(D_1^{\varepsilon}, N),\ k = 0, 1, 2, 3 \tag{31}$$

for three distinct values of $\varepsilon \neq 0$. From (30) for $k = 0, 1, 2$ and by means of (13), (14), (16), (17), (19) and (20) we get

$$\int_M T_M dM = \int_N T_N dN, \quad \int_M T_M^2 dM = \int_N T_N^2 dN \tag{32}$$

$$\int_M |\rho|_M^2 \, dM = \int_N |\rho|_N^2 \, dN, \quad \int_M |R|_M^2 \, dM = \int_N |R|_N^2 \, dN. \tag{33}$$

Since the Riemannian manifold M is locally Ricci we obtain

$$T_M = constant \tag{34}$$

The relations (32), by means of (34), imply

$$T_M = T_N = constant \tag{35}$$

which yields

$$\int_M |\nabla T|_M^3 \, dM = \int_N |\nabla T|_N^3 \, dN = 0 \tag{36}$$

The equalities (33) by virtue of (35) give

$$\int_M (T \mid R \mid^2)_M dM = \int_N (T \mid R \mid^2)_N dN, \quad \int_M (T \mid \rho \mid^2)_M dM = \int_N (T \mid \rho \mid^2)_N dN \tag{37}$$

From (30) for $k = 3$ for $\nu = 0, 1, 2$ and (31) for $k = 3$ which by means of (15), (18) and (21), taking under the consideration (36) and (37), we obtain the following relations

$$\int_M [26 \mid \nabla\rho \mid^2 + 7 \mid \nabla R \mid^2 - 35 \mid \rho \mid^3 + 20L_1 - 8L_2 + 24L_3] dM =$$

$$= \int_N [26 \mid \nabla\rho' \mid^2 + 7 \mid \nabla R' \mid^2 - 35 \mid \rho' \mid^3 + 20L_1' - 8L_2' + 24L_3'] dN \tag{38}$$

$$\int_M [\alpha_1(n) \mid \nabla\rho \mid^2 + \alpha_2(n) \mid \nabla R \mid^2 - \alpha_3(n) \mid \rho \mid^3 + \alpha_4(n)L_1 - \alpha_5(n)L_2 + \alpha_6(n)L_3] dM =$$

$$\begin{aligned} = \int_N [\alpha_1(n) \mid \nabla\rho' \mid^2 + \alpha_2(n) \mid \nabla R' \mid^2 - \alpha_3(n) \mid \rho' \mid^3 + \alpha_4(n)L_1' - \alpha_5(n)L_2' + \\ + \alpha_6(n)L_3'] dM \end{aligned} \tag{39}$$

$$\int_M [\beta_1(n) \mid \nabla\rho \mid^2 + \beta_2(n) \mid \nabla R \mid^2 - \beta_3(n) \mid \rho \mid^3 + \beta_4(n)L_1 - \beta_5(n)L_2 + \beta_6(n)L_3] dM =$$

$$=\int_N [\beta_1(n) \mid \nabla\rho' \mid^2 +\beta_2(n) \mid \nabla R' \mid^2 -\beta_3(n) \mid \rho' \mid^3 +\beta_4(n)L'_1 - \beta_5(n)L'_2+ \\ +\beta_6(n)L'_3]dM \tag{40}$$

$$\int_M [\gamma_1(n,\varepsilon) \mid \nabla\rho \mid^2 +\gamma_2(n,\varepsilon) \mid \nabla R \mid^2 -\gamma_3(n,\varepsilon) \mid \rho \mid^3 +\gamma_4(n,\varepsilon)L_1 - \gamma_5(n,\varepsilon)L_2+ \\ +\gamma_6(n,\varepsilon)L_3]dM = \\ =\int_{n,\varepsilon} [\gamma_1(n,\varepsilon) \mid \nabla\rho' \mid^2 +\gamma_2(n,\varepsilon) \mid \nabla R' \mid^2 -\gamma_3(n,\varepsilon) \mid \rho' \mid^3 +\gamma_4(n,\varepsilon)L'_1 - \gamma_5(n,\varepsilon)L'_2+ \\ +\gamma_6(n,\varepsilon)L'_3]dM \tag{41}$$

where without (') and with (')we mean quantities for M and N respectively and

$$\alpha_1(n) = -(5680n + 980),\ \alpha_2(n) = -104n + 1078,\ \alpha_3(n) = 280n + 49 \tag{42}$$

$$\alpha_4(n) = -(800n + 392),\ \alpha_5(n) = 320n - 1372,\ \alpha_6(n) = 360n + 49 \tag{43}$$

$$\beta_1(n) = -(52n^2 - 1026n + 8036,\ \beta_2(n) = -140n^2 + 149n - 1568 \tag{44}$$

$$\beta_3(n) = 720n^2 + 1112n - 28616,\ \beta_4(n) = -800n^2 + 1192n - 18421 \tag{45}$$

$$\beta_5(n) = 160n^2 - 1532n + 26246,\ \beta_6(n) = -480n^2 + 627n - 4708 \tag{46}$$

$$\gamma_1(n,\varepsilon) = 392 - 1470\varepsilon^2 - 2480n,\ \gamma_2(n,\varepsilon) = 49 - 290n \tag{47}$$

$$\gamma_3(n,\varepsilon) = 245 + 98\varepsilon - 1680n,\ \gamma_4(n,\varepsilon) = 382 + 80n \tag{48}$$

$$\gamma_5(n,\varepsilon) = 98 - 147\varepsilon + 320n,\ \gamma_6(n,\varepsilon) = 147 - 960n \tag{49}$$

The equations (38), (39), (40) and other three of the type (41) for three distinct values of $\varepsilon \neq 0$ form an homogenous linear system of six equations with six unknowns.

$$X_1 = \left(\int_M \mid \nabla\rho \mid^2 dM - \int_N \mid \nabla\rho' \mid^2 dN\right),\ X_2 = \left(\int_M \mid \nabla R \mid^2 dM - \int_N \mid \nabla R' \mid^2 dN\right)$$

$$X_3 = \left(\int_M \mid \rho \mid^3 dM - \int_N \mid \rho' \mid^3 dN\right),\ X_4 = \left(\int_M L_1 dM - \int_N L'_1 dN\right)$$

$$X_5 = \left(\int_M L_2 dM - \int_N L'_2 dN\right),\ X_6 = \left(\int_M L_3 dM - \int_N L'_3 dN\right)$$

If we choose the three distinct values of $\varepsilon \neq 0$, say ε_1, ε_2, ε_3, such that

$$\begin{vmatrix} 26 & 7 & -35 & 20 & -8 & 24 \\ \alpha_1(n) & \alpha_2(n) & \alpha_3(n) & \alpha_4(n) & \alpha_5(n) & \alpha_6(n) \\ \beta_1(n) & \beta_2(n) & \beta_3(n) & \beta_4(n) & \beta_5(n) & \beta_6(n) \\ \gamma_1(n,\varepsilon_1) & \gamma_2(n,\varepsilon_1) & \gamma_3(n,\varepsilon_1) & \gamma_4(n,\varepsilon_1) & \gamma_5(n,\varepsilon_1) & \gamma_6(n,\varepsilon_1) \\ \gamma_1(n,\varepsilon_2) & \gamma_2(n,\varepsilon_2) & \gamma_3(n,\varepsilon_2) & \gamma_4(n,\varepsilon_2) & \gamma_5(n,\varepsilon_2) & \gamma_6(n,\varepsilon_2) \\ \gamma_1(n,\varepsilon_3) & \gamma_2(n,\varepsilon_3) & \gamma_3(n,\varepsilon_3) & \gamma_4(n,\varepsilon_3) & \gamma_5(n,\varepsilon_3) & \gamma_6(n,\varepsilon_3) \end{vmatrix} \neq 0 \tag{50}$$

then the homogenous linear system has only the unique trivial solution

$$X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = 0 \tag{51}$$

From (51) we have

$$X_1 = \int_M \mid \nabla\rho \mid^2 dM - \int_N \mid \nabla\rho' \mid^2 dN = 0 \tag{52}$$

Since the manifold (M, g) is Ricci, that means

$$\nabla\rho = 0 \Rightarrow \nabla\rho' = 0 \tag{53}$$

which means the manifold (N, h) is Ricci. □

Now we prove the theorem

Theorem 4.2 *Let (M, g) and (N, h) be two compact Riemannian manifolds with the properties $Sp(M, \Delta_k) = Sp(N, \Delta_k)$ for $k = 0, 1$. If (M, g) is Ricci flat, so is (N, h).*

Proof. From the assumption we have

$$Sp(M, \Delta_0) = Sp(N, \Delta_0),\ Sp(M, \Delta_1) = Sp(N, \Delta_1) \tag{54}$$

which imply

$$\alpha_2(M, \Delta_0) = \alpha_2(N, \Delta_0),\ \alpha_2(M, \Delta_1) = \alpha_2(N, \Delta_1) \tag{55}$$

and

$$\dim M = \dim N = n \tag{56}$$

The equalities (55) yield

$$\int_M \left(5T^2 - 2 \mid \rho \mid^2 + 2 \mid R \mid^2\right) dM = \int_N \left(5T'^2 - 2 \mid \rho' \mid^2 + 2 \mid R' \mid^2\right) dN \tag{57}$$

$$\int_M [(5n-60)T^2 - (2n-180) \mid \rho \mid^2 + (2n-30) \mid R \mid^2]\, dM =$$

$$\int_N [(5n-60)T'^2 - (2n-180) \mid \rho' \mid^2 + (2n-30) \mid R' \mid^2]\, dN = \tag{58}$$

From (57) and (58) we conclude

$$\int_M \left[(10n+15)T^2 + 150 \mid \rho \mid^2\right] dM = \int_N \left[(10n+15)T'^2 + 150 \mid \rho' \mid^2\right] dN \tag{59}$$

Since the manifold (M, g) is Riicci flat we obtain

$$T = 0 \quad and \quad \rho = 0 \tag{60}$$

and therefore (59), by means of (60), implies

$$\int\limits_N \left[(10n+15)T'^2 + 150 \mid \rho' \mid^2\right] dN = 0 \tag{61}$$

which gives

$$T' = 0 \quad and \quad \rho' = 0$$

and hence (N, h) is Ricci flat. □

5

Let (M, J, g) be a compact Käler manifold of complex dimension n. Let (U, φ) be a chart on M with complex coordinates $(z^1, \ldots, z^n)$. Unless otherwise stated, Greek indices $\alpha, \beta, \gamma, \ldots$, run from 1 to n, while Latin capitals $A, B, C, \ldots$, run through $1, \ldots, n, \bar{1}, \ldots, \bar{n}$. We set

$$Z_\alpha = \partial/\partial z^\alpha, \quad Z_{\bar{\alpha}} = \bar{Z}_\alpha = \partial/\partial \bar{z}^\alpha.$$

If g is a Hermitian metric, then we have

$$g_{AB} = g(Z_A, Z_B).$$

Then the metric g in local coordinate system has the following components

$$g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \quad g_{\alpha\bar{\beta}} \neq 0, \quad g_{\bar{\alpha}\beta} \neq 0,$$

and therefore the metric ds^2 can be written

$$ds^2 = 2\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta \ .$$

It is known that necessary and sufficient conditions for g to be a Käler metric, are the following

$$\partial g_{\alpha\bar{\beta}}/\partial z^\gamma = \partial g_{\gamma\bar{\beta}}/\partial z^\alpha \quad or \quad \partial g_{\alpha\bar{\beta}}/\partial \bar{z}^\gamma = \partial g_{\alpha\bar{\gamma}}/\partial z^\beta \tag{62}$$

The components ρ_{AB} of the Ricci tensor field ρ are given by

$$\rho_{\alpha\bar{\beta}} = -\sum_\gamma \Gamma^\gamma_{\alpha\gamma}/\partial z^\beta, \ \rho_{\alpha\bar{\beta}} = \bar{\rho}_{\alpha\bar{\beta}} \ , \ \rho_{\alpha\beta} = \rho_{\bar{\alpha}\bar{\beta}} = 0 \tag{63}$$

where $\Gamma^\gamma_{\alpha\gamma}$ are the Chistoffel's symbols of the Levi-Civita connection defined by the metric tensor g.

To every Käler manifold (M, J, g) we can associate an exterior 2-form φ which can be defined as follows

$$\varphi = -2i\sum_{\alpha,\beta} \rho_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \tag{64}$$

which can be written with the form

$$\varphi = -2id\bar{d}\ln G \tag{65}$$

where G is the determinant of the matrix $\left(g_{\alpha\bar{\beta}}\right)$.

From the (64) we can obtain the theorem

Theorem 5.1 *Let (M, J, g) be a compact Käler manifold. (M, J, g) is Ricci (resp. flat Ricci), if and only if, the exterior 2-form φ is parallel (resp. zero).*

Now we can prove the following theorem

Theorem 5.2 *Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be two compact Käler manifolds with the property $Sp(M_1, \Delta_k) = Sp(M_2, \Delta_k)$, $k = 1, 2$. If the restricted linear holonomy group of M_1 is contained in $SU(n)$, then the same is true for the restricted holonomy group of M_2.*

Proof. From the property of the restricted holonomy group of M_1, which is contained in $SU(n)$, we conclude that the manifold M_1 is flat Ricci.

From the relations

$$Sp(M_1, \Delta_0) = Sp(M_1, \Delta_0) \quad \textit{and} \quad Sp(M_1, \Delta_1) = Sp(M_2, \Delta_1)$$

we obtain that M_2 is flat Ricci and therefore its restricted linear holonomy group of M_2 is contained in $SU(n)$. □

Let E be a complex vector bundle over the Käler compact manifold (M, J, g). For each integer $i \geq 0$, we have the i-th Chern class

$$c_1(E) \in H^1(M, IR).$$

We assume that the fibre of E is the $\mathbf{C}^\Gamma$ and the the structure group is the $GL(\Gamma, \mathbf{C})$. Let $\mathbf{P}$ be its associate principal fibre bundle.

We define first polinomial functions

$$f_0, f_1, \ldots, f_\Gamma$$

on the Lie algebra $gl(\Gamma, \mathbf{C})$ by the relation

$$\det\left(\lambda I_\Gamma - \frac{1}{2\pi\sqrt{-1}}X\right) = \sum_{k=0}^{r} f_k(X)\lambda^{\Gamma-k} \quad \textit{for} \quad X \in gl(\Gamma, \mathbf{C}).$$

These are invariant by $ad\left(gl(\Gamma, \mathbf{C})\right)$. Let w be a connection on $\mathbf{P}$ and Ω its curvature form. It is known that there exist a unique closed $2k$-form γ_k on M such that

$$p^*(\gamma_k) = f_k(\Omega)$$

where $p : \mathbf{P} \to M$ is the projection. The cohomology class determinated by γ_k is independent of the choice of the connection w.

Therefore the k-th Chern class $c_k(E)$ of the complex vector bundle E over M is represented by the closed $2k$-form γ_k defined above.

The first Chern class $c_1(E)$ can be represented by the closed 2-form

$$\gamma_1 = -\frac{1}{2\pi\sqrt{-1}} \sum_{i,j=1}^{n} \rho_{ij} dz^i \wedge d\bar{z}^j$$

If $\rho \gtreqless 0$, then $\gamma_1 \gtreqless 0$.

If ρ is parallel, then γ_1 is parallel and conversely.

Theorem 5.3 *Let (M, J) and (M', J') be two compact complex manifolds. We assume that the first Chern class of M is zero. If there are two Käler metrics g and g' on M and M' respectively with the properties $Sp(M, \Delta_k) = Sp(M', \Delta_k)$, $k = 0, 1$, then (M', J') has its first Chern Class equal to zero.*

Proof. It is known that the first Chern class γ_1 of M is given by

$$\gamma_1 = -\frac{1}{2\pi\sqrt{-1}} \sum_{i,j=1}^{n} \rho_{ij} dz^i \wedge d\bar{z}^j \tag{66}$$

Since $\gamma_1 = 0$, we conclude that the Ricci tensor field $\rho = 0$. From the assumption and theorem 4.2 we conclude that the Ricci tensor field $\rho' = 0$, which by means of (66), we obtain $\gamma_1' = 0$, which is the first Chern class of M'. □

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