THE HIGHER-ORDER LAGRANGE SPACES: THEORY OF SUBSPACES

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Abstract

The notion of Lagrange space of order k, k$\in N^*$, was recently introduced by author together with Gh. Atanasiu in some papers [5]. It was studied as a natural extension of that of Lagrange space expounded in the books [2,3].

In the present lecture at the prof. Gr. Tsagas wokshop, from the "Aristotel University of Thessaloniki", I should like to give a shost survey of this interesting geometrical theory, as well as introduction in the study of subspace of these spaces, important in applications.

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1 The Lagrange spaces $L^{(k)n}$

A Lagrange space of order k is a pair $L^{(k)n} = (M, L)$, where:
a. M is a C^∞-real, n-dimensional manifold;
b. $L : Osc^k M \longrightarrow R$ is a differentiable Lagrangian of order k;
c. The d-tensor field

$$g_{ij}(x, y^{(1)}, ..., y^{(k)}) = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}} \tag{1}$$

satisfies the following conditions

$$\text{rank } (\text{g}_{\text{ij}}) = \text{n} \quad (1.1)'$$

and the quadratic form

$$g_{ij}\xi^i\xi^j \tag{2}$$

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has constant signature on $Osc^k M$.

Of course, the indices i,j,h,...run one the set $\{1,\ldots,n\}$ and $(x^i, y^{(1)i}, \ldots, y^{(k)i})$ are the canonical coordinates of a generic point $u = (x, y^{(1)}, \ldots, y^{(k)})$ from the total space of the osculator bundleof order k, $(Osc^k M, \pi, M)$.

Over the paracompact manifold M there exist the Lagrange spaces of order k.

Let us consider a smooth curve $c : [0,1] \longrightarrow M$ and the integral of action $I(c)$ of the Lagrangean $L(x, y^{(1)}, ..., y^{(k)})$ of a space $L^{(k)n}$. Then the varational problem for the functional I(c) leads to the Euler-Lagrange equation

$$\begin{cases} E_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt}(\frac{\partial L}{\partial y^{(1)i}}) + ... + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}(\frac{\partial L}{\partial y^{(k)i}}) = 0 \\ y^{(1)i} = \frac{dx^i}{dt}, ..., y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{dt^k} \end{cases} \tag{3}$$

Of course, $E_i(L)$ is a d-covector field.

Let E_c^k be the energy of order k of $L^{(k)n}$,[]. Thus we have the formula

$$\frac{dE_c^k(L)}{dt} = -E_i(L)\frac{dx^i}{dt} \tag{4}$$

Therefore we can afirme:*the energy of order k is conserved along of the soloution curves of the Euler-Lagrange equation.* This results belongs to some scientists as: M. de Leon, D. Krupka et al.[1].

A theory of Neother symetries can be find in the paper [4].

The following two important theorems we given by Miron-Atanasiu [1,5]:

In a Lagrange space $L^{(k)n} = (M, L)$ there exists a k-spray S on $Osc^k M$, depending on the fundamental function L, only. It is given by

$$S = y^{(1)i}\frac{\partial L}{\partial x^i} + \cdots + ky^{(k)i}\frac{\partial L}{\partial y^{(k-1)i}} - (k+1)G^i\frac{\partial L}{\partial y^{(k)i}} \tag{5}$$

where G^i are the coefficients:

$$(k+1)G^i = \frac{1}{2}g^{ij}\{\Gamma(\frac{\partial L}{\partial y^{(k)i}}) - \frac{\partial L}{\partial y^{(k-1)i}}\}$$

And the second theorem:

In a Lagrange space $L^{(k)n} = (M, L)$ the canonical nonlinear connection N, has the dual coefficients

$$\underset{(1)}{M^i_j} = \frac{\partial G^i}{\partial y^{(k)i}}, \underset{(a)}{M^i_j} = \frac{1}{a}\{S \underset{(a-1)}{M^i_j} + \underset{(1)}{M^i_s}\underset{(a-1)}{M^s_j}\}, (a = 2...k) \tag{6}$$

Consequently the canonical nonlinear connection $\underset{0}{N} = N$, determines the $\mathcal{J}$-vertical distribution $\underset{(1)}{N}, \cdots, \underset{(k-1)}{N}$ such that the following direct sum of linear spaces holds:

$$T_u(Osc^k M) = \underset{(0)}{N}(u) \oplus \underset{(1)}{N}(u) \oplus \cdots \oplus \underset{(k-1)}{N}(u) \oplus \underset{(k)}{V}(u), \ \forall u \in Osc^k M \tag{7}$$

The local adapted basis to this direct decomposition is

$$\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \cdots, \frac{\delta}{\delta y^{(k)i}}\} \quad (8)$$

and its dual is

$$\{\delta x^i, \delta y^{(1)i}, \cdots, \delta y^{(k)i}\} \quad (1.8)'$$

where

$$\begin{cases} \delta x^i = dx^i, \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M^i_j}\, dx^j, \cdots, \\ \delta y^{(k)i} = dy^{(k)i} + \underset{(1)}{M^i_j}\, dy^{(k-1)j} + \cdots + \underset{(k)}{M^i_j}\, dx^j \end{cases} \quad (9)$$

It is remarkable that the autoparallel curves of the canonical connection $\underset{0}{N}$ are given by

$\frac{\delta y^{(1)i}}{dt} = \cdots = \frac{\delta y^{(k)i}}{dt} = 0$, where

$y^{(1)i} = \frac{dx^i}{dt}, \cdots, y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{dt^k}$

About the notion of N-linear connection we have an important result:

In the Lagrange space $L^{(k)n}$=(M,L) there exists an unique canonical metrical N-connection, whose coefficients are given by the generalised Christoffel symbols

$$\begin{cases} L^m_{ij} = \frac{1}{2} g^{ms}(\frac{\delta g_{is}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^s}), \\ \underset{(a)}{C^m_{ij}} = \frac{1}{2} g^{ms}(\frac{\delta g_{is}}{\delta y^{(a)}j} + \frac{\delta g_{sj}}{\delta y^{(a)}i} - \frac{\delta g_{ij}}{\delta y(a)^s}), (a = 1, \cdots, k) \end{cases} \quad (10)$$

Of course the canonical metrical N-linear connection with the coefficients $C\Gamma(N) = (L^i_{jh}, \underset{(1)}{C^i_{jh}}, \cdots, \underset{(k)}{C^i_{jh}})$, has the following properties:

a. It is metrical:

$g_{ij|h} = 0, g_{ij} \overset{(1)}{|}_h = \cdots = g_{ij} \overset{(k)}{|}_h = 0$

b. $L^i_{jh} = L^i_{hj}, \underset{(a)}{C^i_{jh}} = \underset{(a)}{C^i_{hj}}, (a = 1, \cdots, k)$

c. Its covariant d-tensor of curvature $R_{ijhm}, \underset{(a)}{P_{ijhm}}, \underset{(ab)}{S_{ijhm}}$ are skewsymmetric in the first two indices i and j.

Let ω^i_j be the 1-forms connection of $C\Gamma(N)$:

$$\omega^i_j = L^i_{jh} dx^h + \underset{(1)}{C^i_{jh}}\, \delta y^{(1)h} + \cdots + \underset{(k)}{C^i_{jh}}\, \delta y^{(k)h} \quad (11)$$

We can prove:

Theorem 1 *The structure equations of the canonical metrical N-connection $C\Gamma(N)$ of the Lagrange space of order k, $L^{(k)n}$, are given by*

$$\begin{cases} d(dx^i) - dx^m \wedge \omega^i_m = -\overset{(0)}{\Omega^i} \\ d(\delta y^{(a)i}) - \delta y^{(a)m} \wedge \omega^i_m = -\overset{(a)}{\Omega^i}\ (a = 1, \cdots, k) \\ d\omega^i_j - \omega^m_j \wedge \omega^i_m = -\Omega^i_j \end{cases} \quad (12)$$

where $\overset{(0)}{\Omega}{}^i, \overset{(a)}{\Omega}{}^i$ *are the 2-forms of torsion and* Ω^i_j *is 2-form of curvature:*

$$\Omega^i_j = \frac{1}{2}R_j{}^i{}_{pq}dx^p \wedge dx^q + \sum_{b=1}^{k} \underset{(b)}{P}{}_j{}^i{}_{pq} dx^p \wedge \delta y^{(b)a} + \sum_{a,b=1}^{k} \underset{(ab)}{S}{}_j{}^i{}_{pq} \delta y^{(a)p} \wedge \delta y^{(b)q}. \quad (13)$$

If we put

$$\Omega_{ij} = g_{ip}\Omega^p_j \quad (14)$$

we can see that $\Omega_{ij} + \Omega_{ji} = 0$.

The Bianchi identities of $C\Gamma(N)$ can be obtained from (1.12) applying the operator of exterior differentiation.

2 The Riemannian (k-1)n contact model of the space $L^{(k)n}$

Let N be the canonical nonlinear connection of the Lagrange space of order k ,$L^{(k)n}$.It allows to determine the adapted cobasis (1.8)' and to find the N-lift of the fundamental g_{ij}.This is

$$G = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^{(1)i} \otimes \delta y^{(1)j} + \cdots + g_{ij}\delta y^{(k)i} \otimes \delta y^{(k)j} \quad (1)$$

It follows, that the distributions$N_0, N_1, \cdots, N_{k-1}, V_k$ are ortogonal two by two.We can prove:

Theorem 2 *With respect to the canonical metrical N-connection we have*

$$D_X G = 0 \quad \forall X \in \chi(Osc^k M) \quad (2)$$

Of course G is a pseudo-Riemannian structure on the manifold $Osc^k M$.

Let us consider the almost (k-1)n-contact structure determined by N: $F : \chi(Osc^k M) \longrightarrow \chi(Osc^k M)$ defined by

$$F(\frac{\delta}{\delta x^i}) = -\frac{\partial}{\partial y^{(k)i}}, F(\frac{\delta}{\delta y^{(a)i}}) = 0, \ (a = 1, ..., k-1),$$
$$F(\frac{\partial}{\partial y^{(k)i}}) = \frac{\delta}{\delta x^i}, \ (i = 1, ..., n). \quad (3)$$

If we take a local basis $\underset{(1)}{\xi_i}$ *in* $\underset{(1)}{N}, ..., \underset{(k-1)}{\xi_i}$ *in* $\underset{(k-1)}{N}$ and the coresponding cobasis $\overset{(1)}{\eta}{}^i, ..., \overset{(k-1)}{\eta}{}^i$ *,we obtain* : $F(\underset{(a)}{\xi_i}) = 0, \overset{(a)}{\eta}{}^i (\underset{(b)}{\xi_j}) = \delta^a_b \delta^i_j$ (a,b=1,...,k-1)

$$F(X) = -X + \sum_{i=1}^{n}\sum_{a=1}^{k-1} \overset{(a)}{\eta}{}^i (X) \underset{(a)}{\xi_i}, \ \forall X \in \chi(Osc^k M)$$
$$F^3 + F = 0 \quad (4)$$

We have the following theorem:

Theorem 3 *1. The set* $\{F, \underset{(1)}{\xi_i}, ..., \underset{(k-1)}{\xi_i}, \overset{(1)}{\eta^i}, ..., \overset{(k-1)}{\eta^i}, G)$ *is a pseudo-Riemannian almost (k-1)n contact structure on* $Osc^k M$*, (with* $y^{(1)} \neq 0$*), determined by the fundamental function L of the Lagrange space* $L^{(k)n}$*=(M,L).*

2. The canonical metrical N-connection D of the space $L^{(k)n}$ *is compatible with this structure, i.e.*

$$DF = 0, \quad DG = 0$$

3. The $\mathcal{J}$*-vertical distributions* $\underset{(1)}{N}, ..., \underset{(k-1)}{N}$ *are parallel with respect to D.*

The manifold $Osc^k M$ (with $y^{(1)} \neq 0$),endowed with the previous structure defin the pseudo-Riemannian almost (k-1)n contact model of the space $L^{(k)n}$.

Consequently, the geometry of the Lagrange space of order k, $L^{(k)n}$ can be studied by means of the above mentioned model.

3 Subspace in a Lagrange space of order k

We will study the general principle of the geometry of subspace in a Lagrange space of order k, $L^{(k)n}$=(M,L), determining the main induced geometrical object fields, as connections etc. This theory is a natural extension of that of subspaces in a usualy Lagrange space when k=1.

Let $\tilde{M}$ be a real m-dimensional manifold, (1<m<n) immmersed in the manifold M, through the immersion $i : \tilde{M} \longrightarrow M$. Locally i can be given in the form

$$x^i = x^i(u^1, ..., u^m), \quad rank\ (\frac{\partial x^i}{\partial u^\alpha}) = m \tag{1}$$

The indices $\alpha, \beta, \gamma, ...$ run over the set 1,...,m.

If i is an embedding, then we identifies $\tilde{M}$ to $i(\tilde{M})$ and say that $\tilde{M}$ is a submanifold of the manifold M.

The embeding $i : \tilde{M} \longrightarrow M$ determine an immersion

$$Osc^k i : Osc^k \tilde{M} \longrightarrow Osc^k M$$

given by

$$\begin{cases} x^i = x^i(u^1, ..., u^m), \quad rank\ (\frac{\partial x^i}{\partial u^\alpha}) = m \\ y^{(1)i} = \frac{\partial x^i}{\partial u^\alpha} v^{(1)\alpha} \\ \cdots \\ k y^{(k)i} = \frac{\partial y^{(k-1)i}}{\partial u^\alpha} v^{(1)\alpha} + ... + \frac{\partial y^{(k-1)i}}{\partial u^{(k-1)\alpha}} v^{(k)\alpha} \end{cases} \tag{2}$$

Now, let us consider a Lagrange space of order k, $L^{(k)n} = (M, L)$ having g_{ij} in (1.1) as fundamental tensor field. The restriction $\tilde{L}$ of the Lagrangian L to the manifold $Osc^k \tilde{M}$ is as follows

$$\tilde{L}(u, v^1, ..., v^{(k)}) = L(x(u), y^{(1)}(u, v^{(1)}), ..., y^{(k)}(u, v^{(1)}, .., v^{(k)}) \tag{3}$$

Theorem 4 *The pair $\tilde{L}^{(k)m} = (\tilde{M}, \tilde{L})$ is a Lagrange spaces of order k.*

The fundamental tensor field of this spsace is

$$\tilde{g}_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 \tilde{L}}{\partial v^{(k)\alpha}\partial v^{(k)\beta}} \tag{4}$$

The d-vector fields

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}(\alpha = 1, ..., m) \tag{5}$$

are independent. Therfore we can determine a frame

$$R = \{w; B^i_\alpha, B^i_{\overline{\alpha}}\} w \in Osc^k \tilde{M}$$

$(\alpha, \beta, \gamma, ... = 1, ..., m; \overline{\alpha}, \overline{\beta}, \overline{\gamma}, ... = 1, ..., n-m)$, where $B^i - \alpha$ are given by

$$g_{ij} B^i_\alpha B^j_{\overline{\alpha}} = 0, \quad g_{ij} B^i_{\overline{\alpha}} B^i_{\overline{\beta}} = \delta_{\overline{\alpha}\overline{\beta}} \tag{6}$$

We denote by $R^* = \{w; B^\alpha_i, B^{\overline{\alpha}}_i\}$ the dual of the frame R. So we have

$$\tilde{g}_{\alpha\beta} B^\alpha_i = g_{ij} B^j_\beta, \quad \delta_{\alpha\beta} B^\beta_i = g_{ij} B^i_\alpha \tag{7}$$

Consequently $\tilde{g}_{\alpha\beta}$ and g_{ij} can be represented in R in the form:

$$\tilde{g}_{\alpha\beta} = B^i_\alpha B^j_\beta g_{ij}, \; g_{ij} = \tilde{g}_{\alpha\beta} B^\alpha_i B^\beta_j + \delta_{\overline{\alpha}\overline{\beta}} B^{\overline{\alpha}}_i B^{\overline{\beta}}_j \quad (3.7)'$$

Definition 3.1 A nonlinear connection $\tilde{N}$ in $\tilde{L}^{(k)m}$ is called *induced* by the canonical nonlinear connection if we have:

$$\delta v^{(1)\alpha} = B^\alpha_i \delta y^{(1)i}, ..., \delta v^{(k)\alpha} = B^\alpha_i \delta y^{(k)i} \tag{8}$$

These conditions uniquely determine an induced nonlinear connection on $Osc^k \tilde{M}$.

The adapted cobasis $\{dx^i, \delta y^{(1)i}, ..., \delta y^{(k)i}\}$ is uniquely represented in the frame R in the form:

$$\begin{cases} dx^i = B^i_\alpha du^\alpha \\ \delta y^{(1)i} = B^i_\alpha \delta v^{(1)\alpha} + B^i_{\overline{\alpha}} \underset{(1)}{K^{\overline{\alpha}}_\beta} du^\beta \\ \dots \\ \delta y^{(k)i} = B^i_\alpha \delta v^{(k)\alpha} + B^i_{\overline{\alpha}} \{ \underset{(1)}{K^{\overline{\alpha}}_\beta} \delta v^{(k-1)\beta} + \cdots + \underset{(k)}{K^{\overline{\alpha}}_\beta} du^\beta \} \end{cases} \tag{9}$$

Now we shal construct the components of an operator ∇ of relative covariant differentiation in the algebra of mixed d-tensor fields on $Osc^k \tilde{M}$.

We call a cupling the canonical metrical N-connection $C\Gamma(N)$ of $L^{(k)n}$ to the induced nonlinear connection $\tilde{N}$, an operator $\tilde{D}$ with the property

$$\tilde{D} X^i = D X^i \quad modulo (3.9) \tag{10}$$

consequently, we get

$$\tilde{D}X^i = dX^i + X^j\tilde{\omega}^i_j \quad (3.10)'$$

where

$$\tilde{\omega}^i_j = \tilde{L}^i_{j\alpha}du^\alpha + \sum_{(\alpha=1)}^{(k)} \underset{(a)}{\tilde{C}^i_{j\alpha}}\, \delta v^{(a)\alpha} \quad (3.10)''$$

After this, we define the induced tangent connection on $Osc^k\tilde{M}$ by the N-connection $C\Gamma(N)$, is determined by the operator D^T, given as follows:

$$D^T X^\alpha = B^\alpha_i \tilde{D}X^i, \ for\ X^i = B^i_\alpha X^\alpha \tag{11}$$

Then, we have

$$D^T X^\alpha = dX^\alpha + X^\beta\omega^\alpha_\beta \quad (3.11)'$$

where

$$\omega^\alpha_\beta = L^\alpha_{\beta\gamma}du^\gamma + \sum_{a=1}^{(k)} \underset{(a)}{C^\alpha_{\beta\gamma}}\, \delta v^{(a)\gamma} \quad (3.11)''$$

Finally, the induced normal connection by $C\Gamma(N)$ is given by the operator $D^\perp$ defined by

$$D^\perp X^{\overline{\alpha}} = B^{\overline{\alpha}}_i \tilde{D}X^i, \quad forX^i = B^i_{\overline{\alpha}} X^{\overline{\alpha}} \tag{12}$$

One deduces:

$$D^\perp X^{\overline{\alpha}} = dX^{\overline{\alpha}} + X^{\overline{\beta}}\omega^{\overline{\alpha}}_{\overline{\beta}} \quad (3.12)'$$

where

$$\omega^{\overline{\alpha}}_{\overline{\beta}} = L^{\overline{\alpha}}_{\overline{\beta}\gamma}du^\gamma + \sum_{a=1}^{k} \underset{(a)}{C^{\overline{\alpha}}_{\overline{\beta}\gamma}}\, \delta v^{(a)\gamma} \quad (3.12)''$$

All coefficients, from $\tilde{\omega}^i_j, \omega^\alpha_\beta$ and $\omega^{\overline{\alpha}}_{\overline{\beta}}$ are well determined.

So, a relative covariant differetiation ∇ in the algebra of mixed d-tensor fields is defined by its components $\tilde{D}, D^T$ and $D^\perp$, as follows:

$$\nabla f = df, \nabla X^i = \tilde{D}X^i, \nabla X^\alpha = D^\top X^\alpha, \nabla X^{\overline{\alpha}} = D^\top X^{\overline{\alpha}} \tag{13}$$

For instance, B^i_α is a mixed d-tensor. We get

$$\nabla B^i_\alpha = dB^i_\alpha + B^j_\alpha\tilde{\omega}^i_j - B^i_\beta\omega^\beta_\alpha \tag{14}$$

and for mixed d-tensor $B^i_{\overline{\alpha}}$

$$\nabla B^i_{\overline{\alpha}} = dB^i_{\overline{\alpha}} + B^j_{\overline{\alpha}}\tilde{\omega}^i_j - B^i_{\overline{\beta}}\omega^{\overline{\beta}}_{\overline{\alpha}} \quad (3.14)'$$

Therfore we can determine the Gauss-Weingarten formulae. They are given by

$$\nabla B^i_\alpha = B^i_{\overline{\beta}}\pi^{\overline{\beta}}_\alpha, \quad \nabla B^i_{\overline{\alpha}} = -B^i_\beta\pi^\beta_{\overline{\alpha}} \tag{15}$$

where $\pi^\beta_{\overline{\alpha}} = g^{\beta\gamma}\delta_{\overline{\alpha}\overline{\beta}}\pi^{\overline{\beta}}_\gamma$, and $\pi^{\overline{\beta}}_\alpha$ is well expressed by means of $\tilde{\omega}^i_j, \omega^\alpha_\beta$ and $\omega^{\overline{\alpha}}_{\overline{\beta}}$.

The conditions of integrability of the equations (3.15) leads to the Gauss-Codazzi equations of ∇

Theorem 5 *The Gauss-Codazzi equations of the Lagrange subspaces* $\tilde{L}^{(k)m}$ *in the Lagrange space of order k,* $L^{(k)n}$ *are as follows:*

$$\begin{aligned} & B^i_\alpha B^j_\beta \tilde{\Omega}_{ij} - \Omega_{\alpha\beta} = \Pi_{\beta\overline{\gamma}} \wedge \Pi^{\overline{\gamma}}_\alpha \\ & B^i_{\overline{\alpha}} B^j_{\overline{\beta}} \tilde{\Omega}_{ij} - \Omega_{\overline{\alpha}\overline{\beta}} = \Pi_{\gamma\overline{\beta}} \wedge \Pi^\gamma_{\overline{\alpha}} \\ & -B^i_\alpha B^j_{\overline{\beta}} \tilde{\Omega}_{ij} = \delta_{\overline{\beta}\overline{\gamma}}(d\Pi^{\overline{\gamma}}_\alpha + \Pi^{\overline{\gamma}}_\varphi \wedge \omega^\varphi_\alpha - \Pi^{\overline{\varphi}}_\alpha \wedge \omega^{\overline{\gamma}}_{\overline{\varphi}}) \end{aligned} \tag{16}$$

where $\Pi_{\alpha\overline{\beta}} = g_{\alpha\gamma}\Pi^\gamma_{\overline{\beta}}$ *and d is the operator of exterior differential.*

Remark It is important the particular case m=n-1 of the hiper subspaces $\tilde{L}^{(k)n-1}$ in the Lagrange space of order k, $L^{(k)n}$.

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