A NOTE ON HARMONIC MAPS OF SEMI-RIEMANNIAN MANIFOLDS

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Abstract

In this note we consider a harmonic map f between two semi- Riemannian manifolds and prove some conservation laws when a one-parameter group of diffeomorphisms preserves the energy density of f.

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In classical mechanics, a lot of conservation laws are particular cases of Nother's theorem: *For every one-parameter group of diffeomorphisms on the configuration space of a Lagrangean system, which preserves Langrange function, corresponds a prime integral of Euler - Lagrange equation.*
For some recent results about Nother's theory to see for example, the papers [1][2][3][5] and their references.
Let (M,g) , (N,h) be smooth semi-Riemannian manifolds (without boundary) of any dimensions and let $f : M \to N$ be a smooth map from M to N. We define *the energy density function* $e(f)$ of f by the formula

$$e(f)(x) = \frac{1}{2} Trace_g(f^*h)(x) \quad , \quad x \in M \tag{1}$$

Here $Trace_g(f^*h)$ is the trace of the tensor field f^*h by g , i.e.,

$$e(f)(x) = \frac{1}{2} \sum_{i=1}^{m} h(f_*e_i, f_*e_i) \quad , \quad x \in M \tag{2}$$

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where $(e_i)_{1\leq i\leq m}$ is an orthonormal basis of the tangent space T_xM with respect to g. It is easy to see that $e(f)$ is a smooth function on M.
Taking local coordinates $(x_1,\ldots,x_m)$, $(y_1,\ldots,y_n)$ on neighborhoods U of x and $\tilde{U}$ of $f(x)$ in M,N respectively, and putting $f^a := y_a\circ f$, $a=1,\ldots,n$, then (2) can be expressed as

$$e(f)(x)=\frac{1}{2}\sum_{i,j,a,b} g^{ij}(x)h_{ab}(f(x))\frac{\partial f^a}{\partial x^i}(x)\frac{\partial f^b}{\partial x^j}(x) \tag{3}$$

where (g^{ij}) is the inverse matrix of (g_{ij}), $g_{ij}=g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)$ and $h_{ab}=h\left(\frac{\partial}{\partial y_a},\frac{\partial}{\partial y_b}\right)$
Now we suppose that M is a compact manifold.
Then, the integral

$$E(f):=\int_M e(f)dv_g \tag{4}$$

where dv_g is the volume element of (M,g) is called the *energy* or the *action integral* of f.
We say $f\in C^\infty(M,N)$ is a *harmonic map* if f is a critical point of E at $C^\infty(M,N)$,i.e., for any smooth variation $f_t\in C^\infty(M,N)$ with $-\varepsilon<t<\varepsilon$ of f, and $f=f_0$ we have

$$\frac{d}{dt}\Big|_{t=0}E(f_t)=0$$

It is well known that f is a critical point of E iff f satisfies the Euler-Langrange equation:

$$\sum_{i=1}^m\left(\frac{\partial e(f)}{\partial f^a_{,i}}\right)_{,i} = \frac{\partial e(f)}{\partial f^a} \quad , \quad a=1,\ldots,n \tag{5}$$

where $f^a_{,i}=\frac{\partial f^a}{\partial x_i}$.
Let $\{\sigma_t \mid t\in R\}$ be a one - parameter group of diffeomorphisms of the manifold M. We suppose that

$$e(f\circ\sigma_t)=e(f),\forall t\in R \tag{6}$$

Let $\varphi : M \text{ x } R\to N$ be a smooth map such that $\varphi(x,t)=f(\sigma_t(x))$. If $e(f\circ\sigma_t)$ is independent of t, then we have

$$\frac{\partial e}{\partial\varphi^a}\cdot\frac{\partial\varphi^a}{\partial t}+\frac{\partial e}{\partial(\varphi^a_{,k})}\cdot\frac{\partial(\varphi^a_{,k})}{\partial t}=\frac{\partial e}{\partial t}=0 \tag{7}$$

On the order hand we have

$$\frac{\partial(\varphi^a_{,k})}{\partial t}=\left(\frac{\partial\varphi^a}{\partial t}\right)_{,k} \tag{8}$$

Using (5) and (8), from (7) we obtain

$$\left(\frac{\partial e}{\partial(\varphi^a_{,k})}\right)_{,k} \cdot \frac{\partial \varphi^a}{\partial t} + \frac{\partial e}{\partial(\varphi^a_{,t})} \cdot \left(\frac{\partial \varphi^a}{\partial t}\right)_{,k} = 0 \tag{9}$$

and equivalently

$$\left(\frac{\partial e}{\partial(\varphi^a_{,k})} \cdot \frac{\partial \varphi^a}{\partial t}\right)_{,k} = 0 \tag{10}$$

For $t = 0$ we obtain

$$\left(\frac{\partial e}{\partial(f^a_{,k})} \cdot V(f^a)\right)_{,k} = 0 \tag{11}$$

Now, we denote ξ a vector field on M by

$$V_x(F) = \frac{dF(\sigma_t(x))}{dt}\,|_{t=0}, \quad x \in M, F \in C^\infty(U, R) \tag{12}$$

This vector field V is called the infinitesimal transformation of the one -parameter group of diffeomorphisms $\{\sigma_t/t \in R\}$.
By a straightforward calculation, we have:

$$\frac{\partial e}{\partial(f^a_{,k})} = \frac{1}{2}\left(g^{kj} f^b_{,j} h_{ab} + g^{ik} f^b_{,i} h_{ba}\right) = \sum g^{kj} f^b_{,j} h_{ab} \tag{13}$$

Let ξ be the vector field given, locally, by

$$\xi^k = \sum_{j,b,c} g^{kj} f^b_{,j} h_{bc} \cdot V(f^c) \tag{14}$$

We obtain the following theorem:

Theorem 1 *Let $f : (M, g) \to (N, h)$ be a harmonic map betwen two semi Riemannian manifolds and let $\{(\sigma_t)/t \in R\}$ an one-parameter group of diffeomorphism on M, such that $e(f \circ \sigma_t) = e(f), \forall t \in R$. Then we have the following conservation law:*

$$div(\xi) = 0 \tag{15}$$

where ξ is a vector field on given locally by (14) .

Taking N to be R with its standard metric, a harmonic map f from (M, g) to R is a harmonic function.
Suppose that exist an 1 - parameter group $G = \{\sigma_t/t \in R\}$ of difeomorfisms on M which preserves energy density of f, i.e.

$$e(f \circ \sigma_t) = e(f), \quad , \quad \forall \quad t \in R \tag{16}$$

Let V be the vectorial field induced by G.
In this case we can prove the following theorem:

Theorem 2 *If ξ is a vector field on (M, g) given by*

$$\xi = V(f).gradf \tag{17}$$

we have

$$div\xi = 0 \tag{18}$$

Let V a complete Killing vector field on a semi-Riemann manifold (M, g). Then the 1-parameter group of difeomorphism G generated by V consists of global isometries, i.e.,

$$\sigma_t^*(g) = g, \quad \forall \quad t \in R.$$

Proposition 3 *Let $f : (M, g) \to (N, h)$ be an isometric immersion, that is, $g = f^*h$. If V is a complete Killing vector field on (M, g), we have:*

$$e(f \circ \sigma_t) = e(f) = const, \quad \forall \quad s \in R \tag{19}$$

where $G = \{\sigma_t / t \in R\}$ is the 1 - parameter group defined by V.

Proof. From $\sigma_t^*(g) = g$ and $f^*h = g$ we obtain $(f \circ \sigma_t)^*(h) = f^*(h)$. Consequently,

$$Trace_g(f \circ \sigma_t)^*(h) = Trace_g f^*(h) \tag{20}$$

Therefore, we have

$$e(f \circ \sigma_t) = e(f) = n \tag{21}$$

where n = dim M.

Proposition 4 *Let $f : (M, g) \to (N, h)$ be harmonic isometric immersion and V a complete Killing vector field on M. If $G = \{\sigma_t / t \in R\}$ is the 1 -parameter group defined by V then, $\sigma_t : (M, g) \to (M, g)$ is a harmonic diffeomorfism for any $t \in R$.*

Proof. We have the well known formula

$$\tau(f \circ \sigma_t) = dt\tau(\sigma_t) + Trace\nabla df(d\sigma_t, d\sigma_t) \tag{22}$$

where $\tau(f)$ denote the tension field defined by f. If f is harmonic map we have $\tau(f) = 0$.
Since f is an isometric immersion we obtain from (22) that tangent component of $\tau(f \circ \sigma_t)$ is zero, so that $df\tau(\sigma_t) = 0$. But df is one-to-one, so that $\tau(\sigma_t) = 0$, and σ_t

is harmonic for any $t \in R$.
Finally, let $f : R \to N$ be a geodesic of (N, h) and let $G = \{\sigma_t \mid t \in R$, be a one-parameter group of diffeomorphisms on N such that $e(\sigma_t \circ f) = e(f), \quad \forall \quad t \in R$. Then we can prove that the angle between the geodesic f(t) and the integral paths of the vector field V induced of G must be constant.

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