GENERALIZED GAUGE ASANOV EQUATIONS ON $\mathrm{Osc}^{(2)}(\mathrm{M})$ BUNDLE

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Abstract

The paper introduces the notions of gauge transformations and gauge derivatives, and gives the detailed form of the generalized Einstein Yang-Mills equations for the osculator bundle $Osc^{(2)}(M)$ of a differentiable manifold M ([7],[8]).

AMS Subject Classification: 53C60, 53C80, 81T13, 53C07
Key words: gauge transformations, osculator bundle, Einstein-Yang Mills equations.

1 Generalized gauge transformations of second order

For a certain state of a mechanical system, many physical theories are concerned with determining the evolution of the given state. This evolution is usually gouverned by Euler-Lagrange equations and has to be invariant with respect to coordinate changes. In gauge theories, is also required the invariance under the so-called gauge transformations. In classical gauge theory the physical system is represented by sections in an associate bundle, and has to be invariant with respect to these gauge changes.

Let M be a differentiable manifold, $\dim M = n, (U, \phi)$ a local map at $x \in U$ and $x = (x^i)$ the local coordinates. If (U', ϕ') is another local map at x, then the expression of $\phi'\phi^{-1} :{}^n\to{}^n$ is

$$x'^i = x'^i(x) \tag{1}$$

Let E_2=$\mathrm{Osc}^{(2)}M$ be the osculator bundle of order 2 ([7],[8]), which admits a (locally trivial) fiber bundle structure, locally isomorphical with the 2-jet bundle of M. The total space is characterized in a local map $((\pi^{2^{-1}})(U), h^2)$ at $E_{2,z}$ by the coordinates $z = (x^i, y^i, y^i_*), \pi^2(z) = x$, with the change-rules

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1995,* 21-31

$$\begin{cases} x'^i &= x'^i(x), \\ y'^i &= \frac{\partial x'^i}{\partial x^j} y^j \\ 2y'^i_* &= \frac{\partial y'^i_*}{\partial x^j} y^j + 2\frac{\partial y'^i}{\partial y^j} y^j_* \end{cases} \quad (2)$$

In this case, $\pi^2 : E_2 \to M$ is considered here as bundle of base M, but one can consider also the structure $\pi^2_1 : E_2 \to E_1$, where E_1 is the tangent bundle TM, $\pi^1 : E_1 \to M$ with $\pi^2 = \pi^2_1 \circ \pi^1$.

In classical gauge theories, the gauge transformations are automorphisms of the associated bundle of a principal bundle of Lie group G, which induce the identity on the base manifold. The osculating bundle $Osc^{(2)}(M)$ (denoted hereafter E_2) is an associated bundle of the principal bundle of frames of second order P_2 ([8]). G.S.Asanov ([1]) considers a set of generalized gauge transformations; this concept will be developed in the present approach for E_2.

Definition 1.1. A *gauge transformation* on E_2 is a sequence (f_0, f_1, f_2) of diffeomorphisms, $f_0 : M \to M, f_1 : E_1 \to E_1$ and $f_2 : E_2 \to E_2$, such that

$$\begin{cases} \pi^1 \circ f_0 = f_1 \circ \pi^1 \\ \pi^2_1 \circ f_1 = f_2 \circ \pi^2_1 \end{cases} \quad (3)$$

In a local map at $x \in M, (\pi^{1^{-1}}(U), h_1)$ of coordinates (x^i, y^i) on E_1, since $\pi^1 \circ f_0 = f_1 \circ \pi^1$, the application $h_1 \circ f_1 \circ h_1^{-1} : R^n \times R^n \to R^n \times R^n$ has the expression $z_1 = (x^i, y^i) \to (X^i(x), Y^i(x, y^i)) = \tilde{z}_1$.

Then, relative to the map $(\pi^{2^{-1}}_1(U), h^2_1)$, the application $h^2_1 \circ f_2 \circ h^{2^{-1}}_1$ will have the expression $z_2 = (x^i, y^i_*) \to (\tilde{z}^i_1, Y^i_*(x, y, y_*)) = \tilde{z}_2$.

Thus, a gauge transformation will have the local shape

$$\begin{cases} \tilde{x}^i &= X^i(x), \\ \tilde{y}^i &= Y^i(x, y), \\ \tilde{y}^i_* &= Y^i_*(x, y, y_*), \end{cases} \quad (4)$$

For being a triplet of diffeomorphisms, (4) must have nonvanishing Jacobian, i.e.,

$$\det(\tilde{z}) = \det(\frac{\partial X^i}{\partial x^j}) \cdot \det(\frac{\partial Y^i}{\partial y^j}) \cdot \det(\frac{\partial Y^i_*}{\partial y^j_*}) \neq 0. \quad (5)$$

Since the triple (f_0, f_1, f_2) is globally defined, it satisfies the compatibility conditions with the coordinate changes on E_2

$$\begin{cases} \tilde{x'}^i(X^j)(x) &= X'^i(x'^j(x)) \\ \tilde{y'}^i(x, y') &= Y'^i(x', y') \\ \tilde{y'}^i_*(x, y', y'_*) &= Y'_*(x', y', y'_*) \end{cases} \quad (6)$$

Typical examples of gauge transformations are, e.g.,

1) $\tilde{x}^i = x^i, \tilde{y}^i = Y^i(x, y), \tilde{y}^i_* = Y^i_*(x, y, y_*)$,

2) $\tilde{x}^i = X^i(x), \tilde{y}^i = Y^i_j(x)y^j, \tilde{y}^i_* = Y^i_{*j}(x)y^j_*$,
3) $\tilde{x}^i = X^i(x), \tilde{y}^i = A^i_{j_1}(x)y^{j_1}, \tilde{y}^i_* = A^i_{j_1 j_2}(x)y^{j_1}y^{j_2} + A^i_{j_1}(x)y^{j_1}_*$,
where $A^i_{j_1 j_2}$ are symmetrical, and all obey the conditions (5).
Considering the composition of diffeomorphisms, we infer

Proposition 1.1. *The set of gauge transformations on $E_2 = Osc^{(2)}(M)$ represents a subgroup in $DiffM \times DiffE_1 \times DiffE_2$.*

Let consider the tangent spaces in z to E_2, $T_z(E_2)$, having the rules of change for the local bases ([7])

$$\begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial y'^m_*}{\partial x^i}\frac{\partial}{\partial y'^m_*} + \frac{\partial y'^m}{\partial x^i}\frac{\partial}{\partial y'^m} + \frac{\partial x'^m}{\partial x^i}\frac{\partial}{\partial x'^m} \\ \frac{\partial}{\partial y^i} = \frac{\partial y'^m_*}{\partial y^i}\frac{\partial}{\partial y'^m_*} + \frac{\partial y'^m}{\partial y^i}\frac{\partial}{\partial y'^m} \\ \frac{\partial}{\partial y^i_*} = \frac{\partial y'^m_*}{\partial y^i_*}\frac{\partial}{\partial y'^m_*} \end{cases} \tag{7}$$

and the local changes imposed by (4)

$$\begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial Y^m_*}{\partial x^i}\frac{\partial}{\partial \tilde{y}^m_*} + \frac{\partial Y^m}{\partial x^i}\frac{\partial}{\partial \tilde{y}^m} + X^m_i\frac{\partial}{\partial \tilde{x}^m} \\ \frac{\partial}{\partial y^i} = \frac{\partial Y^m_*}{\partial y^i}\frac{\partial}{\partial \tilde{y}^m_*} + Y^m_i\frac{\partial}{\partial \tilde{y}^m} \\ \frac{\partial}{\partial y^i_*} = Y^m_{*i}\frac{\partial}{\partial \tilde{y}^m_*} \end{cases} \tag{8}$$

where

$$X^m_i = \frac{\partial X^m}{\partial x^i}, Y^m_i = \frac{\partial Y^m}{\partial y^i}, Y^m_{*i} = \frac{\partial Y^m_*}{\partial y^i_*}$$

A special class of geometric objects which occur on E_2 is the one of d-tensors, which can be formally defined like systems of functions $W^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ on E_2, obeying the rules of change ([6])

$$W^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x', y', y'_*) = \frac{\partial x'^{i_1}}{\partial x^{h_1}} \ldots \frac{\partial x'^{i_r}}{\partial x^{h_r}} \cdot \frac{\partial x^{l_1}}{\partial x'^{j_1}} \ldots \frac{\partial x^{l_s}}{\partial x'^{j_s}} \cdot W^{h_1 \ldots h_r}_{l_1 \ldots l_s}(x, y, y_*). \tag{9}$$

The set of d-tensors determines a sub-algebra of the tensor algebra on E_2. We denote $Y^{(1)i}_j = Y^i_j, Y^{(2)i}_j = Y^i_{*j}$ and by $\overline{X}^i_j, \overline{Y}^{(\alpha)i}_j$ the elements of the matrices inverse to X^i_j and $Y^{(\alpha)i}_j, \alpha = \overline{1,2}$ respectively.

Definition 1.2. A *h-d-gauge tensor* is a d-tensor $W^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ on E_2 which satisfies also

$$W^{i_1 \ldots i_r}_{j_1 \ldots j_s}(\tilde{x}, \tilde{y}, \tilde{y}_*) = X^{i_1}_{h_1} \ldots X^{i_r}_{h_r} \cdot \overline{X}^{l_1}_{j_1} \ldots \overline{X}^{l_s}_{j_s} \cdot W^{h_1 \ldots h_r}_{l_1 \ldots l_s}(x, y, y_*). \tag{10}$$

We call v_α-d-gauge tensor, $\alpha = \overline{1,2}$, a d-tensor $W^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ on E_2 which satisfies the additional property

$$W^{i_1\dots i_r}_{j_1\dots j_s}(\tilde{x},\tilde{y},\tilde{y}_*) = Y^{(\alpha)i_1}_{h_1}\dots Y^{(\alpha)i_r}_{h_r}\cdot \overline{Y}^{(\alpha)l_1}_{j_1}\dots \overline{Y}^{(\alpha)l_s}_{j_s}\cdot W^{h_1\dots h_r}_{l_1\dots l_s}(x,y,y_*). \tag{11}$$

For example, $\frac{\partial}{\partial y^i_*}$ is a v_α-d-gauge tensor. Combining these definitions, we can consider h-v_α-d-gauge tensors or v_1-v_2-d-gauge tensors. The set of all types of such tensors will be called the set of d-gauge tensors.

2 Gauge covariant derivatives

A non-linear connection N on $E_2 = Osc^{(2)}(M)$ is determined by giving a splitting in an exact sequence of bundles or, equivalently, by providing a sub-bundle $N(E_2)$ which is supplementary to the vertical bundle $V(E_2) = Ker(\pi^2)^T$, where $(\pi^2)^T : T(E_2) \to T(M)$ is the tangent mapping ([7]). If N is a non-linear connection, then $T(E_2) = N(E_2) \oplus V(E_2)$. The expression of the horizontal lift $\frac{\delta}{\delta x^i} = l_h\left(\frac{\partial}{\partial x^i}\right)$ of $\frac{\partial}{\partial x^i}$ is

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j} - N^j_{*i}\frac{\partial}{\partial y^j_*} \tag{12}$$

and we have $\frac{\delta}{\delta x^i} = \frac{\partial x'^m}{\partial x^i}\frac{\delta}{\delta x'^m}$; N^j_i and N^j_{*i} are called the coefficients of the non-linear connection N.

Let denote by J the natural almost tangent structure on E_2, $J^3 = 0$. In $z \in E_2$ we obtain the following distributions corresponding to the non-linear connection N: $N(E_2) = N_0, N_1 = J(N_0), N_2 = J^2(N_0)$ having respectively the following local bases

$$\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i} = J\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i} - N^j_i\frac{\partial}{\partial y^j_*} \tag{13}$$

and $\frac{\delta}{\delta y^j_*} = J\left(\frac{\delta}{\delta y^i}\right) = \frac{\partial}{\partial y^i_*}$.

These are d-vector fields and $\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^i_*}\right\}$ is called adapted basis in $T_z(E_2)$. The coefficients of the non-linear connection $N(E_2)$ change obeying the rules

$$\begin{cases} N'^i_m\frac{\partial x'^m}{\partial x^j} &= \frac{\partial x'^i}{\partial x^m}N^m_j - \frac{\partial y'^i}{\partial x^j} \\ N'^i_{*m}\frac{\partial x'^m}{\partial x^j} &= \frac{\partial x'^i}{\partial x^m}N^m_{*j} + \frac{\partial y'^i}{\partial x^m}N^m_j - \frac{\partial y'^i_*}{\partial x^j}\end{cases} \tag{14}$$

The associated dual adapted basis is $\{dx^i, \delta y^i, \delta y^i_*\}$, where
$\delta y^i = dy^i + N^i_j x^j$; $\delta y^i_* = dy^i_* + N^i_j dy^j + (N^i_{*j} + N^i_m \cdot N^m_j)dx^j$.

The adapted fields $\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^i_*}\right\}$ are d-tensors but not gauge fields, generally. The adapted basis consists of d-gauge tensors iff

$$\left\{\begin{array}{lcl} \tilde{N}^i_m X^m_j & = & Y^i_m N^m_j - \frac{\partial Y^i}{\partial x^j} \\ \tilde{N}^i_m Y^m_j & = & Y^i_{*m} N^m_j - \frac{\partial Y^i_*}{\partial x^j} \\ \tilde{N}^i_{*m} Y^m_j & = & Y^i_{*m} N^m_{*j} + \frac{\partial Y^i_*}{\partial y^m} N^m_j - \frac{\partial Y^i_*}{\partial x^j} \end{array}\right. \tag{15}$$

Let denote by $V_1 = V_z(E_2)$ and $V_2 = Ker(\pi^2_{1,z})^T$, the so-called vertical distributions locally spanned by $\left\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i_*}\right\}$ and respectively $\left\{\frac{\partial}{\partial y^i_*}\right\}$; let v_1, v_2 be the two corresponding projectors. For a non-linear connection N, we denote by h the projector onto $N_z(E_2) = N_0$. We obtain the following derivation operators ([7])

$D^{(\alpha)h}_X Y = D_{X^h} Y^{v_\alpha}$ and $D^{(\alpha)v_\beta}_X Y = D_{X^{v_\beta}} Y^{v_\alpha}, \beta = \overline{1,2}, \alpha = \overline{0,2}$
with $v_0 = h$, defined locally by

$$\left\{\begin{array}{lcl} D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} & = & L^{(\alpha)m}_{ij} \frac{\delta}{\delta y^{(\alpha)m}} \\ D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} & = & C^{(\alpha)m}_{(\beta)ij} \frac{\delta}{\delta y^{(\alpha)m}}, \alpha = \overline{0,2}, \beta = \overline{1,2}, \end{array}\right. \tag{16}$$

where $y^{(0)i} = x^i, y^{(1)i} = y^i$ and $y^{(2)i} = y^i_*$; their coefficients change by the following rules

$$\left\{\begin{array}{lcl} L^{(\alpha)'i}_{pq} \frac{\partial x'^p}{\partial x^h} \frac{\partial x'^q}{\partial x^m} & = & L^{(\alpha)p}_{hm} \frac{\partial x'^i}{\partial x^p} - \frac{\partial^2 x'^i}{\partial x^h \partial x^m} \\ C^{(\alpha)'i}_{(\beta)pq} \frac{\partial x'^p}{\partial x^h} \frac{\partial x'^q}{\partial x^m} & = & \frac{\partial x'^i}{\partial x^j} C^{(\alpha)i}_{(\beta)hm}. \end{array}\right. \tag{17}$$

We remark that, in particular, the coefficients $L^{(\alpha)}$ can be equal and that $C^{(\alpha)}_{(\beta)}$ are d-tensors which can coincide for $\alpha = \overline{0,2}$. In this case, D will be called M-linear connection.

Also, $D^{(\alpha)h}$ and $D^{(\alpha)v_\beta}$ determine the following covariant derivation operators on d-tensors

$$\left\{\begin{array}{lcl} W^{(\alpha)i_1\ldots i_r}_{j_1\ldots j_s|_m} & = & \frac{\delta W^{i_1\ldots i_r}_{j_1\ldots j_s}}{\delta x^m} + L^{(\alpha)i_1}_{hm} W^{hi_2\ldots i_r}_{j_1\ldots j_s} + \ldots + L^{(\alpha)i_1}_{hm} W^{i_1\ldots i_{r-1}h}_{j_1\ldots j_s} - \\ & & - L^{(\alpha)h}_{j_1 m} W^{i_1\ldots i_r}_{hj_2\ldots j_s} - \ldots - L^{(\alpha)h}_{j_1 m} W^{i_1\ldots i_r}_{j_1\ldots j_{s-1}h} \\ W^{(\alpha)i_1\ldots i_r}_{j_1\ldots j_s} |^{(\beta)}_m & = & \frac{\delta W^{i_1\ldots i_r}_{j_1\ldots j_s}}{\delta y^{(\beta)m}} + C^{(\alpha)i_1}_{(\beta)hm} W^{hi_2\ldots i_r}_{j_1\ldots j_s} + \ldots + C^{(\alpha)i_1}_{(\beta)hm} W^{i_1\ldots i_{r-1}h}_{j_1\ldots j_s} - \\ & & - C^{(\alpha)h}_{(\beta)j_1 m} W^{i_1\ldots i_r}_{hj_2\ldots j_s} - \ldots - C^{(\alpha)h}_{(\beta)j_1 m} W^{i_1\ldots i_r}_{j_1\ldots j_{s-1}h} \end{array}\right.$$

The h-v_β gauge tensorial character is preserved if aditionally we have

$$\left\{\begin{array}{lcl} \tilde{L}^{(\alpha)r}_{pq} & = & \overline{Y}^{(\alpha)i}_p \cdot \overline{X}^j_q \cdot Y^{(\alpha)r}_m \cdot L^{(\alpha)m}_{ij} - \overline{X}^j_q \cdot \overline{Y}^{(\alpha)i}_p \cdot \frac{\delta Y^{(\alpha)r}_i}{\delta x^j} \\ \tilde{C}^{(\alpha)r}_{(\beta)pq} & = & \overline{Y}^{(\alpha)i}_p \cdot \overline{Y}^{(\beta)j}_q \cdot Y^{(\alpha)r}_m \cdot C^{(\alpha)m}_{(\beta)ij} - \overline{Y}^{(\beta)j}_q \cdot \overline{Y}^{(\alpha)i}_p \cdot \frac{\delta Y^{(\alpha)r}_i}{\delta y^{(\beta)j}}, \end{array}\right. \tag{18}$$

where $\alpha = \overline{0,2}, \beta = \overline{1,2}$, $Y^{(0)i}_j = X^i_j, Y^{(1)i}_j = Y^i_j, Y^{(2)i}_j = Y^i_{*j}$ and the overlined coefficients belong to the corresponding inverse matrices. We remark that for $\beta > \alpha, C^{(\alpha)r}_{(\beta)pq}$ become d-gauge tensors.

Definition 2.1. A d-linear connection is said to be *d-linear gauge connection* iff its coefficients satisfy (17) and (18). If its coefficients do not depend on $\alpha \in \overline{0,2}$, then it is called M-gauge connection.

In the following we shall also denote by $d_m^{(\alpha)h} W^{(\alpha)i_1\ldots i_r}_{j_1\ldots j_s}$ and $d_m^{(\alpha)v_\beta} W^{(\alpha)i_1\ldots i_r}_{j_1\ldots j_s}$ the h- and v_β-gauge derivatives of a d-gauge tensor, respectively.

3 Metric d-gauge connections of second order

Let be $g^{(\alpha)}_{ij}(x, y, y_*), \alpha = \overline{0,2}$ a system of d-tensors, symmetric and positively defined, with rank $(g^{(\alpha)}_{ij}) = n$, where $g^{(0)}_{ij}$ is h-d-gauge tensor and $g^{(\beta)}_{ij}$ are v_β-d-gauge tensors, $\beta = \overline{1,2}$; these d-tensors will be called h- and v_β-gauge metrics respectively.

Let N be a d-gauge linear connection. Then

$$G = g^{(0)}_{ij} dx^i \otimes dx^j + g^{(1)}_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + +g^{(2)}_{ij} \delta y^{(2)i} \otimes \delta y^{(2)j}$$

is globally defined on E_2 and is said to define a gauge metric structure on E_2.

Definition 3.1. A d-gauge linear connection D on E_2 is a h- (resp. v_β-) *metric connection* iff

$$g^{(\alpha)\ (\alpha)}_{\ ij|m} = 0 \text{ and } d_m^{(\alpha)h} g^{(\alpha)}_{ij} = 0 \tag{19}$$

(resp. $g^{(\alpha)(\alpha)\ (\beta)}_{\ ij|m} = 0$ and $d_m^{(\alpha)v_\alpha} g^{(\alpha)}_{ij} = 0), \forall \alpha = \overline{0,2}$.

If D is both h- and v_β-metric then we say that D is a d-gauge metric connection.

Theorem 3.1. *The following d-linear connection ([7])*

$$\begin{cases} L^{(\alpha)m}_{ij} = \frac{1}{2} g^{(\alpha)ms} \{ \frac{\delta g^{(\alpha)}_{sj}}{\delta x^i} + \frac{\delta g^{(\alpha)}_{is}}{\delta x^j} - \frac{\delta g^{(\alpha)}_{ij}}{\delta x^s} \} \\ C^{(\alpha)m}_{(\beta)ij} = \frac{1}{2} g^{(\alpha)ms} \{ \frac{\delta g^{(\alpha)}_{sj}}{\delta y^{(\beta)i}} + \frac{\delta g^{(\alpha)}_{is}}{\delta y^{(\beta)j}} - \frac{\delta g^{(\alpha)}_{ij}}{\delta y^{(\beta)s}} \}, \alpha = \overline{0,k}, \beta = \overline{1,2} \end{cases} \tag{20}$$

is a symmetric d-linear gauge connection.

4 Einstein-Yang Mills equations of second order

Let $L_0(x, y, y_*)$ be a Lagrangian defined on a compact set $\Omega \subset^{3n}$, a non-linear gauge connection $N = \{N^i_j, N^i_{*j}\}$, and a metric gauge structure G on E_2 defined by the metric d-gauge fields $g^{(\alpha)}_{ij}(x, y, y_*), \alpha = \overline{0,2}$.

Let Φ be a gauge field, that in applications belongs usually to the bundle of linear connections. In the present context, L_0 depends on $z = (x, y, y_*)$ through Φ and its derivatives $\frac{\delta\Phi}{\delta x^i}, \frac{\delta\Phi}{\delta y^i}, \frac{\delta\Phi}{\delta y^i_*}$, i.e.

$$L_0(x, y, y_*) = L_0(\Phi, \frac{\delta\Phi}{\delta x^i}, \frac{\delta\Phi}{\delta y^i}, \frac{\delta\Phi}{\delta y^i_*}). \tag{21}$$

For Φ varying in Ω, the action $\int_\Omega L_0(x, y^{(1)}, y^{(2)})d\omega$, where $d\omega = dx^1 \wedge \ldots \wedge dy^{(2)n}$ depends on the local coordinates. In order to remove this inconvenience, we consider the Lagrangian density:

$$\mathcal{L}(x, y, y_*) = L_0(x, y, y_*) \cdot \sqrt{g^{(0)}g^{(1)}g^{(2)}}, \tag{22}$$

where $g^{(\alpha)} = \det(g^{(\alpha)}_{ij}), \alpha = \overline{0, 2}$. Also, we remark that

$$\mathcal{L}(x, y, y_*) \quad = \quad \mathcal{L}(x, y', {y'}_*) \cdot \mathcal{J}, \text{ where } \mathcal{J} = \det(\tfrac{\partial x'^i}{\partial x^j}). \tag{23}$$

So that, the action $I(\Phi) = \int_\Omega \mathcal{L}(x, y, y_*)d\omega$ is independent of the coordinates on $E_2 = Osc^{(2)}(M)$. Applying the variational principle, the extremization of action $I(\Phi)$ leads to the following Euler-Lagrange attached to Φ

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\Phi}{\partial x^i})} - \frac{\partial}{\partial y^i}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\Phi}{\partial y^i})} - \ldots - \frac{\partial}{\partial y^i_*}\frac{\partial\mathcal{L}}{\partial(\frac{\partial\Phi}{\partial y^i_*})} = 0. \tag{24}$$

Taking into consideration the gauge transformations, it is more convenient to express (24) in the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\delta}{\delta y^i_*})$. The resulting relations look more complicated, but they evidentiate easier that (24) is invariant with respect to the change of coordinates (2) on E_2.

In (21), L depends on $\frac{\partial\Phi}{\partial x^i}, \frac{\partial\Phi}{\partial y^i}, \frac{\partial\Phi}{\partial y^i_*}$, by means of

$$\frac{\delta\Phi}{\delta x^i} = \frac{\partial\Phi}{\partial x^i} - N^j_i\frac{\partial\Phi}{\partial y^j} - N^j_{*i}\frac{\partial\Phi}{\partial y^j_*}, \qquad \frac{\delta\Phi}{\delta y^i} = \frac{\partial\Phi}{\partial y^i} - N^j_i\frac{\partial\Phi}{\partial y^j_*} \text{ and } \frac{\delta\Phi}{\delta y^j_*} = \frac{\partial\Phi}{\partial y^i_*}.$$

One can easily check that

$$\begin{cases} \dfrac{\partial L}{\partial(\frac{\partial\Phi}{\partial x^i})} &= \dfrac{\partial L}{\partial(\frac{\delta\Phi}{\delta x^i})} \\ \dfrac{\partial L}{\partial(\frac{\partial\Phi}{\partial y^i})} &= \dfrac{\partial L}{\partial(\frac{\delta\Phi}{\delta x^j})}\left(-N^i_j\right) + \dfrac{\partial L}{\partial(\frac{\delta\Phi}{\delta y^i})}\,|_{C_1}, \\ \dfrac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial y^i_*}\right)} &= \dfrac{\partial L}{\partial(\frac{\delta\Phi}{\delta x^j})}\left(-N^i_{*j}\right) + \dfrac{\partial L}{\partial(\frac{\delta\Phi}{\delta y^j})}\,|_{C_1}\left(-N^i_j\right) + \dfrac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta y^i_*}\right)}\,|_{C_1,C_2}. \end{cases}$$

where

$$\begin{cases} \# \,|_{C_1} &= \# \,|_{\frac{\delta\Phi}{\delta x^j}=C_1=const.} \\ \# \,|_{C_1,C_2} &= \# \,|_{\frac{\delta\Phi}{\delta x^j}=C_1=const., \frac{\delta\Phi}{\delta y^j}=C_2=const.} \end{cases}$$

Using these relations in (24), after reducing the terms, one obtains

Theorem 4.1 *The adapted form of the Einstein-Yang Mills equations on $Osc^{(2)}(M)$ for a M-gauge connecton is*

$$\left\{\begin{array}{l} \frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta\Phi} \equiv \frac{\partial L}{\partial\Phi} - \left(\ \delta_i\Phi^i + \delta'{}_i\Phi^{(1)i} + \delta''{}_i\Phi^{(2)i}\right) + \\ \quad +\Phi^j\delta'{}_iN'^i_j + \Phi^{(1)j}\delta''{}_iN^i_j + \Phi^j\delta''{}_iN''^i_j + \Phi^jN'^s_i\delta''{}_sN'^i_j - \\ \quad -\frac{1}{\sqrt{g}}\left(\ \Phi^i\delta_i\sqrt{g} + \Phi^{(1)i}\delta'{}_i\sqrt{g} + \Phi^{(2)i}\delta''{}_i\sqrt{g}\right) + \frac{1}{\sqrt{g}}L\frac{\delta\sqrt{g}}{\delta\Phi} = 0, \end{array}\right. \tag{25}$$

where

$$\Phi \in \left\{g_{ij}, g'{}_{ij}, g''{}_{ij}, L^i_{jk}, C'^i{}_{jk}, C''^i{}_{jk}, N'^i_j, N''^i_j\right\} \tag{26}$$

and we used the notations

$$\left\{\begin{array}{l} \Phi^i = \frac{\partial L}{\partial(\delta_i\Phi)}, \Phi^{(1)i} = \frac{\partial L}{\partial(\delta'{}_i\Phi)}\ |_{C_1}, \Phi^{(12)i} = \frac{\partial L}{\partial(\delta''{}_i\Phi)}\ |_{C_1,C_2}, \\ \qquad \delta_i = \frac{\delta}{\delta x^i}, \delta'{}_i = \frac{\delta}{\delta y^i}, \delta''{}_i = \frac{\delta}{\delta y^i_*} \\ L^i_{jk} = L^{(\alpha)i}_{jk}, C'^i{}_{jk} = C^{(\alpha)i}_{(1)jk}, C''^i{}_{jk} = C^{(\alpha)i}_{(2)jk}, \alpha = \overline{0,2}, \\ \sqrt{g} = \sqrt{g^{(1)}g^{(2)}g^{(3)}}, N'^i_j = N^i_j, N''^i_j = N^i_{*j}. \end{array}\right.$$

Remark. The last left term in (25) is effective only for $\Phi \in \{g_{ij}, g'{}_{ij}, g''{}_{ij}\}$.

From (25) results the invariance of the Euler-Lagrange equations with respect to the coordinate changes (2). Similar calculation proves that the Euler-Lagrange equations are not gauge-invariant in general, but if L is a scalar field which is invariant relative to both (2) and (4), then this becomes true. Therefore, an important problem is to determine the Lagrangian densities which are gauge-invariant. The Utiyama theorem is generalized in ([8]), where is shown that in the osculator bundle associated to the bundle of frames P_n, a gauge-invariant Lagrangian $\mathcal{L}$ depends only on the curvature form of a given connection from the bundle of connections (i.e., $\mathcal{L} = \mathcal{L}_0 \circ \Omega$, where Ω is the curvature form and $\mathcal{L}_0$ is a fixed gauge-invariant Lagrangian density).

For a given gauge non-linear connection $N = \{N'^i_j, N''^i_j\}$ and a M-linear connection $D = \{L^i_{jk}, C'^i{}_{jk}, C''^i{}_{jk}\}$, are derived the following components of the corresponding torsion d-gauge tensors

$$\left\{\begin{array}{l} T^i_{jk} = L^i_{jk} - L^i_{kj}, C'^i{}_{jk}, C''^i{}_{jk}, R^{(1)i}_{(0)jk} = \delta_kN'^i_j - \delta_jN'^i_k, \\ R^{(2)i}_{(0)jk} = \delta_kN''^i_j - \delta_jN''^i_k + N'^i_s\left(\delta_kN'^s_j - \delta_jN'^s_k\right), \\ R^{(1)i}_{(1)jk} = \delta'{}_kN'^i_j - \delta'{}_jN'^i_k, R^{(2)i}_{(2)jk} = \delta''{}_kN'^i_j - \delta''{}_jN'^i_k, R^{(1)i}_{(2)jk} = 0, \\ P^{(1)i}_{(1)jk} = \delta'{}_kN'^i_j - L^i_{kj}, P^{(2)i}_{(1)jk} = \delta'{}_kN''^i_j + N'^i{}_s\delta'{}_kN'^s_j - \delta'{}_jN'^i_k, \\ P^{(1)i}_{(2)jk} = \delta''{}_kN'^i_j, P^{(2)i}_{(2)jk} = \delta''{}_kN''^i_j + N'^i{}_s\delta''{}_kN'^s_j - L^i_{kj}, \\ P^{(2)i}_{(1)(2)jk} = \delta''{}_kN''^i_j - C'^i{}_{kj}, \\ S^i_{(1)jk} = C'^i{}_{jk} - C'^i{}_{kj}, S^i_{(2)jk} = C''^i{}_{jk} - C''^i{}_{kj}, \end{array}\right.$$

and the curvature d-gauge tensors

$$\begin{cases} R^m_{rpq} = \delta_q L^m_{rp} - \delta_p L^m_{rq} + L^j_{rp} L^m_{jr} - L^j_{rq} L^M_{jp} + \sum\limits_{\beta=1}^{2} C^m_{(\beta)rj} R^{(\beta)j}_{(0)pq}, \\ P^m_{(\beta)rpq} = \frac{\delta L^m_{rp}}{\delta y^{(\beta)q}} - C^m_{(\beta)rq|p} + \sum\limits_{\gamma=1}^{2} C^m_{(\gamma)rj} P^{(\gamma)j}_{(\beta)pq}, \beta = \overline{1,2} \\ P^m_{(1)(2)rpq} = \frac{\delta C^m_{(1)rp}}{\delta y^{(2)q}} - C^m_{(2)rq} \mid^{(1)}_p + C^m_{(2)rj} P^{(2)j}_{(1)(2)pq}, \\ S^m_{(\beta)rpq} = \frac{\delta C^m_{(\beta)rp}}{\delta y^{(\beta)q}} - \frac{\delta C^m_{(\beta)rq}}{\delta y^{(\beta)p}} + \\ \qquad + C^j_{(\gamma)rp} C^m_{(\beta)jq} - C^j_{(\gamma)rq} C^m_{(\beta)jp} + R^{(1)j}_{(\beta)pq} C^m_{(\beta)rj}, \beta = \overline{1,2} \end{cases}$$

where $C_{(1)} = C', C_{(2)} = C''$. If D is a d-gauge connection, then the torsions and curvatures described above are d-gauge tensors.

Theorem 4.2. *The Einstein-Yang Mills equations in G. S. Asanov's form have the expressions*

$$\begin{cases} \frac{1}{\sqrt{g}} \frac{\delta \mathcal{L}}{\delta \Phi} \equiv \frac{\partial L}{\partial \Phi} - \left(\Phi^i \mid_i + \Phi^{(1)i} \mid^{(1)}_i + \Phi^{(12)i} \mid^{(2)}_i \right) + \\ \qquad + \left(T^m_{jm} + P^{(1)m}_{(1)jm} + P^{(2)m}_{(2)jm} \right) \Phi^j + \\ \qquad + \left(P^{(1)m}_{(2)jm} - 2C^m_{(1)mj} + S^{(1)m}_{(1)jm} \right) \Phi^{(1)j} + \\ \qquad + \left(S^{(2)m}_{(2)jm} - 2C^m_{(2)mj} \right) \Phi^{(12)j} - \\ - \frac{1}{\sqrt{g}} \left(D^*_j \sqrt{g} + D'^*_j \sqrt{g} + D''^*_j \sqrt{g} \right) \Phi^j + \theta = 0, \end{cases} \tag{27}$$

where

$$\begin{cases} D^*_i \sqrt{g} = \delta_i \sqrt{g} - 3L^m_{mi} \sqrt{g} \\ D'^*_i \sqrt{g} = \delta'_i \sqrt{g} - 3C^m_{(1)mi} \sqrt{g} \\ D''^*_i \sqrt{g} = \delta''_i \sqrt{g} - 3C^m_{(2)mi} \sqrt{g} \end{cases}$$

In (27) we denoted $\theta = \frac{L}{2} A^{ij}$, where for $\Phi \in \{g^{(0)}_{ij}, g^{(1)}_{ij}, g^{(2)}_{ij}\}$, A^{ij} equals respectively $g^{(0)ij}, g^{(1)ij}, g^{(2)ij}$ (the d-gauge tensors which are reciprocal to $g^{(0)}_{ij}$, $g^{(1)}_{ij}$, $g^{(2)}_{ij}$), and for the rest of alternatives for θ, A^{ij} is set null.

We remark that raising and lowering the indices by means of the tensors

$$\{g^{(0)ij}, g^{(1)ij}, g^{(2)ij}; g^{(0)}_{ij}, g^{(1)}_{ij}, g^{(2)}_{ij}\}$$

preserves the gauge character of d-tensor fields. Therefore, the torsions and curvatures of the M-gauge connection D produce scalar gauge fields as follows

$$\begin{cases} L_{(\alpha),\tau} = \tau^m_{ij} \tau^{ij}_m, \\ L_{(\alpha),\varrho} = \varrho^m_{ijl} \varrho^{ijl}_m, \end{cases}$$

where $\{\tau^i_{jk}\}$ is any torsion d-tensor field of D, $\{\varrho^i_{jkl}\}$ curvature of D, and

$$\begin{cases} \tau^{ij}_m = g^{(\alpha)ir} g^{(\alpha)js} g^{(\alpha)}_{mq} \tau^q_{rs}, \\ \varrho^{ijl}_m = g^{(\alpha)ir} g^{(\alpha)js} g^{(\alpha)lt} g^{(\alpha)}_{mq} \varrho^q_{rst}. \end{cases}$$

Thus, a general gauge invariant Lagrangian which depends only on the curvatures and torsions of a metric gauge connection is $\mathcal{L} = L\sqrt{g}$, where L is a linear combination with real coefficients of gauge scalar fields built as above.

In the following, we determine the Einstein-Yang Mills equations for several types of such elementary Lagrangians.

a) For $L = T^m_{ij} T^{ij}_m$, an M-gauge connection D and for $\Phi = L^i_{jk}$, the equations (25) become

$$\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta\Phi} \equiv 2T^i_{jk} = 0$$

and are equivalent to the condition that D is h-symmetrical ([6],[2]).

b) For $L = R^{(1)m}_{(0)ij} R^{(1)ij}_{(0)m}$, an M-gauge connection D and for $\Phi = N^k_i$, we have $\Phi^{(1)i} = \Phi^{(12)i} \equiv 0, \frac{\partial L}{\partial\Phi} = 0, \Phi^j = 2R^{(1)ij}_{(0)k}$.

In this case, the Asanov equations are

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta\Phi} \equiv -D_i\Phi^i + \left(T^m_{jm} + P^{(1)m}_{(1)jm} + P^{(2)m}_{(2)jm}\right)\Phi^j + \\ -\frac{1}{\sqrt{g}}\Phi^j\left(\ D^*_j\sqrt{g} + D'^*_j\sqrt{g} + D''^*_j\sqrt{g}\right) = 0.\end{aligned}$$

The expressions (25) for these n^2 equations in the unknowns $\{N^k_i\}$ become

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta\Phi} \equiv -\delta_i\Phi^i + \Phi^j\left(\delta'_i N'^i_j + \delta''_i N''^i_j + N'^s_i\delta''_s N''^i_j - \frac{\delta_j\sqrt{g}}{\sqrt{g}}\right) = 0 \\ \Leftrightarrow \delta_j R^{(1)ij}_{(0)k} = R^{(1)ij}_{(0)k}\left(\ \partial_s N'^s_j + \partial''_s N''^s_j - \frac{\delta_j g}{2g}\right); i,k = \overline{1,n}\end{aligned}$$

where $\partial_s = \frac{\partial}{\partial x^s}, \partial''_s = \frac{\partial}{\partial y^s_*}$.

c) For $L = P^{(1)m}_{(2)ij} P^{(1)ij}_{(2)m}$, an M-gauge connection D and for $\Phi = N^k_i$, we get $\Phi^i = \Phi^{(1)i} = 0,\ \frac{\partial L}{\partial\Phi} = 0,\ \Phi^{(12)j} = 2P^{(1)m}_{(2)ij}$. In the case of identical metric components, i.e., $g^{(0)}_{ij} = g^{(1)}_{ij} = g^{(2)}_{ij} = g_{ij}$, the Einstein-Yang Mills equations (25) have the form

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta\Phi} \equiv -\delta''_j\Phi^{(12)j} - \frac{\delta''_j\sqrt{g}}{\sqrt{g}}\Phi^{(12)j} = 0 \\ \Leftrightarrow 2\partial''_j P^{(1)ij}_{(2)m} = -P^{(1)ij}_{(2)m}\cdot\partial''_j(ln\sqrt{g}) \\ \Leftrightarrow \partial''_j\Psi^t_{sr} + \Psi^t_{sr}\cdot\partial''_j\left(\ \frac{ln\sqrt{g}}{2} + g^{ir}g^{sj}g_{mt}\right) = 0,\end{aligned}$$

where we denoted $\Psi^t_{sr} = \partial''_s N'^t_r$.

The investigation of the solutions of the Einstein-Yang Mills equations on $Osc^{(2)}(M)$ for certain relevant cases will be the subject of a forecoming paper.

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