A NOTABLE SUBMERSION IN THE HIGHER ORDER GEOMETRY

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Abstract

The geometry of the k-osculator bundle over a smooth manifold M was developed by R.Miron and his school. It was used for the geometrization of the higher order Lagrangians and the prolongation of the Riemannian, Finslerian and Lagrangian structures, ([5]).

In this work we show that the prolongation of a Riemannian metric provides a Riemannian submersion which is notable in some respects. For simplicity we confine ourselves to the case $k = 2$.

First we associate to a Riemannian metric g a nonlinear connection in the 2-oscultor bundle. Using the connection map associated to it ([1]) a prolongation G of g to Osc^2M is constructed in §2. It is shown that the projection map becomes a Riemannian submersion whose vertical subspace in a fixed point splits into two subspaces which are also isometric with the tangent space to M. Some properties of this Riemannian submersion are shown in §3 and 4.

AMS Subject Classification: 53C60
Key words: Riemannian metric and submersion, nonlinear connection, osculator bundle

1 A nonlinear connection of second order associated to a Riemannian metric

Let (M, g) be a smooth Riemannian manifold, of dimension n and $(E = Osc^2M, \pi, M)$ its 2-osculator bundle. Then Osc^2M is a smooth manifold of dimension $3n$.

Let (x^i) be the local coordinates in a local chart (U, φ),$U \subset M$. The local coordinates on $\pi^{-1}(U) \subset E$ will be denoted by $(x^i, y^{(1)i}, y^{(2)i})$. Let $\Gamma^i_{jk}(x)$ be the local coefficients of the Levi-Civita connection ∇.

As $\pi_* : (TE, \tau_E, E) \to (TM, \tau, M)$ is an epimorphism of vector bundles, it results that its kernel is a vector subbundle of the bundle (TE, τ_E, E). This will be denoted by

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1995,* 1-10

VE and will be called the vertical subbundle of the TE. The fibres of VE determine an integrable distribution $V : u \in E \to V_u \subset T_uE$ which has the dimension $2n$, called vertical distribution. A local basis for this distribution is $\{\frac{\partial}{\partial y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\}$.

On every domain of local charts of E we consider the functions:

$$\begin{cases} \underset{(1)}{N^i_j}\,(x, y^{(1)}) = \Gamma^i_{kj}(x) y^{(1)k} \\ \underset{(2)}{N^i_j}\,(x, y^{(1)}, y^{(2)}) = \frac{1}{2}(\frac{\partial \Gamma^i_{jk}}{\partial x^s}(x) - \Gamma^i_{mk}(x)\Gamma^m_{js}(x)) y^{(1)k} y^{(1)s} + \Gamma^i_{kj}(x) y^{(2)k} \end{cases} \tag{1}$$

We set:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N^j_i}\,\frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N^j_i}\,\frac{\partial}{\partial y^{(2)j}}$$

Starting from the transformation law of the coefficients $\Gamma^i_{jk}(x)$, by a long and tedious calculation one shows that under a change of coordinates on E:

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x^1, x^2, ..., x^n); \quad \text{rank} \left\| \frac{\partial \widetilde{\mathrm{x}}^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{j}}} \right\| = \mathrm{n}, \\ \widetilde{y}^{(1)i} = \frac{\partial \widetilde{x}^i}{\partial x^j} y^{(1)j}, \\ 2\widetilde{y}^{(2)i} = \frac{\partial \widetilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2 \frac{\partial \widetilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \ , \end{cases} \tag{2}$$

the local vector fields $\{\frac{\delta}{\delta x^i}\}_{i=\overline{1,n}}$ change as follows: $\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\delta}{\delta \widetilde{x}^j}$.

Thus we obtain that, for each $u \in E$, $\{\frac{\delta}{\delta x^i}\ |_u\}_{i=\overline{1,n}}$ span a subspace $N_0(u)$ of dimension n in T_uE. The map $N_0 : u \in E \to N_0(u) \subset T_uE$ is a distribution of dimension n (generally not integrable). The distribution N_0 is called the horizontal distribution on E. It is supplementary to the vertical distribution, that is, $T_uE = N_0(u) \oplus V(u), \quad \forall u \in E$.

In other words N_0 defines a nonlinear connection N which is clearly derived from g only.

The $\mathcal{F}$ (E)-linear mapping $J : \chi(E) \to \chi(E)$ defined by :

$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^{(1)i}}$, $J(\frac{\partial}{\partial y^{(1)i}}) = \frac{\partial}{\partial y^{(2)i}}$, $J(\frac{\partial}{\partial y^{(2)i}}) = 0$ is a 2-tangent structure, that is, $J^3 = 0$. Let us consider $N_1 = J(N_0)$ and V_2 the distribution locally generated by $\{\frac{\partial}{\partial y^{(2)i}}\}_{i=\overline{1,n}}$. We have three distributions (N_0, N_1, V_2), each of dimension n, such that :

$T_uE = N_0(u) \oplus N_1(u) \oplus V_2(u)$, $\forall u \in E$.

A local basis for the $\mathcal{F}$ (E)-module $\chi(E)$, adapted to the distributions N_0, N_1, V_2, is :

$\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}} = J(\frac{\delta}{\delta x^i}) = \frac{\partial}{\partial y^{(1)i}} - N^j_{(1)i} \frac{\partial}{\partial y^{(2)j}}, \frac{\partial}{\partial y^{(2)i}}\}$

The dual basis is $\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}$ with :

$$\begin{cases} \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M^i_j}\, dx^j, \\ \delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{M^i_j}\, dy^{(1)j} + \underset{(2)}{M^i_j}\, dx^j, \end{cases} \tag{3}$$

where:

$$\begin{cases} \underset{(1)}{M_j^i}(x,y^{(1)}) = \underset{(1)}{N_j^i}(x,y^{(1)}), \\ \underset{(2)}{M_j^i}(x,y^{(1)},y^{(2)}) = \underset{(2)}{N_j^i}(x,y^{(1)},y^{(2)}) + \underset{(1)}{N_m^i}(x,y^{(1)})\,\underset{(1)}{N_j^m}(x,y^{(1)}). \end{cases} \tag{4}$$

Notice that $\frac{\partial \underset{(1)}{N_j^i}}{\partial y^{(1)k}} = \frac{\partial \underset{(2)}{N_j^i}}{\partial y^{(2)k}} = \Gamma^i_{jk}$.

2 Prolongation of second order of a Riemannian metric. A notable Riemannian submersion.

By a caracterisation of nonlinear connections in the k-osculator bundle given in [1], to give the nonlinear connection N_0 is equivalent to give a connection map i.e. a π-morphism of vector bundles $K = (\overset{(1)}{K}, \overset{(2)}{K}) : (TE, \tau_E, E) \to (TM \oplus TM, \tau \oplus \tau, M)$, where (TM, τ, M) is the tangent bundle over M which verifies:

$$\overset{(2)}{K} \circ J = \overset{(1)}{K}, \quad \overset{(2)}{K} \circ J^2 = \pi_*. \tag{1}$$

For $X_u = \overset{(0)i}{X} \frac{\partial}{\partial x^i}|_u + \overset{(1)i}{X} \frac{\partial}{\partial y^{(1)i}}|_u + \overset{(0)i}{X} \frac{\partial}{\partial y^{(2)i}}|_u \in T_uE$ the map K is given by:

$$\begin{aligned} \overset{(1)}{K}_u X_u &= (\overset{(1)}{X^i} + \underset{(1)}{M_j^i} \overset{(0)}{X^j}) \frac{\partial}{\partial x^i}|_{\pi(u)}, \\ \overset{(2)}{K}_u X_u &= (\overset{(2)}{X^i} + \underset{(1)}{M_j^i} \overset{(1)}{X^j} + \underset{(2)}{M_j^i} \overset{(0)}{X^j}) \frac{\partial}{\partial x^i}|_{\pi(u)}, \end{aligned} \tag{2}$$

where $\underset{(1)}{M_j^i}$ and $\underset{(2)}{M_j^i}$ are taken from (1.4).

Using the connection map K we define a Riemannian metric G on Osc^2M which prolonges g as Sasaki metric on TM does.

For every $u \in E$ we define $G_u : T_uE \times T_uE \to R$ by:

$$\begin{aligned} G_u(X_u, Y_u) &= g_{\pi(u)}(\pi_{*,u}X_u, \pi_{*,u}Y_u) + g_{\pi(u)}(\overset{(1)}{K}_u X_u, \overset{(1)}{K}_u Y_u) + \\ &+ g_{\pi(u)}(\overset{(2)}{K}_u X_u, \overset{(2)}{K}_u Y_u). \end{aligned} \tag{3}$$

Thus we get a Riemannian metric on Osc^2M. Indeed, since the mappings $\pi_{*,u}, \overset{(1)}{K}_u, \overset{(2)}{K}_u$: $T_uE \to T_{\pi(u)}$ are linear and $g_{\pi(u)}$ is bilinear, it results that G_u is bilinear. It is clear that $G_u(X_u, X_u) \geq 0$. If for $X_u \in T_uE$ we have $G_u(X_u, X_u) = 0$, then $g_{\pi(u)}(\pi_{*,u}X_u, \pi_{*,u}X_u) = 0$, $g_{\pi(u)}(\overset{(1)}{K}_u X_u, \overset{(1)}{K}_u X_u) = 0$ and $g_{\pi(u)}(\overset{(2)}{K}_u X_u, \overset{(2)}{K}_u X_u) = 0$

from which it follows: $\pi_{*,u}X_u = \overset{(1)}{K}_u X_u = \overset{(2)}{K}_u X_u = 0$. Using (2.2), a direct calculation shows that X_u takes the form

$$X_u = \overset{(0)}{X^i} \frac{\delta}{\delta x^i} |_u + (\overset{(1)}{K}_u X_u)^i \frac{\delta}{\delta y^{(1)i}} |_u + (\overset{(2)}{K}_u X_u)^i \frac{\partial}{\partial y^{(2)i}} |_u . \tag{4}$$

Now it is clear that the previous equations imply $X_u = 0$.

Proposition 2.1. *1. The distributions N_0, N_1, V_2 are mutual orthogonal with respect to G*
2.The mappings:
$\pi_{*,u} : (N_0(u), G_u |_{N_0(u)}) \to (T_{\pi(u)}, g_{\pi(u)})$,
$\overset{(1)}{K}_u$: $(N_1(u), G_u |_{N_1(u)} \to (T_{\pi(u)}, g_{\pi(u)})$,
$\overset{(2)}{K}_u$: $(V_2(u), G_u |_{V_2(u)} \to (T_{\pi(u)}, g_{\pi(u)})$ *are linear isometries.*

Proof. 1. If $X_u \in N_0(u)$ and $Y_u \in N_1(u)$ by (2.4) we have $\overset{(1)}{K}_u X_u = \overset{(2)}{K}_u X_u = 0$, $\pi_{*,u}Y_u = \overset{(2)}{K}_u Y_u = 0$ and by (2.3) one gets $G_u(X_u, Y_u) = 0$. On proceeds similarly for the rest.
2. By (2.4) it folows that $X_u \in N_0(u)$ if and only if $\overset{(1)}{K}_u X_u = \overset{(2)}{K}_u X_u = 0$ and similarly for Y_u. Hence for $X_u, Y_u \in N_0(u)$, by (2.3) one obtains $G_u(X_u, Y_u) = g_{\pi(u)}(\pi_{*,u}X_u, \pi_{*,u}Y_u)$. For $X_u, Y_u \in N_1(u)$ we have $\pi_{*,u}X_u = \pi_{*,u}Y_u = 0$ and $\overset{(2)}{K}_u X_u = \overset{(2)}{K}_u Y_u = 0$. By (2.3) one gets that $\overset{(1)}{K}$ is a linear isometry.
Corollary 2.1. *The projection map $\pi : (Osc^2M, G) \to (M, g)$ is a Riemannian submersion.*

Notice that the Riemannian submersion π has a special feature: every vertical subspace $Ker\pi_{*,u}$ splits into two subspace $N_1(u)$ and $V_2(u)$ of the same dimension n each of them being isometric with $(T_{\pi(u)}M, g_{\pi(u)})$. This feature has several implications on the geometry of the Riemannian submersion π. Some of then will be pointed in the next sections.

3 Some brackets. An expression of the Levi-Civita connection of G.

Next we establish the brackets for two vector fields on the total space E, by using geometrical objects on base M. Using these brackets we express the Levi-Civita connection of the Riemannian manifold (E, G).

We denote by $\chi^{N_0}(E)$ the $\mathcal{F}(E)$-module of the sections of vector bundle $(N_0E, \tau_E |_{N_0E}, E)$. $\chi^{N_0}(E)$ is just the $\mathcal{F}(E)$-module of horizontal vector fields on E. $\chi^{N_1}(E)$ and $\chi^{V_2}(E)$ are denote $\mathcal{F}(E)$-modules of the sections of vector bundles $(N_1E, \tau_E |_{N_1E}, E)$ and $(V_2E, \tau_E |_{V_2E})$, respectively.

Proposition 3.1. *If $X, Y \in \chi^{N_1}(E)$ are π projectable vector fields then:*

$$\left\{\begin{array}{l} \pi_*[X,Y] = [\pi_* X, \pi_* Y], \\ \overset{(1)}{K}_u [X,Y]_u = R(\pi_{*,u}X, \pi_{*,u}Y, \overset{(1)}{C}_{\pi(u)}), \\ \overset{(2)}{K}_u [X,Y]_u = \frac{1}{2}\{(\nabla_{\overset{(1)}{C}_{\pi(u)}} R)(\pi_{*,u}X, \pi_{*,u}Y, \overset{(1)}{C}_{\pi(u)}) + \\ \qquad + R(\pi_{*,u}X, \pi_{*,u}Y, \overset{(2)}{C}_{\pi(u)})\}, \end{array}\right. \tag{1}$$

where $\overset{(1)}{C}_{\pi(u)} = \overset{(1)}{K}_u \overset{(1)}{\Gamma}_u$, $\overset{(2)}{C}_{\pi(u)} = \overset{(1)}{K}_u \overset{(2)}{\Gamma}_u$; $\overset{(1)}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(2)i}}$ *and* $\overset{(2)}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}}$ *are the Liouville vector fields.*

Proof. One obtains the previous equalities using that :
$[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = R_p{}^k{}_{ij} z^{(1)p}\frac{\delta}{\delta y^{(1)k}} + \frac{1}{2}(R_p{}^k{}_{ij|s} z^{(1)p} z^{(1)s} + R_p{}^k{}_{ij} z^{(2)p})\frac{\partial}{\partial y^{(2)k}}$,
where $z^{(1)p} = y^{(1)p}$, $2z^{(2)p} = 2y^{(2)p} + \underset{(1)}{M_i^p}\, z^{(1)i}$ and $_{|s}$ denotes the covariant derivative with respect to ∇.

Proposition 3.2 *If $X \in \chi^{N_0}(E)$ is π-projectable and $Y \in \chi^{N_1}(E)$ is $\overset{(1)}{K}$-projectable then:*

$$\left\{\begin{array}{l} \pi_*[X,Y] = 0, \\ \overset{(1)}{K} [X,Y] = \nabla_{\pi_* X} \overset{(1)}{K} Y, \\ \overset{(2)}{K}_u [X,Y]_u = \frac{1}{2}R(\overset{(1)}{C}_{\pi(u)}, \overset{(1)}{K}_u Y, \pi_{*,u}X). \end{array}\right. \tag{2}$$

Proof. These equalities result by a straightforward calculation using:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)j}}\right] = \Gamma^k_{ij}\frac{\delta}{\delta y^{(1)k}} + \frac{1}{2}R_i{}^k{}_{jp} y^{(1)p}\frac{\partial}{\partial y^{(2)k}}$$

.
Much more easier are the proofs of

Proposition 3.3. *If $X \in \chi^{N_0}(E)$ is π-projectable and $Y \in \chi^{V_2}(E)$ is $\overset{(2)}{K}$-projectable then:*

$$\left\{\begin{array}{l} \pi_*[X,Y] = \overset{(1)}{K} [X,Y] = 0, \\ \overset{(2)}{K} [X,Y] = \nabla_{\pi_* X} \overset{(2)}{K} Y. \end{array}\right. \tag{3}$$

Proposition 3.4. *The distributions N_1 and V_2 are integrable.*

We denote by $\widetilde{\nabla}$ the Levi-Civta connection of the Riemannian manifold (E, G). This is uniquely determined by: $2G(\widetilde{\nabla}_X Y, Z) = XG(Y,Z) + YG(Z,X) - ZG(X,Y) +$

$G([X,Y],Z)+G([Z,X],Y)+G([Z,Y],X), \quad \forall X,Y,Z\in\chi(E)$
For the proofs of the following propositions we refer to [2].

Proposition 3.5.

1. If $X,Y\in\chi^{N_0}(E)$ are π-projectable then for any $u\in E$:

$(\widetilde{\nabla}_X Y)_u = (\ell_h)_{\pi(u),u}(\nabla_{\pi_{ast}X}\pi_{ast}Y)_{\pi(u)} + \frac{1}{2}(\ell_{v_2})_{\pi(u),u}R(\pi_{*,u}X,\pi_{*,u}Y,\overset{(1)}{C}_{\pi(u)})$
$+\frac{1}{2}(\ell_{v_2})_{\pi(u),u}(\nabla_{\overset{(1)}{C}_{\pi(u)}} R(\pi_{*,u}X,\pi_{*,u}Y,\overset{(1)}{C}_{\pi(u)}) + R(\pi_{*,u}X,\pi_{*,u}Y,\overset{(2)}{C}_{\pi(u)}).$

2. If $X\in\chi^{N_0}(E)$ is π-projectable and $Y\in\chi^{N_1}(E)$ is $\overset{(1)}{K}$-projectable then:
$(\widetilde{\nabla}_X Y)_u = \frac{1}{2}(\ell_h)_{\pi(u),u}R(\overset{(1)}{C}_{\pi(u)},\overset{(1)}{K}_u Y,\pi_{*,u}X) + (\ell_{v_1})_{\pi(u),u}(\nabla_{\pi_* X}\overset{(1)}{K} Y)_u +$
$\frac{1}{2}(\ell_{v_2})_{\pi(u),u}R(\overset{(1)}{C}_{\pi(u)},\overset{(1)}{K}_u Y,\pi_{*,u}X) \quad \forall u\in E.$

3. If $X\in\chi^{N_0}(E)$ is π-projectable and $Y\in\chi^{V_2}(E)$ is $\overset{(2)}{K}$-projectable then:
$(\widetilde{\nabla}_X Y)_u = \frac{1}{2}(\ell_h)_{\pi(u),u}((\nabla_{\overset{(1)}{C}_{\pi(u)}} R)(\overset{(1)}{C}_{\pi(u)},\overset{(2)}{K}_u Y,\pi_{*,u}X) + R(\overset{(2)}{C}_{\pi(u)},\overset{(2)}{K}_u Y,\pi_{*,u}X))+$
$\frac{1}{2}(\ell_{v_1})_{\pi(u),u}(R(\pi_{*,u}X,\overset{(2)}{K}_u Y,\overset{(1)}{C}_{\pi(u)})) + (\ell_{v_2})_{\pi(u),u}(\nabla_{\pi_* X}\overset{(2)}{K} Y)_{\pi(u)}, \quad \forall u\in E.$

Proposition 3.6.

1. If $X\in\chi^{N_1}(E)$ is $\overset{(1)}{K}$-projectable and $Y\in\chi^{N_0}(E)$ is π-projectable then:
$(\widetilde{\nabla}_X Y)_u = \frac{1}{2}(\ell_h)_{\pi(u),u}R(\overset{(1)}{C}_{\pi(u)},\overset{(1)}{K}_u X,\pi_{*,u}Y) + \frac{1}{2}(\ell_{v_2})_{\pi(u),u}R(\overset{(1)}{C}_{\pi(u)},\overset{(1)}{K}_u X,\pi_{*,u}Y)$
$\forall u\in E$.

2. If $X\in\chi^{V_2}(E)$ is $\overset{(2)}{K}$-projectable and $Y\in\chi^{N_0}(E)$ is π-projectable then:
$(\widetilde{\nabla}_X Y)_u = \frac{1}{2}(\ell_h)_{\pi(u),u}R(\overset{(1)}{C}_{\pi(u)},\overset{(2)}{K}_u X,\pi_{*,u}Y) + \frac{1}{2}(\ell_{v_1})_{\pi(u),u}R(\overset{(2)}{K}_u X,\pi_{*,u}Y,\overset{(1)}{C}_{\pi(u)}$
$), \quad \forall u\in E$

3. If $X\in\chi^{V_2}(E)$ is $\overset{(2)}{K}$-projectable and $Y\in\chi^{N_1}(E)$ is $\overset{(1)}{K}$-projectable then:
$(\widetilde{\nabla}_X Y)_u = \frac{1}{2}(\ell_h)_{\pi(u),u}R(\overset{(1)}{C}_{\pi(u)},\overset{(2)}{K}_u X,\overset{(1)}{K}_u Y), \quad \forall u\in E.$

Here $(\ell_h)_{\pi(u),u}$, $(\ell_{v_1})_{\pi(u),u}$ and $(\ell_{v_2})_{\pi(u),u} : T_{\pi(u)}M \to T_uE$, denote the horizontal and the vertical lifts.
We have $\pi_{*,u}\circ(\ell_h)_{\pi(u),u} = \overset{(1)}{K}_u\circ(\ell_{v_1})_{\pi(u),u} = \overset{(2)}{K}_u\circ(\ell_{v_2})_{\pi(u),u} = 1_{T_{\pi(u)}M}$.

Proposition 3.7.

1. If $X,Y\in\chi^{N_1}(E)$ are $\overset{(1)}{K}$-projectable then $\widetilde{\nabla}_X Y = 0$.

2. If $X,Y\in\chi^{V_2}(E)$ are $\overset{(2)}{K}$-projectable then $\widetilde{\nabla}_X Y = 0$.

4 Geodesics

Let $I\subset R,\ 0\in I$ be an open interval and $c: I\to c(t)\in M$ be a smooth parametrized curve on M such that if $(U,\phi=(x^i))$ is a local chart in M then $c(I)\subset U$. The curve c is expressed in local coordinates by $c(t)=(x^i(t))$

Let $v : t \in I \to v(t) \in T_{c(t)}M$ be a vector field along of curve c: $v(t) = v^i(t)\frac{\partial}{\partial x^i} |_{c(t)}$.

We define a smooth parametrized curve $\ell_2(c) : t \in I \to (x^i(t), \frac{1}{1!}\frac{dx^i}{dt} |_t, \frac{1}{2!}\frac{d^2x^i}{dt^2} |_t)$ on E and then $(\ell_2(v))(t) = v^i(t)\frac{\partial}{\partial x^i} |_{(\ell_2(c))(t)} + \frac{1}{1!}\frac{dv^i}{dt} |_t \frac{\partial}{\partial y^{(1)i}} |_{(\ell_2(c))(t)} + \frac{1}{2!}\frac{d^2v^i}{dt^2} |_t \frac{\partial}{\partial y^{(2)i}} |_{(\ell_2(c))(t)}$ is a vector field along of curve $\ell_2(c)$.

Lemma 4.1. *If v, v_1 and v_2 are vector fields along of curve c and $f \in \mathcal{F}(M)$ then:*
1. $\ell_2(v_1 + v_2) = \ell_2(v_1) + \ell_2(v_2)$,
2. $(\ell_2(fv))(t) = (f \circ c)(t)(\ell_2(v))(t) + \frac{1}{1!}\frac{df}{dt} |_t v^i(t)\frac{\partial}{\partial y^{(1)i}} |_{(\ell_2(c))(t)} + \frac{1}{2!}(\frac{d^2f}{dt^2} |_t v^i(t) + 2\frac{df}{dt} |_t \frac{dv^i}{dt} |_t)\frac{\partial}{\partial y^{(2)i}} |_{(\ell_2(c))(t)}$.

Remark 4.1. $\pi_*(\ell_2(v)) = v$.

Lemma 4.2. *For a vector field v along of curve c and $f \in \mathcal{F}(M)$, we have:*
$\overset{(\alpha)}{K}(\ell_2((f \circ c)v)) = \sum_{i=0}^{2} \frac{1}{i!}\frac{d^if}{dt^i} \overset{(\alpha-i)}{K}(\ell_2(v)), \quad \alpha \in \{1, 2\}$.

Proposition 4.1. *Let M be a smooth manifold with a linear connection ∇. For a vector field v along of curve c there are well-defined two vector fields $\frac{\overset{(\alpha)}{\nabla}}{dt}v : t \in I \to \frac{\overset{(\alpha)}{\nabla}}{dt}v |_t \in T_{c(t)}M$ along of curve c ($\alpha \in \{1, 2\}$) which satisfy:*
1. $\frac{\overset{(\alpha)}{\nabla}}{dt}(v_1 + v_2) = \frac{\overset{(\alpha)}{\nabla}}{dt}v_1 + \frac{\overset{(\alpha)}{\nabla}}{dt}v_2$;
2. $\frac{\overset{(\alpha)}{\nabla}}{dt}(fv) = \sum_{i=0}^{2} C_\alpha^i \frac{d^if}{dt^i}\frac{\overset{(\alpha-i)}{\nabla}}{dt}v$; (the Leibniz formula)
3. If v is the restiction of a vector field $Y \in \chi(M)$ then:
$\frac{\overset{(\alpha)}{\nabla}}{dt}v = \nabla^\alpha_{\dot{c}}Y \quad (\alpha \in \{1, 2\})$.

Proof. We define $\frac{\overset{(\alpha)}{\nabla}}{dt}v |_t = \alpha! \overset{(\alpha)}{K}_{(\ell_2(c))(t)}(\ell_2(v))(t)$ $(\frac{\overset{(0)}{\nabla}}{dt}v \overset{def}{=} \pi_*(\ell_2(v)) = v)$
By the lemmas 4.1 and 4.2 we obtain 1. and 2. A straightforward calculation gives: $\nabla^\alpha_{\dot{c}}v |_t = \alpha! \overset{(\alpha)}{K}_{(\ell_2(c))(t)}(\ell_2(v))(t)$ that is $\nabla^\alpha_{\dot{c}}v = \frac{\overset{(\alpha)}{\nabla}}{dt}v$ and 3. is proved.

Proposition 4.2. *Let $c : I \to M$ be a smooth parametrized curve in M. Then c is a geodesic on M if and only if the component of $\ell_2(\dot{c})$ in N_1 vanishes.*

Proof. The curve c is geodesic on M if and only if $\frac{\overset{(1)}{\nabla}}{dt}\dot{c} = 0$ equivalently, $\overset{(1)}{K}_{(\ell_2(c))(t)}(\ell_2(\dot{c}))(t) = 0$ and $(v_1)_{(\ell_2(c))(t)}(\ell_2(\dot{c}))(t) = 0$. Hence the component of $\ell_2(\dot{c})$ in N_1 vanishes.

On each domain $\pi^{-1}(U)$ of local chart $(\pi^{-1}(U), \Phi = (x^i, y^{(1)i}, y^{(2)i}))$ on E, we consider the system of functions :
$G^i(x, y^{(1)}, y^{(2)}) = \frac{1}{3}(2 \underset{(1)}{M^i_j}(x, y^{(1)})y^{(2)j} + \underset{(2)}{M^i_j}(x, y^{(1)}, y^{(2)})y^{(1)j}) =$

$= \frac{1}{3}(2 \underset{(1)}{N_j^i}(x, y^{(1)})z^{(2)j} + \underset{(2)}{N_j^i}(x, y^{(1)}, y^{(2)})z^{(1)j}).$

Proposition 4.3. *The map* $S : u \in E \to S_u = y^{(1)i}\frac{\partial}{\partial x^i}|_u + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}|_u - 3G^i(u)\frac{\partial}{\partial y^{(2)i}}$ *is a vector field on* E *(it will be called the canonical spray).*

Proof. For $u \in E$
$S_u = z^{(1)i}\frac{\delta}{\delta x^i}|_u + 2z^{(2)i}\frac{\delta}{\delta y^{(1)i}}|_u + (2 \underset{(1)}{N_j^i}(u)z^{(2)j} + \underset{(2)}{N_j^i}(u)z^{(1)j} - 3G^i(u))\frac{\partial}{\partial y^{(2)i}}|_u =$
$z^{(1)i}\frac{\delta}{\delta x^i}|_u + 2z^{(2)i}\frac{\delta}{\delta y^{(1)i}}|_u \in T_uE$. Therefore for each $u \in E$, S_u belong to T_uE, that is, S is a vector field on E.

Proposition 4.4. *A smooth curve* $\widetilde{c} : t \in I \to \widetilde{c}(t) = (x^i(t), y^{(1)i}(t), y^{(2)i}(t))$ *on* E *is an integral curve for the canonical spray* S *if and only if:*
$\widetilde{c} = \ell_2(\pi \circ \widetilde{c})$ *and* $\frac{\overset{(2)}{\nabla}}{dt}\ell_2(\pi \circ \dot{\widetilde{c}}) = 0$

Proof. $\widetilde{c}$ is an integrable curve for S if and only if $\dot{\widetilde{c}}(t) = S_{\widetilde{c}(t)}$, $\forall t \in I$
$\dot{\widetilde{c}}(t) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{dy^{(1)i}}{dt}\frac{\partial}{\partial y^{(1)i}} + \frac{dy^{(2)i}}{dt}\frac{\partial}{\partial y^{(2)i}} =$
$(\ell_2(\pi \circ \dot{\widetilde{c}}))^i\frac{\delta}{\delta x^i} + (\frac{\overset{(1)}{\nabla}}{dt}\ell_2(\pi \circ \dot{\widetilde{c}}))^i\frac{\delta}{\delta y^{(1)i}} + (\frac{\overset{(2)}{\nabla}}{dt}\ell_2(\pi \circ \dot{\widetilde{c}}))^i\frac{\partial}{\partial y^{(2)i}},$
According to these considerations we obtain that $\widetilde{c}$ is integral curve for S if and only if

$$\begin{cases} (\ell_2(\pi \circ \dot{\widetilde{c}}))^i = z^{(1)i}, \\ (\frac{\overset{(1)}{\nabla}}{dt}\ell_2(\pi \circ \dot{\widetilde{c}}))^i = 2z^{(2)i}, \\ \frac{\overset{(2)}{\nabla}}{dt}\ell_2(\pi \circ \dot{\widetilde{c}}) = 0. \end{cases}$$

The first two conditions are equivalent with : $y^{(1)i}(t) = \frac{1}{1!}\frac{dx^i}{dt}$ and $y^{(2)i}(t) = \frac{1}{2!}\frac{d^2x^i}{dt^2}$.
These imply the following expression for $\widetilde{c}$: $\widetilde{c} = (x^i, \frac{1}{1!}\frac{dx^i}{dt}, \frac{1}{2!}\frac{d^2x^i}{dt^2}) = \ell_2(\pi \circ \widetilde{c})$.

Corollary 4.1. *For the curve* $c : t \in I \to c(t) \in M$ $\ell_2(c)$ *is an integral curve for* S *if and only if the components of* $\frac{d}{dt}(\ell_2(c))$ *in* V_2 *vanishes.*

Let $\widetilde{c} : t \in I \to \widetilde{c}(t) = (x^i(t), y^{(1)i}(t), y^{(2)i}(t))$ on Osc^2M be a smooth curve and $X(t) = \dot{\widetilde{c}}(t) = \frac{dx^i}{dt}\frac{\delta}{\delta x^i}|_{\widetilde{c}(t)} + \frac{\overset{(1)}{\nabla} y^{(1)i}}{dt}\frac{\delta}{\delta y^{(1)i}}|_{\widetilde{c}(t)} + \frac{\overset{(2)}{\nabla} y^{(2)i}}{dt}\frac{\partial}{\partial y^{(2)i}}|_{\widetilde{c}(t)} = E + U_1 + U_2$ the tangent vector field along of $\widetilde{c}$.

Proposition 4.5. *Let* $\widetilde{c}$ *be a horizontal curve on* Osc^2M. *If* $\widetilde{c}$ *is a geodesic on* Osc^2M *then its projection* $\pi \circ \widetilde{c}$ *is a geodesic on* M.

Proof. If $\widetilde{c}$ is a horizontal curve then $\dot{\widetilde{c}} = E$, $U_1 = U_2 = 0$ and consequently $\widetilde{c}$ is a geodesic if and only if $\widetilde{\nabla}_E E = 0$. According to the Proposition 3.5.(1) $H\widetilde{\nabla}_E E$ is π-projectable and $\pi_*\widetilde{\nabla}_E E = \nabla_{\pi_* E}\pi_* E$. Since $\pi_* E = \frac{d}{dt}(\pi \circ \widetilde{c})$ we get $\widetilde{\nabla}_E E = 0$ if

and only if $\nabla_{\frac{d}{dt}(\pi\circ\widetilde{c})}\frac{d}{dt}(\pi\circ\widetilde{c})=0$, that is $\pi\circ\widetilde{c}$ is geodesic on M.
Notice the result from the previous proposition is true for any Riemannian submersion.

Proposition 4.6. *A smooth curve $\widetilde{c}$ on Osc^2M which satisfies $E=U_2=0$ is a geodesic on Osc^2M.*

Proof. As $E=U_2=0$, it results that $\widetilde{\nabla}_X X=\widetilde{\nabla}_{U_1}U_1$. According to the Proposition 3.7.(1) $\widetilde{\nabla}_{U_1}U_1=0$. and so the curve $\widetilde{c}$ is a geodesic on Osc^2M.

Proposition 4.7. *A smooth curve $\widetilde{c}$ on Osc^2M which satisfies $E=U_1=0$ is a geodesic on Osc^2M.*

Proof. As $E=U_1=0$ it results that $\widetilde{\nabla}_X X=\widetilde{\nabla}_{U_2}U_2$. According to Proposition 3.7.(2) $\widetilde{\nabla}_{U_2}U_2=0$. Hence the curve $\widetilde{c}$ is a geodesic on Osc^2M. **Proposition 4.8.** *A smooth curve $\widetilde{c}$ in fibre ($E=0$) is a geodesic on Osc^2M if and only if:* $\widetilde{\nabla}_{U_1}U_2+\widetilde{\nabla}_{U_2}U_1=0$.

Proof. $\widetilde{c}$ is a vertical curve. The condition $\widetilde{\nabla}_X X=0$ are equivalent with $\widetilde{\nabla}_{U_1}U_2+\widetilde{\nabla}_{U_2}U_1=0$ ($E=0$, $\widetilde{\nabla}_{U_1}U_1=0$ and $\widetilde{\nabla}_{U_2}U_2=0$)

Definition 4.1. The submersion $\pi:(Osc^2M,G)\to(M,g)$ is called *totally geodesic* if each vertical curve is a geodesic on Osc^2M.

Proposition 4.9. *The submersion $\pi:(Osc^2M,G)\to(M,g)$ is totally geodesic if and only if the Riemannian manifold (M,g) is locally flat.*

Proof. Taking into account Proposition 4.8 we prove that the submersion π is totally geodesic if and only if for each curve $\widetilde{c}:t\in I\to\widetilde{c}(t)\in Osc^2M$ with $\dot{\widetilde{c}}(t)=U_1(t)+U_2(t)$ we have $\widetilde{\nabla}_{U_1}U_2+\widetilde{\nabla}_{U_2}U_1=0$. According to the Proposition 3.6.(3) $\widetilde{\nabla}_{U_1}U_2+\widetilde{\nabla}_{U_2}U_1=0$, which is equivalent with
$\frac{1}{2}R(\overset{(1)}{C}_{(\pi\circ\widetilde{c})(t)},\overset{(1)}{K}_{\widetilde{c}(t)}U_1,\overset{(2)}{K}_{\widetilde{c}(t)}U_2)+\frac{1}{2}R(\overset{(1)}{C}_{(\pi\circ\widetilde{c})(t)},\overset{(1)}{K}_{\widetilde{c}(t)}U_1,\overset{(2)}{K}_{\widetilde{c}(t)}U_2)=0\quad\forall U_1,U_2$
and so $R=0$

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