Some classes of Ricci solitons on Lorentzian α-Sasakian manifolds

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Abstract. In this paper, we mainly focus on some classes of Ricci solitons on a Lorentzian α-Sasakian manifold M and obtain some important results which classify such manifolds. We provide the necessary condition for which such a manifold M is Einstein-like. Also, we find that an η-Ricci soliton on M is steady if and only if M satisfies the curvature condition $\varphi.P = 0$. Finally, we give an example which verifies our results.

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1 Introduction

The notion of Ricci soliton, which is a natural generalization of Einstein manifolds, was appeared by Hamilton in 1988 [11]. This notion is a fixed point of Hamilton's Ricci flow defined by $\frac{\partial}{\partial t}g = -2S$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphims and scaling. Also, it models the formation of singularities in the Ricci flow.

In the framework of the contact geometry, Ricci solitons have been studied by many mathematicians in some different classes of contact geometry, since Sharma applied Ricci solitons to K-contact manifolds [23]. For example, Blaga obtained that there is no Ricci soliton on a Lorentzian para-Sasakian manifold with the potential vector field ξ satisfying conditions $R(\xi, X).S = 0$ and $S.R(\xi, X) = 0$ in [5]. Patra proved that if a metric of para-Sasakian manifold is a Ricci soliton, then either it is Einstein or η-Einstein manifold in [17]. Later, Ghosh studied Ricci solitons on Kenmotsu manifold and showed that if the metric of a Kenmotsu manifold represents a Ricci soliton, then it is expanding in [10]. For the recent works on Ricci solitons, we refer to ([14], [18], [29], [30]) and references therein.

A Riemannian manifold (M, g) is called a Ricci soliton if the following condition is satisfied for arbitrary vector fields X, Y on M

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

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where $\pounds_V g$ denotes the Lie-derivative of the metric tensor g along vector field V, S is the Ricci tensor of M and λ is a constant. A Ricci soliton on M is denoted by (g, V, λ). The vector field V is called the potential vector field of the Ricci soliton. If V is the gradient of a potential function $-f$ (i.e., $V = -\nabla f$), then the Ricci soliton is called gradient. Also, a Ricci soliton is called shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

In 2009, Cho and Kimura introduced the concept of η-Ricci soliton as a generalization of classical Ricci soliton [8]. Calin and Crasmareanu treated this concept on Hopf hypersurfaces in complex space forms in 2012 [7]. Inspired by these works, η-Ricci solitons have been studied in many contexts: Sasakian [19], quasi-Sasakian [26], Kenmotsu [25], nearly Kenmotsu [1], ϵ-Kenmotsu [12], para-Sasakian [15], [21], para-Kenmotsu [3], Lorentzian para-Sasakian [4], [20], ϵ-almost paracontact metric manifolds [6] and many others.

A Riemannian manifold (M, g) is called an η-Ricci soliton if there exists a vector field V satisfying

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{1.2}$$

where $\pounds_V g$ denotes the Lie-derivative of the metric tensor g along vector field V, S is the Ricci tensor of M and λ, μ are real constants. An η-Ricci soliton on M is denoted by (g, V, λ, μ). The vector field V is called the potential vector field of the η-Ricci soliton. If $\mu = 0$, then η-Ricci soliton reduces to Ricci soliton. An η-Ricci soliton is called shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

On the other hand, in 2016, Nurowski and Randall introduced the generalized Ricci soliton which is another generalization of Ricci solitons [16]. According to them, the generalized Ricci soliton equation in a Riemannian manifold (M, g) is defined as follows:

$$(\pounds_V g)(X, Y) = -2c_1 V^\star(X)V^\star(Y) + 2c_2 S(X, Y) + 2\lambda g(X, Y), \tag{1.3}$$

where $\pounds_V g$ denotes the Lie-derivative of the metric tensor g along vector field V, S is the Ricci tensor of M, $V^\star$ is the dual 1-form of the vector field V and c_1, c_2, λ are real constants.

Motivated by these circumstances, we examine some classes of Ricci solitons on Lorentzian α-Sasakian manifolds. The present paper is organized as follows. In section 2, we give some basic definitions, notations and formulas. In last section, we give our main results. We find that the vector field V of the Ricci soliton (g, V, λ) on a Lorentzian $\alpha-$Sasakian manifold M is Killing under some conditions. Also, we obtain some necessary conditions for a Lorentzian α-Sasakian manifold M to be Einstein, Einstein-like.

2 Preliminaries

In this section, we recall some fundamental notations and formulas from [9], [13], [27] and [28].

A differentiable manifold M of dimension $(2n + 1)$ is said to be a Lorentzian α-Sasakian manifold if it admits a structure (φ, ξ, η, g) and a Lorentzian metric g

satisfies the following relations:

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.1}$$
$$\eta(\xi) = -1, \tag{2.2}$$
$$\varphi\xi = 0, \tag{2.3}$$
$$\eta \circ \varphi = 0, \tag{2.4}$$
$$\eta(X) = g(X, \xi) \tag{2.5}$$
$$g(\varphi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.6}$$
$$g(\varphi X, Y) = g(X, \varphi Y) \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where ξ is a contravariant vector field of type $(0, 1)$, 1-form η is the g-dual of ξ of type $(1, 0)$ and φ is a tensor field of type $(1, 1)$ on M. Also, such a manifold satisfies:

$$\nabla_X \xi = \alpha\varphi X, \tag{2.8}$$
$$(\nabla_X \eta)Y = \alpha g(\varphi X, Y), \tag{2.9}$$
$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi + \eta(Y)\varphi X), \tag{2.10}$$

where ∇ indicates the operator of covariant differentiation with respect to the Lorentzian metric g and $\alpha \in \mathbb{R}$. Throughout the paper, we will consider Lorentzian α-Sasakian manifold with $\alpha \neq 0$.

A Lorentzian α-Sasakian manifold M is called η-Einstein if there exists two real constants a and b such that the Ricci tensor field S of M satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y).$$

If the constant b is equal to zero, then M is called Einstein. Also, the Ricci tensor S of a Lorentzian α-Sasakian manifold M is said to be η-parallel if it satisfies

$$(\nabla_X S)(\varphi Y, \varphi Z) = 0$$

such that

$$(\nabla_X S)(\varphi Y, \varphi Z) = \nabla_X S(\varphi Y, \varphi Z) - S(\nabla_X \varphi Y, \varphi Z) - S(\varphi Y, \nabla_X \varphi Z)$$

for any $X, Y, Z \in \Gamma(TM)$.

For a Lorentzian α-Sasakian manifold, we also have the followings:

$$R(X, Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y), \tag{2.11}$$
$$R(\xi, X)Y = \alpha^2(g(X, Y)\xi - \eta(Y)X), \tag{2.12}$$
$$R(\xi, X)\xi = \alpha^2(X + \eta(X)\xi), \tag{2.13}$$
$$S(X, \xi) = 2n\alpha^2\eta(X), \tag{2.14}$$
$$Q\xi = 2n\alpha^2\xi, \tag{2.15}$$
$$S(\xi, \xi) = -2n\alpha^2, \tag{2.16}$$

where R is the Riemann curvature tensor, Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$.

On the other hand, the projective curvature tensor P on a Riemannian manifold M is defined as follows:

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}(S(Y,Z)X - S(X,Z)Y) \tag{2.17}$$

for any $X, Y, Z \in \Gamma(TM)$. It is well known that if the projective curvature tensor P vanishes, then M is called a projectively flat manifold. Also, if $P(X,Y)\xi = 0$, then M is named as a ξ-projectively flat manifold.

A vector field V on a semi-Riemannian manifold (M, g) is called an affine conformal vector field if it satisfies [24]

$$(\pounds_V \nabla)(X,Y) = X(\rho)Y + Y(\rho)X - g(X,Y) grad\rho, \tag{2.18}$$

where $\pounds_V$ is the Lie-derivative with respect to V and ρ is a smooth function on M. Also if ρ in (2.18) is constant, then V is called an affine vector field.

In [24], Sharma and Duggal prove that a vector field V on a semi-Riemannian manifold (M, g) is affine conformal if and only if

$$(\pounds_V g)(X,Y) = 2\rho g(X,Y) + K(X,Y), \quad \nabla K = 0, \tag{2.19}$$

where K is a symmetric parallel tensor of type $(0, 2)$.

3 Main Results

In this section, we give our main results which are obtained in this paper.

We introduce the following definition analogous to the Einstein-like para-Sasakian manifolds [22]:

Definition 3.1. A Lorentzian α-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is called Einstein-like if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\varphi X, Y) \tag{3.1}$$

for some real constants a, b, c and for $X, Y \in \Gamma(TM)$.

Now, we begin with the following:

Proposition 3.1. *If M is a Lorentzian α-Sasakian manifold admitting a generalized Ricci soliton with the potential vector field ξ, then M is an Einstein-like manifold provied $c_2 \neq 0$.*

Proof. Since η is the dual 1-form of the structure vector field ξ, from (1.3) we have

$$(\pounds_\xi g)(X,Y) = -2c_1\eta(X)\eta(Y) + 2c_2 S(X,Y) + 2\lambda g(X,Y), \tag{3.2}$$

for any $X, Y \in \Gamma(TM)$. It follows from (2.8), one immediately has

$$(\pounds_\xi g)(X,Y) = 2\alpha g(X, \varphi Y). \tag{3.3}$$

In view of (3.2) and (3.3), we get

$$S(X,Y) = -\frac{\lambda}{c_2} g(X,Y) + \frac{c_1}{c_2}\eta(X)\eta(Y) - \frac{\alpha}{c_2} g(\varphi X, Y). \tag{3.4}$$

Thus, we get the requested result. □

Proposition 3.2. *If M is an Einstein-like Lorentzian α-Sasakian manifold, then M has η-parallel Ricci tensor.*

Proof. From (3.1), by a straightforward calculation we get

$$\begin{aligned}\nabla_X S(Y,Z) &= a(g(\nabla_X Y,Z)+g(\nabla_X Z,Y)) \\ &\quad +b((\nabla_X\eta(Y))\eta(Z)+(\nabla_X\eta(Z))\eta(Y)) \\ &\quad +c(g(\nabla_X\varphi Y,Z)+g(\nabla_X Z,\varphi Y))\end{aligned} \tag{3.5}$$

for any $X,Y,Z\in\Gamma(TM)$. Similarly, using (3.1) we also have

$$S(\nabla_X Y,Z) = ag(\nabla_X Y,Z)+b\eta(\nabla_X Y)\eta(Z)+cg(\varphi(\nabla_X Y),Z) \tag{3.6}$$
$$S(Y,\nabla_X Z) = ag(Y,\nabla_X Z)+b\eta(Y)\eta(\nabla_X Z)+cg(\varphi Y,\nabla_X Z). \tag{3.7}$$

From (3.5)-(3.7) and using (2.9), (2.10) we acquire

$$\begin{aligned}(\nabla_X S)(Y,Z) &= b\alpha(g(X,\varphi Y)\eta(Z)+g(X,\varphi Z)\eta(Y)) \\ &\quad c\alpha(g(X,Y)\eta(Z)+g(X,Z)\eta(Y)).\end{aligned} \tag{3.8}$$

Also, taking $X=\varphi X$ and $Y=\varphi Y$ in (3.8) gives

$$(\nabla_X S)(\varphi Y,\varphi Z)=0,$$

which completes the proof. □

Now, suppose that M is a Lorentzian α-Sasakian manifold admitting an η-Ricci soliton (g,V,λ,μ) such that the vector field V is the gradient Df of a smooth function f on M, that is, $V=Df$. Then, from (1.2) we have

$$\nabla_X Df = -QX-\lambda X-\mu\eta(X)\xi \tag{3.9}$$

for any $X\in\Gamma(TM)$. Taking covariant derivative of (3.9) with respect to Y and keeping in mind (2.8) we find that

$$\begin{aligned}\nabla_Y\nabla_X Df &= -\nabla_Y QX-\lambda\nabla_Y X-\mu\eta(\nabla_Y X)\xi-\mu\alpha g(X,\varphi Y)\xi \\ &\quad -\mu\alpha\eta(X)\varphi(Y).\end{aligned} \tag{3.10}$$

Interchanging the roles of X and Y in (3.10), we obtain

$$\begin{aligned}\nabla_X\nabla_Y Df &= -\nabla_X QY-\lambda\nabla_X Y-\mu\eta(\nabla_X Y)\xi-\mu\alpha g(Y,\varphi X)\xi \\ &\quad -\mu\alpha\eta(Y)\varphi(X).\end{aligned} \tag{3.11}$$

Also, we have

$$\nabla_{[X,Y]}Df = -Q[X,Y]-\lambda[X,Y]-\mu\eta([X,Y])\xi. \tag{3.12}$$

With the help of (3.10), (3.11) and (3.12), we arrive at

$$R(X,Y)Df = (\nabla_X Q)Y-(\nabla_Y Q)X-\mu\alpha(\eta(Y)\varphi(X)-\eta(X)\varphi(Y)). \tag{3.13}$$

Putting $X = \xi$ in (3.13) and keeping in mind (2.12) we get

$$\alpha^2(Y(f)\xi - \xi(f)Y) = (\nabla_\xi Q)Y - (\nabla_Y Q)\xi - \mu\alpha\varphi(Y)). \tag{3.14}$$

Taking inner product of (3.14) with ξ yields

$$\alpha^2(-Y(f) - \xi(f)\eta(Y)) = g((\nabla_\xi Q)Y, \xi) - g(\nabla_Y Q)\xi, \xi). \tag{3.15}$$

It is easy to see that $g((\nabla_\xi Q)Y, \xi) = g(\nabla_Y Q)\xi, \xi)$. Since $\alpha \neq 0$, then the equation (3.15) becomes

$$(-Y(f) - \xi(f)\eta(Y)) = 0. \tag{3.16}$$

Removing Y from both sides in (3.16), we acquire

$$Df = -\xi(f)\xi \tag{3.17}$$

and hence

$$V = -\xi(f)\xi.$$

On the other hand, taking covariant derivative of (3.17) with respect to X gives

$$\nabla_X Df = -X(\xi(f))\xi - \alpha\xi(f)\varphi X. \tag{3.18}$$

for X being tangent to M. Making use of (3.9) and (3.18), we have

$$-X(\xi(f))\xi - \alpha\xi(f)\varphi X = -QX - \lambda X - \mu\eta(X)\xi. \tag{3.19}$$

Operating inner product of (3.19) with ξ and using (2.14), (2.15) provide

$$X(\xi(f)) = -(2n\alpha^2 + \lambda - \mu)\eta(X). \tag{3.20}$$

Again operating inner product of (3.19) with arbitrary vector field Y, one has

$$\begin{aligned} -X(\xi(f))\eta(Y) - \alpha\xi(f)g(\varphi X, Y) \;&=\; -S(X,Y) - \lambda g(X,Y) \\ &\quad\; -\mu\eta(X)\eta(Y). \end{aligned} \tag{3.21}$$

In view of (3.20) and (3.21), we find that

$$S(X,Y) \;=\; -\lambda g(X,Y) - (2n\alpha^2 + \lambda)\eta(X)\eta(Y) - \alpha\xi(f)g(\varphi X, Y).$$

Then, we can state:

Theorem 3.3. *Let M be a Lorentzian α-Sasakian manifold admitting an η-Ricci soliton (g, V, λ, μ) such that V is the gradient Df of a smooth function f on M. Then, the followings are satisfied:*
i) V is pointwise collinear with the structure ξ.
ii) M is an Einstein-like manifold provided that $\xi(f)$ is constant.

Let us assume that M is an Einstein-like Lorentzian α-Sasakian manifold and $V=\xi$. Then, from (3.1) we write

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y)+cg(\varphi X,Y) \tag{3.22}$$

for any $X,Y\in\Gamma(TM)$. With the help of (2.7) and (2.8), we get

$$(\pounds_{\xi}g)(X,Y)=2\alpha g(\varphi X,Y) \tag{3.23}$$

and hence

$$\begin{aligned}&(\pounds_{\xi}g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)\\&=2(a+\lambda)g(X,Y)+2(b+\mu)\eta(X)\eta(Y)+2(c+\alpha)g(\varphi X,Y).\end{aligned}$$

From the last equation it is clear that M admits an η-Ricci soliton (g,ξ,λ,μ) if

$$a+\lambda=0,\qquad b+\mu=0,\qquad c+\alpha=0. \tag{3.24}$$

Thus, we are ready to give the following:

Proposition 3.4. *Let M be an Einstein-like Lorentzian α-Sasakian manifold with defined by (3.1). Then, the data $(g,\xi,-a,-b)$ is an η-Ricci soliton on M.*

The next theorem gives an important characterization for a Lorentzian $\alpha-$Sasakian manifold.

Theorem 3.5. *Let M be a Lorentzian α-Sasakian manifold admitting a η-Ricci soliton (g,V,λ,μ) such that V is an affine conformal vector field. Then, V is an affine vector field and (M,g,V,λ,μ) is an η-Einstein manifold.*

Proof. Let the data (g,V,λ,μ) be an η-Ricci soliton on Lorentzian α-Sasakian manifold M. Then, from (1.2) and (2.19) we have

$$S(X,Y)=-(\rho+\lambda)g(X,Y)-\mu\eta(X)\eta(Y)-\frac{1}{2}K(X,Y). \tag{3.25}$$

for any $X,Y\in\Gamma(TM)$. Since V is an affine conformal vector field, from (2.19) symmetric tensor K is parallel respect to ∇, namely $\nabla K=0$. Then, it follows that

$$\nabla^2K(X,Y;Z,W)-\nabla^2K(X,Y;W,Z)=0$$

and hence

$$K(R(X,Y)Z,W)+K(Z,R(X,Y)W)=0 \tag{3.26}$$

for any $X,Y,Z,W\in\Gamma(TM)$. Putting $X=W=\xi$ in (3.26) and using (2.12), (2.13) yield

$$\alpha^2(g(Y,Z)K(\xi,\xi)-\eta(Z)K(Y,\xi)+\eta(Y)K(Z,\xi)+K(Y,Z))=0.$$

Since $\alpha\neq0$, the above equation takes the form

$$K(Y,Z)=-K(\xi,\xi)g(Y,Z)+\eta(Z)K(Y,\xi)-\eta(Y)K(Z,\xi). \tag{3.27}$$

Substituting Z for ξ in (3.27), one has

$$K(Y,\xi) = -\eta(Y)K(\xi,\xi). \tag{3.28}$$

From (3.27) and (3.28), we derive

$$K(Y,Z) = -K(\xi,\xi)g(Y,Z). \tag{3.29}$$

On the other hand, utilizing the parallelism of the symmetric tensor K, we have

$$(\nabla_X K)(Y,Z) = 0 \tag{3.30}$$

such that

$$(\nabla_X K)(Y,Z) = \nabla_X K(Y,Z) - K(\nabla_X Y, Z) - K(Y, \nabla_X Z) \tag{3.31}$$

for any $X, Y, Z \in \Gamma(TM)$. Using (3.29) in (3.31) and by means of (3.30) we find

$$X(K(\xi,\xi))g(Y,Z) = 0. \tag{3.32}$$

Taking $Y = Z = \xi$ in (3.32) gives that $X(K(\xi,\xi)) = 0$. This implies that $K(\xi,\xi)$ is a constant. Furthermore, making use of (3.25) and (3.29) we get

$$S(X,Y) = -(\rho + \lambda - \frac{1}{2}K(\xi,\xi))g(X,Y) - \mu\eta(X)\eta(Y), \tag{3.33}$$

which means that M is an η-Einstein manifold. Also, if we put $X = Y = \xi$ in (3.33), we obtain $\rho = \mu + \frac{1}{2}K(\xi,\xi) - \lambda - 2n\alpha^2$ which implies that ρ is a constant. Then, V is an affine vector field. Thus, the proof is completed. □

As an immediate consequence of Theorem 3.5, we can give the following.

Corollary 3.6. *Let M be a Lorentzian α-Sasakian manifold admitting a Ricci soliton (g, V, λ) such that V is an affine conformal vector field. Then, the vector field V is affine and (M, g, V, λ) is an Einstein manifold.*

Theorem 3.7. *Let $(M, \varphi, \xi, \eta, g)$ be a non-projectively flat Lorentzian α-Sasakian manifold admitting an η-Ricci soliton (g, ξ, λ, μ) and the curvature condition $Q.P = 0$ is satisfied. Then, the η-Ricci soliton (g, ξ, λ, μ) is steady if and only if M satisfies the curvature condition $\varphi.P = 0$.*

Proof. It follows from (1.2) that we have

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{3.34}$$

for any $X, Y \in \Gamma(TM)$. Using the definition of Lie derivative and from (2.8), we have

$$(\pounds_\xi g)(X,Y) = 2\alpha g(X, \varphi Y). \tag{3.35}$$

From (3.34) and (3.35), we acquire

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y) - \alpha g(\varphi X, Y). \tag{3.36}$$

Setting $X = Y = \xi$ in (3.35) and using (2.16), one has $\mu = \lambda + 2n\alpha^2$. Using this fact in (3.36), we arrive at

$$S(X,Y) = -\lambda g(X,Y) - (\lambda + 2n\alpha^2)\eta(X)\eta(Y) - \alpha g(\varphi X, Y)$$

which yields

$$QX = -\lambda X - (\lambda + 2n\alpha^2)\eta(X)\xi - \alpha\varphi X \tag{3.37}$$

for any $X, Y \in \Gamma(TM)$. On the other hand, suppose that the manifold M satisfies the condition $(Q.P)(X,Y)Z = 0$, namely

$$Q(P(X,Y)Z) - P(QX,Y)Z - P(X,QY)Z - P(X,Y)QZ = 0 \tag{3.38}$$

for any $X, Y, Z \in \Gamma(TM)$. Substituting (3.37) into (3.38) provides

$$\begin{aligned} 2\lambda P(X,Y)Z + (\lambda + 2n\alpha^2)\Big\{&\eta(X)P(\xi,Y)Z + \eta(Y)P(X,\xi)Z \\ &+\eta(Z)P(X,Y)\xi - \eta(P(X,Y)Z)\xi\Big\} - \alpha(\varphi.P)(X,Y)Z = 0, \end{aligned} \tag{3.39}$$

where $(\varphi.P)(X,Y)Z = \varphi(P(X,Y)Z) - P(\varphi X,Y)Z - P(X,\varphi Y)Z - P(X,Y)\varphi Z = 0$. Also, making use of the equations (2.11)-(2.15) and after a straight forward computation, we obtain

$$P(\xi,Y)Z = \alpha^2 g(Y,Z)\xi - \frac{1}{2n}S(Y,Z)\xi, \tag{3.40}$$

$$P(X,\xi)Z = -\alpha^2 g(X,Z)\xi + \frac{1}{2n}S(X,Z)\xi, \tag{3.41}$$

$$P(X,Y)\xi = 0 \tag{3.42}$$

and

$$\begin{aligned} \eta(P(X,Y)Z) = &-\alpha^2\eta(Y)g(X,Z) + \alpha^2\eta(X)g(Y,Z) - \frac{1}{2n}S(Y,Z)\eta(X) \\ &+\frac{1}{2n}S(X,Z)\eta(Y). \end{aligned} \tag{3.43}$$

Putting the equalities (3.40)-(3.43) in (3.39) gives

$$2\lambda P(X,Y)Z = \alpha(\varphi.P)(X,Y)Z. \tag{3.44}$$

Since M is a non-projectively flat Lorentzian α-Sasakian manifold, this completes the proof. □

Theorem 3.8. *Let M be a Lorentzian α-Sasakian manifold admitting a Ricci soliton (g, V, λ) and $R.Q = 0$. If V is orthogonal to ξ, then V is a Killing vector field and M is an Einstein manifold.*

Proof. Let us suppose that a Lorentzian α-Sasakian manifold satisfies the condition $(R(X,Y).Q)Z = 0$, that is,

$$R(X,Y)QZ - Q(R(X,Y)Z) = 0 \tag{3.45}$$

for any $X, Y, Z \in \Gamma(TM)$, where Q denotes the Ricci operator defined by $S(X,Y) = g(QX,Y)$. Taking $X = \xi$ in (3.45), we write

$$R(\xi, Y)QZ - Q(R(\xi, Y)Z) = 0. \tag{3.46}$$

Using the equalities (2.12), (2.14) and (2.15) in (3.46) we get

$$\alpha^2(S(Y,Z)\xi - 2n\alpha^2\eta(Z)Y) - 2n\alpha^4 g(Y,Z)\xi + \alpha^2\eta(Z)QY = 0. \tag{3.47}$$

If we take the inner product of (3.47) with ξ and using (2.2), (2.14), then we obtain

$$-\alpha^2 S(Y,Z) + 2n\alpha^4 g(Y,Z) = 0. \tag{3.48}$$

Therefore, we have

$$-\alpha^2(S(Y,Z) - 2n\alpha^2 g(Y,Z)) = 0,$$

which implies that

$$S(Y,Z) = 2n\alpha^2 g(Y,Z). \tag{3.49}$$

This shows that M is an Einstein manifold. On the other hand, since (g, V, λ) is a Ricci soliton on M, the equation (1.1) can be written as

$$2S(X,Y) = -(\pounds_V g)(X,Y) - 2\lambda g(X,Y). \tag{3.50}$$

Substituting $X = Y = \xi$ in (3.50) and keeping in mind (2.16) yield

$$-4n\alpha^2 = -2g(\nabla_\xi V, \xi) + 2\lambda. \tag{3.51}$$

If V is orthogonal to ξ and use the fact that $\nabla_\xi \xi = 0$, we get

$$\lambda = -2n\alpha^2. \tag{3.52}$$

Using (3.49) and (3.52) in (3.50), we find $(\pounds_V g)(X,Y) = 0$. This means that the vector field V is Killing on M. Hence, we get the requested result. □

Example 3.2. [2] We consider the three-dimensional Riemannian manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$ and the linearly independent vector fields

$$e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_3 = \alpha \frac{\partial}{\partial z},$$

where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$ and α is a constant. Let g be the Lorentzian metric defined by

$$g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0$$
$$g(e_1,e_1) = g(e_2,e_2) = 1, g(e_3,e_3) = -1.$$

and is given by

$$g = \frac{1}{(e^z)^2}(dy)^2 - \frac{1}{\alpha^2}(dz)^2.$$

Also, let η, φ be the 1-form and the $(1,1)$-tensor field, respectively defined by

$$\eta(Z) = g(Z, e_3), \quad \varphi(e_1) = -e_1, \quad \varphi(e_2) = -e_2, \quad \varphi(e_3) = 0$$

for any $Z \in \Gamma(TM)$. By direct calculations, we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\alpha e_1 \quad and \quad [e_2, e_3] = -\alpha e_2.$$

On the other hand, using Koszul's formula for the Riemannian metric g, we get:

$$\nabla_{e_1} e_1 = -\alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1 \tag{3.53}$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2, \tag{3.54}$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \tag{3.55}$$

Therefore, the manifold M is a 3-dimensional Lorentzian α-Sasakian manifold. Using the equations (3.53), (3.54) and (3.55), we find

$$\begin{aligned}
&R(e_1, e_2)e_3 = 0, && R(e_1, e_3)e_2 = 0, && R(e_2, e_3)e_1 = 0,\\
&R(e_1, e_2)e_2 = \alpha^2 e_1, && R(e_1, e_2)e_1 = -\alpha^2 e_2, && R(e_1, e_3)e_3 = -\alpha^2 e_1,\\
&R(e_1, e_3)e_1 = -\alpha^2 e_3, && R(e_2, e_3)e_3 = -\alpha^2 e_2, && R(e_3, e_2)e_2 = \alpha^2 e_3
\end{aligned}$$

which yields $S(e_1, e_1) = 0, S(e_2, e_2) = 0, S(e_3, e_3) = -2\alpha^2$ and $S(e_i, e_j) = 0$ for all $i, j = 1, 2, 3$ $(i \neq j)$. In this case, the manifold M defines an η-Ricci soliton with potential vector field $e_3 = \xi$ which satisfies (1.1) for $\lambda = -\alpha$ and $\mu = 2\alpha^2 - \alpha$.

Also, we remark that this example verifies Proposition 3.1 and Proposition 3.2.

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