On lifts of Metallic structures related to Weil functors

G. F. Wankap Nono, D. Dehainsala, A. Ntyam and A. Maloko Mavamou

Abstract. The aim of this paper is to study the lift of metallic structures on a pseudo-Riemannian manifold M, to its Weil bundle $T^A M$ (where A is a Weil algebra) and we establish some properties. In particular, we investigate the parallelism, half parallelism and anti-half parallelism of the complete-lift related to Weil functors of the distributions associated to the classical metallic structure.

M.S.C. 2010: Primary 53C15, 53C25, 58A32; Secondary 58A05.
Key words: Weil bundles; lifts; Metallic-structure; integrability; parallelism.

1 Introduction

The differential geometry of polynomial structures on manifolds has been defined by S.I. Goldberg, K. Yano and N.C. Petridis in [10], [11] and [26]. A particular case of polynomial structures called metallic structures, was defined by C.E. Hreţcanu and M. Crâşmăreanu, M. Ozkan and F. Yilmaz in [15] and [24], as a generalization of golden structures on manifolds. Some properties of the concepts of submanifolds of Riemannian manifolds endowed with metallic Riemannian structures were investigated by many authors such as C.E. Hreţcanu and A.M. Blaga [16]-[17]. Being inspired by the metallic proportions introduced by V.W. de Spinadel in [5]-[6], the concepts of metallic manifolds were studied in [5]-[6] as an affinor structure ζ on M satisfying $\zeta^2 - a\zeta - b\delta_M = 0$, where a and b are fixed positive integers and δ_M is the identity affinor structure on M. The members of metallic proportions play an important role to establish the connections between Mathematics and architecture, music... For example Golden and Silver mean have being seen in the sacred art of Egypt, India, China and other different ancient civilizations.

Product preserving bundle functors on manifolds still called Weil functors were classified by [7]. Indeed this author has shown in particular that the set of equivalence classes of such functors are in bijection with the set of equivalence classes of Weil algebras. These functors were used by many authors (ex. [8], [19]) to present some lifts of various geometric objects (smooth functions, tensor fields, linear connections on manifolds, etc.)

Differential Geometry - Dynamical Systems, Vol.24, 2022, pp. 215-231.

In this paper, we present some lifts (related to a Weil bundle functor) of Metallic structures on a smooth pseudo-Riemannian manifold M and we establish some properties. This generalize the results of [2] to the Weil bundles.

The paper has seven sections and is organized as follows: In section 2, we review the notion of Weil algebra, Weil functor (in terms of its algebraic description). We recall in section 3, some basic definitions and properties of the Metallic-structure. The section 4 is devoted to the lift of the Metallic-structure to Weil bundles and studying some properties of these lifts. In section 5 and 6 We discuss of integrability, parallelism, half parallelism and anti-half parallelism of Metallic-structure and the associated distributions to Weil bundles. We finish the last section by studying the lifts of Metallic pseudo-Riemannian structures on a smooth manifold M.

2 Algebraic description of Weil functor

2.1 Weil algebra

A *Weil algebra* is a finite-dimensional quotient of the algebra of germs $\mathcal{E}_p = C_p^\infty(\mathbb{R}^p, \mathbb{R})$ $(p \in \mathbb{N})$. We denote by $\mathcal{M}_p$ the ideal of germs vanishing at 0 is the maximal ideal of the local algebra $\mathcal{E}_p$.

Equivalently, a Weil algebra is a real commutative unital algebra such that $A = \mathbb{R} \cdot 1_A \bigoplus N$, where N is a finite dimensional ideal of nilpotent elements.

Example 2.1. (1) The algebra of real numbers $\mathbb{R} = \mathbb{R} \cdot 1_{\mathbb{R}} \bigoplus \{0\} = \mathcal{E}_p/\mathcal{M}_p$ is a Weil algebra.

(2) The algebra of dual numbers $\mathbb{D} = \mathbb{R} \cdot 1 \bigoplus \varepsilon\mathbb{R} = \mathcal{E}_p/\mathcal{M}_p^2$ (where ε is such that $\varepsilon^2 = 0$) is a Weil algebra.

(3) The algebra of jet $J_0^r(\mathbb{R}^p, \mathbb{R}) = \mathbb{R} \cdot 1 \bigoplus J_0^r(\mathbb{R}^p, \mathbb{R})_0 = \mathcal{E}_p/\mathcal{M}_p^{r+1}$ is a Weil algebra.

2.2 The Weil functor $T^A : \mathcal{M}f \to \mathcal{FM}$

The category of finite dimensional differential manifolds and mappings of class C^∞ will be denoted by $\mathcal{M}f$.
The category of fibered manifolds and fibered manifolds morphisms will be denoted by $\mathcal{FM}$.
The category of sets and mappings will be denoted $\mathcal{E}ns$.

Let us recall this construction of Weil functors based on [25]. For a Weil algebra $A = \mathbb{R}\cdot 1_A \bigoplus N$ and any point x of a manifold M, let $C_x^\infty(M, \mathbb{R})$ and $Hom(C_x^\infty(M, \mathbb{R}), A)$ be respectively the algebra of germs on x of smooth functions and the set of algebra homomorphisms from $C_x^\infty(M, \mathbb{R})$ into A. One defines a functor $T^A : \mathcal{M}f \to \mathcal{E}ns$ by:

$$T^A M := \bigcup_{x \in M} Hom(C_x^\infty(M, \mathbb{R}), A) \text{ and } \left(T^A f\right)_x (\varphi_x) := \varphi_x \circ f_x^*,$$

for a manifold M and $f \in C^\infty(M, M')$, where $f_x^* \in Hom\left(C_{f(x)}^\infty(M', \mathbb{R}), C_x^\infty(M, \mathbb{R})\right)$ is the pull-back algebra homomorphism defined by

$$f^*\left(germ_{f(x)}(h)\right) = germ_x(h \circ f)$$

Now, let $q_{A,M} : T^A M \to M, \left(T^A M\right)_x \ni \varphi \mapsto x$; hence $\left(T^A M, M, q_{A,M}\right)$ is a well-defined fibered manifold. Indeed let $c = (U, u^i)$, $1 \le i \le m$ be a chart of M; then the map

$$\begin{array}{rccl} \phi_c : & (q_{A,M})^{-1}(U) & \longrightarrow & U \times N^m \\ & \varphi_x & \longmapsto & \left(x, \varphi_x \left(germ_x(u^i - u^i(x))\right)\right) \end{array}$$

is a local trivialization of $T^A M$. Given another manifold M' and a smooth map $f : M \to M'$, $T^A f$ is a fibered map. Indeed for charts $c = (U, u, m)$, $c' = (W, w, m')$ of M, M' such that $f(U) \subset W$, $\phi_{c'} \circ T^A f \circ \phi_c^{-1}$ is the map

$$\begin{array}{rcl} U \times N^m & \to & W \times N^{m'} \\ (x, n_i) & \mapsto & (f(x), n'_j) \end{array}$$

where $n'_j = \sum\limits_{\alpha \in \mathbb{N}^m \setminus \{0\}} \frac{1}{\alpha!} D_\alpha (w^j \circ f \circ u^{-1})(u(x)) n_1^{\alpha_1} \ldots n_m^{\alpha_m}$, $1 \le j \le m'$ with $D_\alpha F^j = \frac{\partial^{|\alpha|} F^j}{(\partial x^1)^{\alpha_1} \ldots (\partial x^m)^{\alpha_m}}$.

$T^A : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor called the *Weil functor* associated to A.

Example 2.2. If $A = J_0^r(\mathbb{R}^p, \mathbb{R})$, then T^A is equivalent to the functor T_p^r of (p, r)-velocities, and if $A = \mathbb{D}$, then $T^A = T$ the tangent bundle functor.

3 Metallic pseudo-Riemannian manifolds

Let M be a smooth manifold and $\mathcal{T}_q^p(M)$ the $C^\infty(M)$-module of tensor fields of (p, q)-type on M. An element of $\mathcal{T}_1^1(M)$ is usually called an affinor structure on M. Let $\Gamma(TM)$ be the $C^\infty(M)$-module of all vector fields on M.

Definition 3.1. ([10]-[11]-[26]) An affinor structure ζ on M defines a polynomial structure if it satisfies the following algebraic equation

$$Q(X) = X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \delta = 0 \tag{3.1}$$

where $\zeta^{n-1}(x)$, $\zeta^{n-2}(x)$, $\cdots$, $\zeta(x)$ and δ are linearly independent for every $x \in M$ and δ is the identity transformation affinor. The polynomial $Q(X)$ is called the structure polynomial.

Remark 3.2. In particular, an almost product structure ρ (resp. an almost complex structure ν) is an affinor which satisfies the algebraic equation $Q(X) = X^2 - \delta = 0$ (resp. $Q(X) = X^2 + \delta = 0$). When $Q(X) = X^2$ (resp. $Q(X) = X^2 - X - \delta$), we have the notion of almost tangent structure τ (resp. Golden-structure ξ) (See [3]).

Definition 3.3. ([15]-[24])Let (M, g) be a pseudo-Riemannian manifold. A Metallic-structure on (M, g) is a given non-null affinor structure ζ of class C^∞ on M which verifies the following equation

$$\zeta^2 - a\zeta - b\delta_M = 0 \tag{3.2}$$

where δ_M is the identity transformation affinor structure on M and a and b are positive integers.

We say that the metric g is ζ-compatible if we have the following equality

$$g(\zeta X, Y) = g(X, \zeta Y) \tag{3.3}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. If we substitute Y into ζY in (4.4), then equation (4.4) may also be written as

$$g(\zeta X, \zeta Y) = g(\zeta^2 X, Y) = g((a\zeta + b\delta_M)X, Y) = ag(\zeta X, Y) + bg(X, Y). \tag{3.4}$$

Definition 3.4. ([15]-[24]) A Metallic pseudo-Riemannian manifold is triple (M, g, ζ), where (M, g) is a pseudo-Riemannian manifold, ζ is a Metallic-structure on (M, g) and g is ζ-compatible. Hence, the pair (g, ζ) is named the Metallic pseudo-Riemannian structure on M.

Definition 3.5. ([1]) Let F be a smooth map from a Metallic pseudo-Riemannian manifold (M, g, ζ) to a Metallic pseudo-Riemannian manifold (N, h, ξ). Then F is called a Metallic map if the following condition is satisfied.

$$dF \circ \zeta = \xi \circ dF. \tag{3.5}$$

In [24], the authors proved the following proposition which show the connection between almost product structure and Metallic-structure on M.

Proposition 3.1. *([24])Let (M, g) be a pseudo-Riemannian manifold.*

(i) Any Metallic-structure ζ on M induces two almost product structures on M defined as follows

$$\rho_- = -\frac{1}{2\sigma_{a,b} - a}(2\zeta - a\delta_M) \quad \textit{and} \quad \rho_+ = \frac{1}{2\sigma_{a,b} - a}(2\zeta - a\delta_M). \tag{3.6}$$

(ii) Conversely, any almost product structure ρ on M induces two Metallic-structures on M defined as follows

$$\zeta_- = \frac{1}{2}\Big(a\delta_M - (2\sigma_{a,b} - a)\rho\Big) \quad \textit{and} \quad \zeta_+ = \frac{1}{2}\Big(a\delta_M + (2\sigma_{a,b} - a)\rho\Big). \tag{3.7}$$

Let (M, g, ζ) be a Metallic pseudo-Riemannian manifold. According to [24]-[3], we define these two operators

$$\phi = \frac{1}{2\sigma_{a,b} - a}((\sigma_{a,b} - a)\delta_M + \zeta) \quad \text{and} \quad \psi = \frac{1}{2\sigma_{a,b} - a}(\sigma_{a,b}\delta_M - \zeta) \tag{3.8}$$

where the metallic proportions $\sigma_{a,b} = \dfrac{a + \sqrt{a^2 + 4b}}{2}$ is the positive root of the algebraic equation $t^2 - at - b = 0$ and where a and b are two fixed positive integers. We can easily have these following equalities

$$\phi^2 = \phi, \;\; \psi^2 = \psi, \;\; \psi \circ \phi = \phi \circ \psi = 0 \;\; \text{and} \;\; \psi + \phi = \delta_M. \tag{3.9}$$

This means that, ϕ and ψ are projection operators splitting the tangent bundle TM into two complementary parts, and define two globally complementary distributions Φ and Ψ of TM (see [24]) as follows

$$\begin{aligned}\Phi &= \bigcup_{x\in M}\{X_x \in T_xM:\ \ \zeta(X_x) = \sigma_{a,b}X_x\}\\ \Psi &= \bigcup_{x\in M}\{X_x \in T_xM:\ \ \zeta(X_x) = (a-\sigma_{a,b})X_x\}.\end{aligned}$$

The projection operators ϕ and ψ verify this following equalities

$$\zeta = \sigma_{a,b}\phi + (a-\sigma_{a,b})\psi \tag{3.10}$$

$$\zeta\circ\phi = \phi\circ\zeta = \sigma_{a,b}\phi \ \text{ and } \ \zeta\circ\psi = \psi\circ\zeta = (a-\sigma_{a,b})\psi. \tag{3.11}$$

4 Lifts of Metallic-structures to Weil bundles

Let M be a smooth manifold and A a Weil algebra. For a weil functor $T^A : \mathcal{M}_f \to \mathcal{F}M$, let $\kappa : T^A\circ T \to T\circ T^A$ be the canonical natural isomorphism associated to the exchange isomorphism $\mathbb{D}\otimes_{\mathbb{R}} A \cong A\otimes_{\mathbb{R}}\mathbb{D}$ see ([19]). Using the definition of composed functors $T^A\circ T$ and $T\circ T^A$, one can check that locally

$$\kappa_M : \left(u^i, \dot{u}^i, u^i_\alpha, \left(\dot{u}^i\right)_\alpha\right) \mapsto \left(u^i, u^i_\alpha, \dot{u}^i, \left(\dot{u}^i\right)_\alpha\right)$$

For a given affinor ζ (resp. vector field X) on M, one define an affinor structure (resp. a vector field) on T^AM called the complete lift of ζ (resp. X) from M to T^AM and denoted ζ^c (resp. X^c) as follows (see [8]-[19])

$$\zeta^c := \kappa_M\circ T^A\zeta\circ\kappa_M^{-1} \ \ (\text{resp. } \ X^c := \kappa_M\circ T^AX). \tag{4.1}$$

The authors of [8]-[19]-[22] showed that

$$\alpha X^c + \beta Y^c = (\alpha X+\beta Y)^c, \quad [X^c, Y^c] = ([X,Y])^c \ \text{ and } \ \zeta^c(X^c) = (\zeta(X))^c \tag{4.2}$$

for all affinor structure ζ, all vector fields X, Y on M and all reals α, β.

From relation (4.1), we easily show that

$$\zeta^c\circ\xi^c = (\zeta\circ\xi)^c \tag{4.3}$$

for all $\zeta,\ \xi\in\mathcal{T}^1_1(M)$. When $\zeta=\xi$, equation (4.3) becomes

$$(\zeta^c)^2 = (\zeta^2)^c. \tag{4.4}$$

Hence, we have this following result

Proposition 4.1. *Let ζ be an affinor on M. The following assertions are equivalent*

(i) ζ is a Metallic-structure on M.

(ii) ζ^c *is a Metallic-structure on* T^AM.

(iii) $a\delta_{T^AM} - \zeta^c$ *is a Metallic-structure on* M^A.

Proof. It comes from, equations (4.2) and (4.4) and the fact that $(\delta_M)^c = \delta_{T^AM}$. □

Proposition 4.2. *Let* (M, g, ζ) *a Metallic pseudo-Riemannian manifold.*

(i) The Metallic-structure ζ^c *on* T^AM *is an isomorphism on* $T_\varphi(T^AM)$, *for every* $\varphi \in T^AM$.

(ii) The Metallic-structure ζ^c *on* T^AM *is invertible and its inverse* $(\zeta^c)^{-1} = \frac{1}{b}\zeta^c - \frac{a}{b}\delta_{T^AM}$ *is not a Metallic-structure on* T^AM, *but it satisfies the equation*

$$b\Big((\zeta^c)^{-1}\Big)^2 + a(\zeta^c)^{-1} - \delta_{T^AM} = 0.$$

Remark 4.1. (See [8]) Let $\nu \in \mathcal{T}_1^1(M)$ be an almost complex (resp. almost product) structure on M. Then ν^c and $-\nu^c$ are almost complex (resp. almost product) structures on T^AM. If τ is an almost tangent structure on M, then τ^c (resp. -τ^c) is also an almost tangent structure on T^AM.

The following proposition shows the relationship between Metallic and product structures on T^AM.

Proposition 4.3. *Let* M *be a smooth manifold.*

(i) If ζ *is the Metallic-structure on* M, *then the Metallic-structure* ζ^c *on* T^AM *induces two almost product structures on* T^AM *defined as follows*

$$\rho_-^c = -\frac{1}{2\sigma_{a,b} - a}(2\zeta^c - a\delta_{T^AM}) \quad \text{and} \quad \rho_+^c = \frac{1}{2\sigma_{a,b} - a}(2\zeta^c - a\delta_{T^AM}).$$

(ii) Conversely, if ρ *is an almost product structure on* M, *then the almost product* ρ^c *on* T^AM *induces two Metallic-structure on* T^AM *defined as follows*

$$\zeta_-^c = \frac{1}{2}\Big(a\delta_{T^AM} - (2\sigma_{a,b} - a)\rho^c\Big) \quad \text{and} \quad \zeta_+^c = \frac{1}{2}\Big(a\delta_{T^AM} + (2\sigma_{a,b} - a)\rho^c\Big).$$

According to M. Ozkan in [24], we have the following remark

Remark 4.2. (a) If τ is an almost tangent structure on M, then its complete lift τ^c induces an affinor υ^c on T^AM defined as follows

$$\upsilon^c = \frac{1}{2}(\delta_{T^AM} + (2\sigma_{a,b} - a)\tau^c)$$

and which is called tangent Metallic-structure on T^AM. This tangent Metallic-structure satisfies the equation

$$(\upsilon^c)^2 - \upsilon^c + \frac{a}{4}\delta_{T^AM} = 0.$$

(b) If ν is the complex structure on M, then its complete lift ν^c induces an affinor μ^c on T^AM defined as follows

$$\mu^c = \frac{1}{2}(\delta_{T^AM} + (2\sigma_{a,b} - a)\nu^c)$$

and which is called complex Metallic-structure on T^AM. This complex Metallic-structure satisfies the equation

$$(\mu^c)^2 - \mu^c + \frac{a^2+2b}{2}\delta_{T^AM} = 0.$$

Example 4.3. (complete lifts of triple structures on M) Let ξ, ρ and ν be three affinors structures on the smooth manifold M such that $\nu = \zeta \circ \rho$. According to [24] and [4], the triple (ζ, ρ, ν) is called almost hyperproduct structure (*ahps*), almost biproduct complex structure (*abpcs*), almost product bicomplex structure (*apbcs*) and almost hypercomplex structure (*ahcs*) on M if ζ, τ and ν verify respectively the following equalities:
$\xi^2 = \rho^2 = \nu^2 = \xi \circ \rho \circ \nu = \delta_M$, $\xi^2 = \rho^2 = -\nu^2 = \xi \circ \rho \circ \nu = \delta_M$,
$-\xi^2 = \rho^2 = \nu^2 = \xi \circ \rho \circ \nu = -\delta_M$ and $\xi^2 = \rho^2 = \nu^2 = \xi \circ \rho \circ \nu = -\delta_M$.
Let

$$\begin{aligned}\widetilde{\xi}^c_- &= \frac{1}{2}(\delta_{T^AM} - (2\sigma_{a,b} - a)\xi^c), \ \ \widetilde{\rho}^c_- = \frac{1}{2}(\delta_{T^AM} - (2\sigma_{a,b} - a)\rho^c) \ \text{ and}\\ \widetilde{\nu}^c_- &= \frac{1}{2}(\delta_{T^AM} - (2\sigma_{a,b} - a)\nu^c)\end{aligned}$$

$$\begin{aligned}(\text{resp. } \widetilde{\xi}^c_+ &= \frac{1}{2}(\delta_{T^AM} + (2\sigma_{a,b} - a)\xi^c), \ \ \widetilde{\rho}^c_+ = \frac{1}{2}(\delta_{T^AM} + (2\sigma_{a,b} - a)\rho^c) \ \text{ and}\\ \widetilde{\nu}^c_+ &= \frac{1}{2}(\delta_{T^AM} + (2\sigma_{a,b} - a)\nu^c))\end{aligned}$$

be the induced structures associated to ξ^c, ρ^c and ν^c respectively (see proposition 4.3). Hence, those induced structures verify this following equality

$$(2\sigma_{a,b} - a)\widetilde{\nu}^c = 2\widetilde{\xi}^c \circ \widetilde{\rho}^c - \widetilde{\xi}^c - \widetilde{\rho}^c + \sigma\delta_{T^AM}$$

and the triple $(\widetilde{\xi}^c_-, \widetilde{\rho}^c_-, \widetilde{\nu}^c_-)$ and $(\widetilde{\xi}^c_+, \widetilde{\rho}^c_+, \widetilde{\nu}^c_+)$ are:

1. (*ahps*) (resp. (*apbcs*)) on T^AM if and only if (ξ, ρ, ν) is an (*ahps*) (resp. (*apbcs*)) on M. In this case, $\widetilde{\nu}^c$ is a Metallic-structure on T^AM.
2. (*abpcs*) (resp. (*ahcs*)) on T^AM if and only if (ξ, ρ, ν) is an (*abpcs*) (resp. (*ahcs*)) on M. In this case, $\widetilde{\nu}^c$ is a complex Metallic-structure on T^AM.

5 Integrability of Metallic-structures to bundles of near points

The aims of this section is to investigate the condition of integrability of the Metallic-structure ζ^c on T^AM and its associated distributions. Let (M, ζ) be a Metallic manifold and A a given Weil algebra. We recall that ζ is integrable if its Nijenhuis tensor field

$$N_\zeta(X,Y) = \zeta^2[X,Y] + [\zeta X, \zeta Y] - \zeta[\zeta X, Y] - \zeta[X, \zeta Y]$$

vanishes identically for all vector fields X, Y in M. According to the above, we have the following definition

Definition 5.1. The Metallic-structure ζ^c on T^AM is integrable if

$$N_{\zeta^c}(X^c, Y^c) = 0$$

for all vector fields X, Y in M .

Proposition 5.1. *A Metallic-structure ζ^c on T^AM is integrable if and only if the Metallic-structure ζ on M is integrable.*

Proof. It comes from the fact that

$$N_{\zeta^c}(X^c, Y^c) = (N_\zeta(X,Y))^c \text{ by using relations (4.2) and (4.3)}$$

for all vector fields X, Y in M. □

From relations (4.2) and (4.3), we can construct the complete lift of these two projection operators ϕ and ψ on M as follows

$$\phi^c = \frac{1}{2\sigma_{a,b} - a}((\sigma_{a,b} - a)\delta_{T^AM} + \zeta^c) \text{ and } \psi^c = \frac{1}{2\sigma_{a,b} - a}(\sigma_{a,b}\delta_{T^AM} - \zeta^c) \tag{5.1}$$

which satisfy these following equalities

$$(\phi^c)^2 = \phi^c, \ (\psi^c)^2 = \psi^c, \ \phi^c \circ \psi^c = \psi^c \circ \phi^c = 0 \ \text{ and } \ \phi^c + \psi^c = \delta_{T^AM} \tag{5.2}$$

$$\phi^c \circ \zeta^c = \zeta^c \circ \phi^c = \sigma_{a,b}\phi^c \ \textit{ and } \ \psi^c \circ \zeta^c = \zeta^c \circ \psi^c = (a - \sigma_{a,b})\psi^c. \tag{5.3}$$

$$\zeta^c = \sigma_{a,b}\phi^c + (a - \sigma_{a,b})\psi^c. \tag{5.4}$$

Therefore, ϕ^c and ψ^c are projection operators splitting the tangent bundle of smooth manifold T^AM into two complementary parts, and define two globally complementary distributions $T^A\Phi$ and $T^A\Psi$ of the tangent bundle $T(T^AM)$.

Let's recall this result of M. Ozkan and F. Yilmaz

Proposition 5.2. *([24])Let (M, ζ) be the Metallic-manifold.*
The distribution Φ (resp. Ψ) is integrable if and only if the vector field $[\phi X, \phi Y]$ (resp. $[\psi X, \psi Y]$ pertains $\Gamma(\Phi)$ (resp. $\Gamma(\Psi)$) for all vector fields X, Y in M.

From the above result, we have this following definition

Definition 5.2. The distribution $T^A\Phi$ (resp. $T^A\Psi$) is integrable if the vector field $[\phi^cX^c, \phi^cY^c]$ (resp. $[\psi^cX^c, \psi^cY^c]$) belongs to $\Gamma(T^A\Phi)$ (resp. $\Gamma(T^A\Psi)$) for all vector fields X and Y in M.

Hence, we have these following results

Proposition 5.3. *Let (M,ζ) be the Metallic-manifold.*
The distribution $T^A\Phi$ (resp. $T^A\Psi$) is integrable if and only if Φ (resp. Ψ) is integrable.

Proof. By using relations (4.2) and (4.3), the proof will come from the fact that

$$\psi^c[\phi^c X^c, \phi^c Y^c] = (\psi[\phi X, \phi Y])^c \quad (resp.\ \ \phi^c[\psi^c X^c, \psi^c Y^c] = (\phi[\psi X, \psi Y])^c)$$

for all vector fields X and Y in M. □

Proposition 5.4. *Let (M,ζ) be the Metallic-manifold.*
The distribution $T^A\Phi$ (resp. $T^A\Psi$) is integrable if and only if

$$N_{\zeta^c}(\phi^c X^c, \phi^c Y^c) \in \Gamma(T^A\Phi) \ \ (resp. \quad N_{\zeta^c}(\psi^c X^c, \psi^c Y^c) \in \Gamma(T^A\Psi)) \tag{5.5}$$

for all vector fields X, Y in M.

Proof. For all vector fields X and Y in M, one has

$$\begin{aligned}
\psi^c N_{\zeta^c}(\phi^c X^c, \phi^c Y^c) &= \psi^c(\zeta^c)^2[\phi^c X^c, \phi^c Y^c] + \psi^c[\zeta^c \circ \phi^c X^c, z^c \circ \phi^c Y^c] \\
&- \psi^c\zeta^c[\zeta^c \circ \phi^c X^c, \phi^c Y^c] - \psi^c\zeta^c[\phi^c X^c, \zeta \circ \phi^c Y^c] \\
&= (a-\sigma_{a,b})^2\psi^c[\phi^c X^c, \phi^c Y^c] + \sigma_{a,b}^2\psi^c[\phi^c X^c, \phi^c Y^c] \\
&- \sigma_{a,b}(a-\sigma_{a,b})\psi^c[\phi^c X^c, \phi^c Y^c] - \sigma_{a,b}(a-\sigma_{a,b})\psi^c[\phi^c X^c, \phi^c Y^c] \\
&= (2\sigma_{a,b}-a)^2\psi^c[\phi^c X^c, \phi^c Y^c].
\end{aligned}$$

With the same manner, $\phi^c N_{\zeta^c}(\psi^c X^c, \psi^c Y^c) = (2\sigma_{a,b}-a)^2\phi^c[\psi^c X^c, \psi^c Y^c]$. Hence, the proof is completed. □

Proposition 5.5. *Let ζ be a Metallic-structure on M and ζ^c its complete lift. We have these following equalities*

$$\begin{aligned}
N_{\zeta^c}(X^c, Y^c) &= (2\sigma_{a,b}^2 - a\sigma_{a,b})N_{\phi^c}(X^c, Y^c) + (2\sigma_{a,b}^2 - 3a\sigma_{a,b} + a^2)N_{\psi^c}(X^c, Y^c) \\
&= (2\sigma_{a,b}-a)^2 N_{\phi^c}(X^c, Y^c)
\end{aligned}$$

for all vector fields X, Y in M.

Proof. For all vector fields X, Y in M, one has

$$\begin{aligned}
N_{\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c}(X^c,Y^c) &= (\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c)^2[X^c,Y^c] \\
&+ [(\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c)X^c,(\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c)Y^c] \\
&- (\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c)[(\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c)X^c,Y^c] \\
&- (\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c)[X^c,(\sigma_{a,b}\phi^c+(a-\sigma_{a,b})\psi^c)Y^c] \\
&= \sigma_{a,b}N_{\phi^c}(X^c,Y^c)+(a-\sigma_{a,b})^2N_{\psi^c}(X^c,Y^c) \\
&+ \sigma_{a,b}(a-\sigma_{a,b})([\phi^cX^c,\psi^cY^c]-\phi^c[\psi^cX^c,Y^c] \\
&- \phi^c[X^c,\psi^cY^c]) \\
&+ \sigma_{a,b}(a-\sigma_{a,b})([\psi^cX^c,\phi^cY^c]-\psi^c[\phi^cX^c,Y^c] \\
&- \psi^c[X^c,\phi^cY^c]) \\
&= \sigma_{a,b}N_{\phi^c}(X^c,Y^c)+(a-\sigma_{a,b})^2N_{\psi^c}(X^c,Y^c) \\
&+ \sigma_{a,b}(a-\sigma_{a,b})(-N_{\phi^c}(X^c,Y^c)+[\phi^cX^c,Y^c] \\
&- \phi^c[X^c,Y^c]) \\
&+ \sigma_{a,b}(a-\sigma_{a,b})(-N_{\psi^c}(X^c,Y^c)+[\psi^cX^c,Y^c] \\
&- \psi^c[X^c,Y^c]) \text{ since } \phi^c+\psi^c=\delta_{T^AM} \\
&= (2\sigma_{a,b}^2-a\sigma_{a,b})N_{\phi^c}(X^c,Y^c) \\
&+ (2\sigma_{a,b}^2-3a\sigma_{a,b}+a^2)N_{\psi^c}(X^c,Y^c).
\end{aligned}$$

Therefore

$$\begin{aligned}
N_{\zeta^c}(X^c,Y^c) &= (2\sigma_{a,b}^2-a\sigma_{a,b})N_{\phi^c}(X^c,Y^c)+(2\sigma_{a,b}^2-3a\sigma_{a,b}+a^2)N_{\psi^c}(X^c,Y^c) \\
&= (2\sigma_{a,b}-a)^2N_{\phi^c}(X^c,Y^c) \text{ since } \phi^c+\psi^c=\delta_{T^AM}.
\end{aligned}$$

□

Proposition 5.6. *Let ζ be a Metallic-structure on M and ζ^c its complete lift. The following assertions are equivalent:*

(i) ζ^c is integrable.

(ii) Both the distribution $T^A\Phi$ and $T^A\Psi$ are integrable.

Proof. Let X and Y be two vector fields on M. We have

$$\begin{aligned}
\phi^cN_{\zeta^c}(\psi^cX^c,\psi^cY^c)+\psi^cN_{\zeta^c}(\phi^cX^c,\phi^cY^c) &= (2\sigma_{a,b}-a)^2\phi^c[(\delta_{T^AM}-\phi^c)X^c \\
&, (\delta_{T^AM}-\phi^c)Y^c] \\
&+ (2\sigma_{a,b}-a)^2(\delta_{T^AM}-\phi^c)[\phi^cX^c,\phi^cY^c] \\
&= (2\sigma_{a,b}-a)^2(\phi^c)^2[X^c,Y^c] \\
&+ (2\sigma_{a,b}-a)^2[\phi^cX^c,\phi^cY^c] \\
&- (2\sigma_{a,b}-a)^2\phi^c[\phi^cX^c,Y^c] \\
&- (2\sigma_{a,b}-a)^2\phi^c[X^c,\phi^cY^c] \\
&= (2\sigma_{a,b}-a)^2N_{\phi^c}(X^c,Y^c) \\
&= N_{\zeta^c}(X^c,Y^c).
\end{aligned}$$

Hence, the proof is completed . □

Corollary 5.7. *Let ζ be a Metallic-structure on M and ζ^c its complete lift on T^AM. the following assertions are equivalent:*

(a) ζ^c is integrable.

(b) Both $T^A\Phi$ and $T^A\Psi$ are integrable.

(c) Both Φ and Ψ are integrable.

(d) ζ is integrable.

Theorem 5.8. *Let ρ be the almost product on a smooth manifold M.*
The almost product ρ^c on T^AM is integrable if an only if the associated Metallic-structure ζ^c_+ (resp. ζ^c_-) is integrable.

Proof. Let $X,\ Y \in \mathfrak{X}(M)$, ρ^c an almost product structure on T^AM and $\zeta^c_- = \frac{1}{2}(\delta_{T^AM} - (2\sigma_{a,b} - a)\rho^c)$ (resp. $\zeta^c_+ = \frac{1}{2}(\delta_{T^AM} + (2\sigma_{a,b} - a)\rho^c)$) the induce Metallic-structure on T^AM. By straightforward calculations from the definitions of the Nijenhuis tensors of ζ and ρ, one has:

$$N_{\zeta^c_-}(X^c, Y^c) = N_{\zeta^c_+}(X^c, Y^c) = \frac{(2\sigma_{a,b} - a)^2}{4} N_{\rho^c}(X^c, Y^c).$$

Hence, the proof follows. □

Conversely, we have this following theorem

Theorem 5.9. *Let ζ be the Metallic-structure on smooth manifold M.*
The Metallic-structure ζ^c on T^AM is integrable if and only if the associated almost product ρ^c_+ (resp. ρ^c_-) is integrable.

6 Parallelism, half parallelism and anti-half parallelism of Metallic structures on T^AM

In this section, we discuss parallelism, half parallelism and anti half parallelism of the distributions associated with the Metallic-structure on weil bundles T^AM. We recall that, a distribution $\mathcal{D}$ on M is called parallel with respect to the linear connection ∇ if the vector field ∇_XY pertains to $\mathcal{D}$ for any vector fields $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D})$. Let ζ be an affinor structure on M. For all vector fields X and Y in M, let's put

$$\triangle\zeta(X, Y) = \zeta(\nabla_XY) - \zeta(\nabla_YX) - \nabla_{\zeta X}Y + \nabla_Y\zeta X.$$

We recall this following definition from [24].

Definition 6.1. ([24])Let (M, ζ) be a Metallic manifold.

(d1) The distribution Φ (resp. Ψ) on M is called half-parallel with respect to the linear connection ∇ if

$$\triangle\zeta(X, Y) \in \Gamma(\Phi) \quad (resp.\ \Gamma(\Psi)), \tag{6.1}$$

for all vector fields $X \in \Gamma(\Phi)$ (resp. $\Gamma(\Psi)$) and $Y \in \Gamma(TM)$.

(d2) The distribution Φ (resp. Ψ) on M is called anti-half parallel with respect to the linear connection ∇ if

$$\triangle\zeta(X,Y) \in \Gamma(\Psi) \quad (resp.\ \Gamma(\Phi)), \tag{6.2}$$

for all vector fields $X \in \Gamma(\Phi)$ (resp. $\Gamma(\Psi)$) and $Y \in \Gamma(TM)$.

Let ∇ be a linear connection on M. Its complete-lift ∇^c is a unique linear connection on T^AM which satisfies this equality

$$\nabla^c_{X^c}Y^c = (\nabla_X Y)^c \tag{6.3}$$

for all vector fields X and Y in M ([19]).

From the above, we have these following definitions

Definition 6.2. Let ∇ be a linear connection on a Metallic-manifold (M,ζ). The distribution $T^A\Phi$ (resp. $T^A\Psi$) is parallel with respect to linear connection ∇^A if

$$\nabla^c_{X^c}Y^c \in \Gamma(T^A\Phi) \ \ (\text{resp.}\ \Gamma(T^A\Psi))$$

for all vector fields $X \in \Gamma(\Phi)$ (resp. $\Gamma(\Psi)$) and $Y \in \Gamma(TM)$.

Definition 6.3. Let ∇ be a linear connection on a Metallic-manifold (M,ζ) .

(d1) The distribution $T^A\Phi$ (resp. $T^A\Psi$) on T^AM is called half-parallel with respect to the linear connection ∇^c if

$$\triangle\zeta^c(X^c,Y^c) \in \Gamma(T^A\Phi) \quad (resp.\ \Gamma(T^A\Psi)), \tag{6.4}$$

for all vector fields $X \in \Gamma(\Phi)$ (resp. $\Gamma(\Psi)$) and $Y \in \Gamma(TM)$.

(d2) The distribution $T^A\Phi$ (resp. $T^A\Psi$) on T^AM is called anti-half parallel with respect to the linear connection ∇ if

$$\triangle\zeta^c(X^c,Y^c) \in \Gamma(T^A\Psi) \quad (resp.\ \Gamma(T^A\Phi)), \tag{6.5}$$

for all vector fields $X \in \Gamma(\Phi)$ (resp. $\Gamma(\Psi)$) and $Y \in \Gamma(TM)$.

Let ∇ be a linear connection on a Metallic-manifold (M,ζ). According to [3]-[19], we can associate to the pair (ζ^c,∇^c) two other linear connections $\overset{Sc^c}{\nabla}$ and $\overset{V^c}{\nabla}$ on T^AM called respectively Schouten and Vrănceanu connections, and defined as follows

$$\overset{Sc^c}{\nabla}_{X^c}Y^c = r^c(\nabla^c_{X^c}r^cY^c) + s^c(\nabla^c_{X^c}s^cY^c)$$

$$\overset{V^c}{\nabla}_{X^c}Y^c = r^c(\nabla^c_{r^cX^c}r^cY^c) + s^c(\nabla^c_{s^cX^c}s^cY^c) + r^c[s^cX^c,r^cY^c] + s^c[r^cX^c,s^cY^c]$$

for all vector fields X and Y in M.

Hence, we have the following results

Theorem 6.1. *The Metallic-structure ζ^c on T^AM is parallel with respect to Schouten and Vrănceanu connections.*

Proof. From the linearity of ∇^c and the relations (5.3)-(5.5), one has

$$\begin{array}{rcl}
(\overset{Sc^c}{\nabla}_{X^c}\zeta^c)Y^c & = & \overset{Sc^c}{\nabla}_{X^c}\zeta^c Y^c - \zeta^c \overset{Sc^c}{\nabla}_{X^c} Y^c \\
& = & \phi^c(\nabla^c_{X^c}\phi^c \circ \zeta^c Y^c) + \psi^c(\nabla^c_{X^c}\psi^c \circ \zeta^c Y^c) - \zeta^c \circ \phi^c(\nabla^c_{X^c}\phi^c Y^c) \\
& - & \zeta^c \circ \psi^c(\nabla^c_{X^c}\psi^c Y^c) \\
& = & \sigma_{a,b}\phi^c(\nabla^c_{X^c}\phi^c Y^c) + (a-\sigma_{a,b})\psi^c(\nabla^c_{X^c}\psi^c Y^c) - \sigma_{a,b}\phi^c(\nabla^c_{X^c}\phi^c Y^c) \\
& - & (a-\sigma_{a,b})\psi^c(\nabla^c_{X^c}\psi^c Y^c) \\
& = & 0.
\end{array}$$

With the same manner, $(\overset{V^c}{\nabla}_{X^c}\zeta^c)Y^c = 0$. □

Theorem 6.2. *The projection operator ϕ^c (resp. ψ^c) is parallel with respect to Schouten and Vrănceanu connections.*

Proof. It comes from relations (5.3)-(5.5) and the linearity of ∇^c and the bracket of vector fields on T^AM. □

Proposition 6.3. *Let ∇ be a linear connection on a Metallic-manifold (M, ζ). The distribution Φ (resp. Ψ) is parallel with respect to a fixed linear connection ∇ on M if and only if $T^A\Phi$ (resp. $T^A\Psi$) is parallel with respect to linear connection ∇^c on T^AM.*

Proof. Let $X \in \Gamma(\Phi)$ (resp. $\Gamma(\Psi)$) and Y be a vector field in M. From relations (6.3) and (5.1) in this order, on has

$$\psi^c\nabla^c_{X^c}Y^c = (\psi\nabla_X Y)^c \quad (resp.\ \ \psi^c\nabla^c_{X^c}Y^c = (\psi\nabla_X Y)^c).$$

Hence

$$\psi^c(\nabla^c_{X^c}Y^c) = 0 \Leftrightarrow \psi(\nabla_X Y) = 0 \quad (resp.\ \ \phi^c(\nabla^c_{X^c}Y^c) = 0 \Leftrightarrow \phi(\nabla_X Y) = 0).$$

□

Theorem 6.4. *The distribution $T^A\Phi$ (resp. $T^A\Psi$) is parallel with respect to the Schouten and Vrănceanu connections for every linear connection ∇^c on T^AM.*

Proof. Let ∇^c be a linear connection on T^AM, $Y \in \Gamma(\Phi)$ and $X \in \Gamma(TM)$. From relations (3.9) and (3.10), We easily have $\phi Y = Y$ and $\psi Y = 0$. Hence

$$\begin{array}{rcl}
\overset{sc^c}{\nabla}_{X^c}Y^c & = & \phi^c(\nabla^c_{X^c}\phi^c Y^c) + \psi^c(\nabla^c_{X^c}\psi^c Y^c) \\
& = & \phi^c(\nabla^c_{X^c}(\phi Y)^A) + \psi^c(\nabla^c_{X^c}(\psi Y)^c) \\
& = & \phi^c(\nabla^c_{X^c}Y^c) \in \Gamma(T^A\Phi) \\
\text{and } \overset{v^c}{\nabla}_{X^c}Y^c & = & \phi^c\left(\nabla^c_{(\phi X)^c}Y^c + [\psi X, Y]^c\right) \in \Gamma(T^A\Phi)
\end{array}$$

It can be proved analogously that the distribution $T^A\Psi$ is parallel with respect to the Schouten and Vrănceanu connections for a linear connection ∇^c. □

Proposition 6.5. *Let ζ be a Metallic-structure, parallel with respect to a linear connection ∇ on M. Then ζ^c is parallel with respect to linear connection ∇^c on T^AM if and only if*

$$(\nabla^c_{X^c}\zeta^c)Y^c = 0$$

for all vector fields X and Y in M.

Proposition 6.6. *Let ∇ be a linear connection on a Metallic manifold (M, ζ) and ∇^c its complete lift on (T^AM, ζ^c). The the distribution Φ (resp. Ψ) on M is half parallel with respect to ∇ if and only if the distribution $T^A\Phi$ (resp. $T^A\Psi$) on T^AM is also half parallel with respect to ∇^c.*

Proof. It comes from relation (6.1) and equality (6.3). □

Proposition 6.7. *Let ∇ be a linear connection on a Metallic manifold (M, ζ) and ∇^c its complete lift on (T^AM, ζ^c). The distribution $T^A\Phi$ (resp. $T^A\Psi$) on T^AM is anti-half parallel with respect to ∇^c.*

Proof. Let $X \in \Gamma(\Phi)$ and $Y \in \Gamma(TM)$. From relations (5.5) and (3.10), we have $\phi \circ \zeta = \sigma_{a,b}\phi$ and $\zeta X = \sigma_{a,b}X$. Hence

$$\phi^c\left(\zeta^c(\nabla^c_{X^c}Y^c) - \zeta^c(\nabla^c_{Y^c}X^c) - \nabla^c_{(\zeta X)^c}Y^c + \nabla^c_{Y^c}(\zeta X)^c\right) = 0$$

since ∇^c is linear. Therefore

$$\triangle\zeta^c(X^c, Y^c) \in \Gamma(T^A\Psi)$$

and $T^A\Phi$ is anti-half parallel with respect to ∇^c. $T^A\Psi$ is also anti-half parallel with respect to ∇^c by using the same method. □

Proposition 6.8. *Let ∇ be a fixed linear connection on Metallic-manifold (M, ζ). The the distribution Φ (resp. Ψ) is half parallel with respect to Schouten and Vrănceanu connections if and only if so is $T^A\Phi$ (resp. $T^A\Psi$).*

7 Lifts of Metallic pseudo-Riemannian structures on M

Let M be a smooth manifold and A a weil algebra. Let $\beta : A \to \mathbb{R}$ be a linear function such that the symmetric form $A \times A \to \mathbb{R}$, $(a, b) \mapsto \beta(ab)$ is non-singular. For all element $g \in \mathcal{T}^0_2M$, one defines

$$g^{(\beta)} = \beta \circ T^Ag \circ (\kappa_M^{-1} \times \kappa_M^{-1}) \tag{7.1}$$

It is an element of $\mathcal{T}^0_2(T^AM)$ called the β-lift of g to T^AM. The authors of [8] have proved that if g is pseudo-Riemannian metric on M, then $g^{(\beta)}$ is also a pseudo-Riemannian metric on T^AM and that if ∇ is the Riemannian connection of g, then ∇^c is also the Riemannian connection of $g^{(\beta)}$.

Hence, we have this following proposition

Proposition 7.1. *If the triple (M, g, ζ) is a Metallic pseudo-Riemannian manifold, then so is the triple $(T^AM, g^{(\beta)}, \zeta^c)$.*

Proof. From the above, $g^{(\beta)}$ is a pseudo-Riemannian metric on T^AM and by proposition 4.1 ζ^c is a Metallic-structure on T^AM. Using relations (4.1) and (7.1), one has

$$\begin{aligned} g^{(\beta)}(\zeta^c X^c, Y^c) &= \beta \circ T^A g \circ (\kappa_M^{-1} \times \kappa_M^{-1})(\zeta^c X^c, Y^c) \\ &= \beta \circ T^A g(T^A(\zeta X), T^A Y) \\ &= \beta \circ T^A \Big(g(\zeta X, Y) \Big) \\ &= \beta \circ T^A \Big(g(X, \zeta Y) \Big) \text{ since } \zeta \text{ is } g\text{-compatible} \\ &= g^{(\beta)}(X^c, \zeta^c Y^c) \end{aligned}$$

for all vector fields X and Y in M. Hence, ζ^c is $g^{(\beta)}$-compatible and the proof is completed. □

Corollary 7.2. *Let (M, g, ζ) be a pseudo-Riemannian manifold. For all vector fields in M, we have*

(a) $g^{(\beta)}(\phi^c X^c, Y^c) = g^{(\beta)}(X^c, \phi^c Y^c)$ *(resp.* $g^{(\beta)}(\psi^c X^c, Y^c) = g^{(\beta)}(X^c, \psi^c Y^c)$*): This means that the projection operators ϕ^c and ψ^c are $g^{(\beta)}$-symmetric*

(b) $g^{(\beta)}(\phi^c X^c, \psi^c Y^c) = 0$*: This means that the distribution $T^A\Phi$ and $T^A\Psi$ are $g^{(\beta)}$-orthogonal.*

(c) $N_{\zeta^c}(\zeta^c X^c, Y^c) = N_{\zeta^c}(X^c, \zeta^c Y^c)$*: This means that the Metallic-structure ζ^c is N_{ζ^c}-symmetric.*

Remark 7.1. If (g, ρ) is a pseudo-Riemannian almost product on M (that is, ρ is a g-symmetric almost product structure on pseudo-Riemannian manifold (M, g)), then the pair $(g^{(\beta)}, \rho^c)$ is also a pseudo-Riemannian almost product on T^AM and the triple $(T^AM, g^{(\beta)}, \zeta_-^c)$ (resp. $(T^AM, g^{(\beta)}, \zeta_+^c)$) is a Metallic pseudo-Riemannian structure on T^AM where ζ_-^c and ζ_+^c are the Metallic-structures on T^AM induced by ρ^c (see proposition 4.3).

Proposition 7.3. *If $f : M \to N$ is a Metallic map between Metallic pseudo-Riemannian manifolds (M, g, ζ) and (N, h, ξ), then $T^Af : T^AM \to T^AN$ is also a Metallic map between Metallic pseudo-Riemannian manifolds $(T^AM, g^{(\beta)}, \zeta^c)$ and $(T^AN, h^{(\beta)}, \xi^c)$.*

Proof. Since f is a Metallic map, then we have

$$df \circ \zeta = \xi \circ df.$$

Hence, one has

$$\begin{aligned} d(T^A f) \circ \zeta^c &= d(T^A f) \circ \kappa_M \circ T^A \zeta \circ \kappa_M^{-1} \\ &= \kappa_N \circ T^A (df) \circ T^A \zeta \circ \kappa_M^{-1} \\ &= \kappa_N \circ T^A (df \circ \zeta) \circ \kappa_M^{-1} \\ &= \kappa_N \circ T^A (\xi \circ df) \circ \kappa_M^{-1} \\ &= \kappa_N \circ T^A \xi \circ \kappa_N^{-1} \circ d(T^A f) \\ &= \xi^c \circ d(T^A f). \end{aligned}$$

This completed the proof. □

References

[1] M.A. Akyol, *Remark on metallic maps between Metallic Riemannian manifolds and constancy of certain maps*, Honam Math. J. 41 (2019), 343-356.

[2] S. Azami, *General natural metallic structure on tangent bundle*, Iran. J. Sci. Technol. Trans. Sci., 42 (2018), 81-88.

[3] M. Crâşmăreanu, C.E. Hreţcanu, *Golden differential geometry*, Chaos Solitons Fractals, 38 (2008), 1229-1238.

[4] V. Cruceanu, *On almost biproduct complex manifolds*, An. St. Univ. Al. I. Cuza, Iasi, Mat. 52 (2006), 5-24.

[5] V. W. de Spinadel, *The metallic means family and multifractal spectra*, Nonlinear Anal. Ser. B, Real World Appl. 36 (1999), 721-745.

[6] V. W. de Spinadel, *The family of metallic means*, Vis. Math. 1 (1999).

[7] D. J. Eck, Product-preserving functors on smooth manifolds. J. Pure Appl. Algebra, 42 (1986), 133-140.

[8] J. Gancarzewicz, W.M. Mikulski, Z. Pogoda, *Lifts of some tensor fields and connections to product preserving functors*, Nagoya Math. J. 135 (1994), 1-14.

[9] A. Gezer, N. Cengiz, A. Salimov, *On integrability of Golden Riemannian structures*, Turkish J. Math. 37 (2013), 693-703.

[10] S.I. Goldberg, N.C. Petridis, *Differentiable solutions of algebraic equations on manifolds*, Kodai Math. Sem. Rep. 25 (1973), 111-128.

[11] S.I. Goldberg, K. Yano, *Polynomial structures on manifolds*, Kodai Math. Sem. Rep. 22 (1970), 199-218.

[12] C. Hreţcanu, *Submanifolds in Riemannian manifold with Golden Structure* , Work- shop on Finsler geometry and its applications, Hungary, 2007.

[13] C.E. Hreţcanu, M. Crâşmăreanu, *On some invariant submanifolds in a Riemannian manifold with golden structure*, An. Stiint. Univ. Al. I. Cuza Iasi, Ser. Mat. (N.S.), 53 (2007), 199-211.

[14] C.E. Hreţcanu, M. Crâşmăreanu, *Applications of the golden ratio on Riemannian manifolds*, Turk J. Math. 33 (2009), 179-191.

[15] C.E. Hreţcanu, M. Crâşmăreanu, *Metallic structures on Riemannian manifolds*, Rev. Un. Mat. Argentina, 54 (2013), 15-27.

[16] C.E. Hreţcanu and A.M. Blaga, *Submanifolds in metallic Riemannian manifolds*, Diffrential Geometry- Dynamical Systems, 20 (2018), 83-97.

[17] C.E. Hreţcanu and A.M. Blaga, *Hemi-slant submanifolds in metallic Riemannian manifolds*, Carpahtian J. Math. 35 (2019), 59-68.

[18] I. Kolář, *Covariant approach to natural transformations of Weil bundles*, Comment. Math. Univ. Carolin. 27 (1986), 723-729.

[19] I. Kolář, J. Slovák, P.W. Michor, *Natural*, perations in differential geometry, Springer-Verlag Berlin Heidelberg, 1993.

[20] A. Morimoto, *Prolongation of G-structures to tangent bundles of higher order*, Nagoya Math. J. 38 (1970), 153-179.

[21] A. Morimoto, *Liftings of tensor fields and connections to tangent bundles of higher order*, Nagoya Math. J. 40 (1970), 99-120.

[22] A. Morimoto, *Prolongation of connections to bundles of infinitely near points*, J. Differential geometry, 11 (1976), 479-498.

[23] W.M. Mikulski, *Product preserving gauge bundle functors on vector bundles*, Colloquium Math, 90 (2001), 277-285.

[24] M. Ozkan, F. Yilmaz, *Metallic structures on differentiable manifolds*, Journal of Science and Arts, 44 (2018), 645-660.

[25] A. Weil, *Théorie des points proches sur les variétés différentiables*, Topologie et Géométrie Différentielle, Colloque du CNRS, Strasbourg, 1953, 97-110.

[26] K. Yano, M. Kon, *Structures on manifolds* , Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 3 (1984).

Author's address:

Georges Florian Wankap Nono
University of Ngaoundere, Faculty of Science,
Department of Mathematics and Computer Science,
P.O. Box 454 Ngaoundéré, Cameroun.
E-mail address: georgywan@yahoo.fr,

Djagwa Dehainsala
University of N'Djamena Tchad,
Faculty of Applied and Exact Sciences, Tchad.
E-mail address: djagwa73@gmail.com

Achille Ntyam
University of Yaoundé I,
Higher Teacher Training College,
P.O. Box 49 Yaoundé, Cameroun.
E-mail address: ntyam_achille@yahoo.fr

Ange Maloko Mavamou
University of Marien Ngouabi Brazzaville, Congo,
Higher Teacher Training College,
Department of Exact Sciences, Congo.
E-mail address: ange.malokomavanga@umng.cg