Curvature properties of homogeneous Finsler space with (α, β)-metric

K. S. Venkatesha, S. K. Narasimhamurthy and K. Chandru

Abstract. In the present paper, we study one of the central problems of Finsler-Geometry, namely the curvature properties of a certain class of homogeneous Finsler spaces. In the first part we describe the existence of some invariant vector fields on a homogeneous Finsler space. In the second part, we derive the formula of S-curvature and prove that homogeneous Finsler space with almost isotropic S-curvature must have vanishing S-curvature. Finally, we deduce the formula of the mean Berwald curvature.

M.S.C. 2010: 22E60, 53C30, 53C60.
Key words: homogeneous Finsler space; G-invariant vector field; special (α, β)-metric; S-curvature; mean Berwald curvature.

1 Introduction

In the literature of Finsler Geometry, one of the recent trending areas of research is the study of geometric properties of special classes of Finsler spaces - such as homogeneous Finsler spaces with (α, β)-metrics $F = \alpha\phi(s)$ - and the curvature of homogeneous Finsler spaces - such as Flag curvature, S-curvature, E-curvature, etc - having in view the usefulness of Finsler geometry in Applied Sciences ([1],[2],[6],[7],[9],[12]).

A Finsler space (M, F) is described as homogeneous space if the group of isometries $I(M, F)$ defined on M. According to S. Deng [10], in a homogeneous space, the manifold M can be expressed as a coset space G/K with (α, β)-metric and the form $F = \alpha\,\phi(s)$, where $s = \frac{\alpha}{\beta}$, α is a G-invariant Riemannian metric and β is G-invariant vector field on G/K. Here, the Lie algebra $\mathfrak{g}$ of G uses the composition of $\mathfrak{h}$ and $\mathfrak{m}$, i.e., $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. It is very interesting to derive geometric quantities on homogeneous Finsler space with different (α, β)-metrics.

In [14], the author derived some curvature properties of homogeneous spaces by applying sectional curvature of left invariant Riemannian metric on a Lie Group. Further, in [18], the authors invented the geometric quantity used to measure the rate of change of the volume form of a Finsler space along the geodesics. Since S-curvature vanishes on Riemannian manifold, it is called as non-Riemannian quantity.

The formula of S-curvature of Finsler spaces with (α, β)-metrics is derived in [5]. Numerous authors ([8],[9],[16],[17],[3],[4]) have studied the curvature properties

Differential Geometry - Dynamical Systems, Vol.24, 2022, pp. 201-214.

of homogeneous Finsler spaces and derived the S-curvature and the mean Berwald curvature for (α, β)-metrics.

In this paper, we study homogeneous Finsler spaces with the special (α, β)-metric $F = \frac{\beta^{m+1}}{\alpha^m}$. In the first part we describe the existance of isometric transform on the manifold and prove that (α, β)-metric is the G-invariant. In the second part, we give an explicit formula for the S-curvature on homogeneous Finsler spaces with special (α, β)-metric. Finally, we derive the formula of the mean Berwald curvature of homogeneous Finsler spaces with special (α, β)-metric.

2 Preliminaries

Here, we have given some definitions and statements of the results which are reference to our coming sections.

Definition 2.1. An n-dimensional connected smooth manifold M is said to be Finsler space if there exist a mapping $F : TM \longrightarrow [0, \infty)$, which satisfies the following conditions.

1. Regular : F is C^∞ on the tangent bundle $TM \setminus \{0\}$.
2. Positive homogeneous : $F(u, \lambda v) = \lambda F(u, v)$, $\lambda \geq 0$.
3. Strong convexity : The $n \times n$ Hessian matrix $(g_{ij}) = \frac{1}{2}[F^2]_{v^i v^j}$ is positive definite at even point on $TM \setminus \{0\}$, where $TM \setminus \{0\}$ represents tangent vectors v is non-zero in the tangent bundle TM.

Definition 2.2. [7] A Finsler metric $F = F(u, v)$ is called as (α, β)-metric if it expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{g_{ij}v^i v^j}$ - Riemannian metric and $\beta = b_i(u)v^i$- is the one form and also $\phi(s)$ satisfies $\phi(s) - s\phi^{'} + (b^2 - s^2)\phi^{''} > 0$, $\forall\ |s| \leq b < b_0$ in $(-b_0, b_0)$.

Example of (α, β)-metrics are Randers metric $F = \alpha + \beta$, Kropina metric $F = \frac{\alpha^2}{\beta}$, Matsumoto metric, infinite series metric etc. In this paper we considered a very interest m^{th} order special (α, β)-metric in the form

$$F = \frac{\beta^{m+1}}{\alpha^m} = s^{m+1} \qquad = \alpha[\phi(s)], \quad \phi(s) = s^{m+1}, \tag{2.1}$$

Definition 2.3. Let (M, F) be a Finsler space. A diffeomorphism $\phi : M \longrightarrow M$ is said to be isometry if $F(\phi(t), d\phi_t(U)) = F(t, U)$, for any $t \in M$ and $U \in T_t M$.

Let $\{e_i\}$ be a basis of an n-dimensional real vector space W and F be a Minkowski norm on W. We denote the volume of a subset in $\mathbb{R}^n$ by Vol and B^n, the open unit ball. The quantity

$$\tau(v) = ln\frac{\sqrt{det(g_{ij}(v))}}{\sigma F}, \quad v \in W \setminus \{0\}, \tag{2.2}$$

where

$$\sigma_F = \frac{Vol(B^n)}{Vol(\zeta)}, \tag{2.3}$$

here,

$$\zeta = \{(v^i) \in \mathbb{R}^n : F(v^i e_i) < 1\}, \tag{2.4}$$

is called the distortion of (W, F). Further, let $\tau(u, v)$ be the distortion of F on $T_u M$. For any tangent vector $v \in T_u(M) \setminus \{0\}$, let $\gamma(t)$ be the geodesic such that $\gamma(0) = u$, $\dot{\gamma}(0) = v$. The rate of change of distortion along the geodesic γ is called the S-curvature and it is denoted by $S(u, v)$, and defined as

$$S(u, v) = \frac{d}{dt}\left[\tau\Big(\gamma(t), \dot{\gamma}(t)\Big)\right]_{t=0}. \tag{2.5}$$

This quantity is positively homogeneous of degree one, i.e., $S(u, \lambda v) = \lambda S(u, v)$, $\lambda > 0$. We can observe that any Riemannian manifold has vanishing S-curvature. Therefore, we can say that S-curvature is a non-Riemannian quantity. Further, there is another quantity that is related to S-curvature, called E-curvature or mean Berwald curvature. The mean Berwald curvature is given by

$$E_{ij}(u, v) = \frac{1}{2}\frac{\partial^2 S(u, v)}{\partial v^i \partial v^j}. \tag{2.6}$$

Definition 2.4. Let G be a Lie group and M is a smooth manifold. If G has a smooth action on M, then G is called a Lie transformation group of M.

Definition 2.5. A connected Finsler space (M, F) is said to be homogeneous Finsler space, if the action of the group of isometries of (M, F), denoted by $I(M, F)$ is transitive on M.

3 Isometry of homogeneous Finsler space with (α, β)-metric

In this section, we have described the isometry of (α, β)-metric by verifying the invariant vector field corresponding to 1-form β on homogeneous Finsler space with special (α, β)-metric

$$F = \frac{\beta^{m+1}}{\alpha^m}, \quad \alpha \neq 0, \tag{3.1}$$

Here, first we prove the following theorem.

Theorem 3.1. *Let (M, α) be a Riemannian space and $\beta = b_i v^i$, a 1-form with $||\beta|| = \sqrt{b_i b^i} < 1$. Then corresponding to β, there exists a smooth vector field U on M with $\alpha(U|_u) < 1$, $\forall\ u \in M$,i.e.,*

$$F(u, v) = \frac{(\langle U|_u, v\rangle)^{m+1}}{\alpha(u, v)^m}, \ \ u \in M, \ \ v \in T_u M, \tag{3.2}$$

where $\langle\, , \rangle$ is the inner product induced by the Riemannian metric α.

Proof. The definition of Riemannian metric can be expressed by the bilinear form as $\langle u, v\rangle = a_{ij}u^i v^j,\ \forall\ u, v \in T_uM$ as an inner product on tangent space T_uM. This inner product generates an inner product on the cotangent space $T_u^*(M)$, and we can define a linear isomorphism between $T_u^*(M)$ and T_uM. Here the 1-form β corresponds to the vector field U on M, given by

$$U|_u = b^i \frac{\partial}{\partial u^i}, \ \ where\ \ b^i = a^{ij} b_j. \tag{3.3}$$

Further we get

$$\langle U|_u, v\rangle = \langle b^i \frac{\partial}{\partial u^i}, v^j \frac{\partial}{\partial u^j}\rangle = b^i v^j a_{ij} = b_j v^j = \beta(v), \tag{3.4}$$

and

$$\alpha(U|_u, v) = ||\beta|| < 1. \tag{3.5}$$

□

Theorem 3.2. *Let (M, F) be a Finsler space with special (α, β)-metric. Then the group of isometries $I(M, F)$ of (M, F) is a closed subgroup of the group of isometries $I(M, \alpha)$ of the Riemannian space (M, α).*

Proof. Let ψ be an isometry of (M, F) and let $u \in M$. Therefore, for every $v \in T_uM$, we have

$$F(u, v) = F(\psi(u), d\psi_u(v)). \tag{3.6}$$

Applying theorem (3.1) we get

$$\frac{(\langle U|_u, v\rangle)^{m+1}}{\alpha(u, v)^m} = \frac{(\langle U|_{\psi(u)},\, d\psi_u(v)\rangle)^{m+1}}{\alpha(\psi(u),\, d\psi_u(v))^m}, \tag{3.7}$$

which gives

$$(\langle U|_u, v\rangle)^{m+1}\alpha(\psi(u), d\psi_u(v))^m = \alpha(u, v)^m(\langle U|_{\psi(u)}, d\psi_u(v)\rangle)^{m+1}, \tag{3.8}$$

by equating α and the corresponding inner product, we get

$$\alpha(u, v) = \alpha(\psi(u), d\psi_u(v))\ \ and\ \ \langle U|_u, v\rangle = (\langle U|_{\psi(u)}, d\psi_u(v)\rangle). \tag{3.9}$$

Therefore ψ is an isometry with respect to the Riemannian metric α and $d\psi_u(U|_u) = U|_{\psi(u)}$. Thus $I(M, F)$ is a closed subgroup of $I(M, \alpha)$. □

In the following theorem we prove the existence of G-invariant vector field with respect to the 1-form β.

Theorem 3.3. *Let $F = \frac{\beta^{m+1}}{\alpha^m}$ be a G-invariant with special (α, β)-metric on G/K. Then α is a G-invariant Riemannian metric and their exists G-invariant vector field U corresponding to the 1-form β.*

Proof. Since F is G-invariant, we have

$$F(v) = F(Ad(k)v), \ \ \forall \ k \in K, \ \ v \in \mathfrak{m}. \tag{3.10}$$

By using theorem (3.1), we get

$$\frac{(\langle U, \, v\rangle)^{m+1}}{\alpha(u, \, v)^m} = \frac{(\langle U, \ Ad(k)v\rangle)^{m+1}}{\alpha(Ad(k)v)^m}, \tag{3.11}$$

By cross multiplication we get,

$$(\langle U, \, v\rangle)^{m+1} \ \alpha(Ad(k)v)^m = (\langle U, \ Ad(k)v\rangle)^{m+1} \ \alpha(u, \, v)^m, \tag{3.12}$$

by equating α and corresponding inner product we get

$$\alpha(u, v) = \alpha(Ad(k)v) \ \ and \ \ \langle U, \, v\rangle = \langle U, \ Ad(k)v\rangle. \tag{3.13}$$

Therefore α is a G-invariant Riemannian metric and $Ad(k)U = U$, hence this shows that the vector field U is also G-invariant. □

4 S-curvature of homogeneous Finsler space with (α, β)-metric $F = \frac{\beta^{m+1}}{\alpha^m}$

In this section, we define the formula of S-curvature of homogeneous Finsler space with special (α, β)-metric $F = \frac{\beta^{m+1}}{\alpha^m}$. The S-curvature measures the rate of change of volume form of a Finsler space along the geodesics. We further define the S-curvature: from the definition (2.3), it is very clear that S-curvature of a Finsler space is directly related with volume forms. There are two important volume forms in Finsler geometry: the Busemann-Hausdorff volume form $dV_{BK} = \sigma_{BK}(u)du$, given by

$$\sigma_{BK}(v) = \frac{Vol(B^n)}{Vol(\zeta)}, \tag{4.1}$$

and the Holmes-Thompson volume form $dV_{KT} = \sigma_{KT}(u)du$, given by

$$\sigma_{KT}(u) = \frac{1}{Vol(B^n)} \int_{\zeta} det(g_{ij})dv. \tag{4.2}$$

But in the case of Riemannian metric $F = \sqrt{g_{ij}(u)v^i v^j}$ both the volume forms are equal to the Riemannian volume form, that is $dV_{KT} = dV_{BK} = \sqrt{det(g_{ij}(u))}du = dV_\alpha$. Then the volume form dV is given by $dV = f(b)dV_\alpha$, where

$$f(b) = \begin{cases} \frac{\int_0^\pi sin^{n-2}t \ dt}{\int_0^\pi \frac{sin^{n-2}t}{\phi(b \ \cos \ t)^n} dt} & if \ dV = dV_{BK} \\ \frac{\int_0^\pi (sin^{n-2})T(b \ \cos \ t) \ dt}{\int_0^\pi sin^{n-2}t \ dt} & if \ dV = dV_{KT}. \end{cases} \tag{4.3}$$

Further, Chern and Shen in [5], obtained the formula for the S-curvature of (α,β)-metric in a local coordinate system in the form,

$$S=\left(2\psi-\frac{f^{'}(b)}{bf(b)}\right)(r_o+s_o)-\frac{\Phi}{2\alpha\Delta^2}(r_{oo}-2\alpha Q s_o), \tag{4.4}$$

where

$$\begin{aligned}
Q &= \frac{\phi^{'}}{\phi-s\phi^{'}},\\
\Delta &= 1+sQ+(b^2-s^2)Q^{'},\\
\psi &= \frac{Q^{'}}{2\Delta},\\
\Phi &= (sQ^{'}-Q)(n\Delta+1+sQ)-(b^2-s^2)(1+sQ)Q^{''},\\
r_{ij} &= \frac{1}{2}(b_{i|j}+b_{j|i}),\quad r_j=b^i r_{ij},\quad r_o=r_i v^i,\quad r_{oo}=r_{ij}v^i v^j,\\
s_{ij} &= \frac{1}{2}(b_{i|j}-b_{j|i}),\quad s_j=b^i s_{ij},\quad s_o=s_i v^i,\quad b^i=b_j a^{ji}.
\end{aligned}$$

Later, they have proved that if the length b is constant, then $r_o+s_o=0$, and the S-curvature of an (α,β)-metric reduces to

$$S=-\frac{\Phi}{2\alpha\Delta^2}\left(r_{oo}-2\alpha Q s_o\right). \tag{4.5}$$

Similarly, the length of b is constant in homogeneous Finsler spaces. Therefore, the S-curvature of a homogeneous Finsler space with (α,β)-metric can be written by the form (4.5) and in a homogeneous space, it is sufficient to compute S-curvature at origin K, as follows:

Suppose w be a G-invariant vector field in $\mathfrak{m}$ corresponding to 1-form β with length $c=|w|$. Also, let $\{w_1,w_2,.....,w_n\}$ be an orthonormal basis of $\mathfrak{m}$ such that $w_n=\frac{w}{c}$. Then there exists a neighborhood N of origin $eK=K$ in G/K, such that the mapping

$$(exp\,u^1w_1,\ exp\,u^2w_2,.........exp\,u^nw_n)K\longmapsto(u^1,\ u^2,\ .....,u^n), \tag{4.6}$$

defines a local coordinate system on N [13] and in [8], it is proved that $\tilde{w}=c\frac{\partial}{\partial u^n}$. Now, we calculate b_i.

$$b_i=\beta(\frac{\partial}{\partial u^i})=\langle\tilde{w},\frac{\partial}{\partial u^i}\rangle=c\langle\frac{\partial}{\partial u^n},\frac{\partial}{\partial u^i}\rangle, \tag{4.7}$$

thus, $b_n=c$ and all other $b_i=0\ \forall\ i\neq n$ at the origin.
Next, by differentiation with respect to u^i and u^j we get

$$\frac{\partial b_j}{\partial u^i} = c\frac{\partial}{\partial u^i}\langle\frac{\partial}{\partial u^n},\frac{\partial}{\partial u^j}\rangle \tag{4.8}$$

$$= c\left(\langle\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^n},\frac{\partial}{\partial u^j}\rangle+\langle\frac{\partial}{\partial u^n},\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}\rangle\right), \tag{4.9}$$

and

$$\frac{\partial b_i}{\partial u^j} = c\frac{\partial}{\partial u^j}\langle\frac{\partial}{\partial u^n},\frac{\partial}{\partial u^i}\rangle \tag{4.10}$$

$$= c\left(\langle\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^n},\frac{\partial}{\partial u^i}\rangle+\langle\frac{\partial}{\partial u^n},\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i}\rangle\right). \tag{4.11}$$

In [8], the formula of Γ^h_{ij} is given by

$$\Gamma^h_{ij}(K)=\frac{1}{2}\left(-\langle[w_i,\,w_j]_{\mathfrak{m}},\,w_h\rangle+\langle[w_h,\,w_i]_{\mathfrak{m}},\,w_j\rangle+\langle[w_h,\,w_j]_{\mathfrak{m}},\,w_i\rangle\right),\quad i\geq j. \tag{4.12}$$

and

$$\begin{aligned}
S_{ij}(K) &= \frac{1}{2}(b_{i|j}-b_{j|i})\\
&= \frac{1}{2}\left(\frac{\partial b_i}{\partial u^j}-b_h\Gamma^h_{ij}-\frac{\partial b_j}{\partial u^i}+b_h\Gamma^h_{ji}\right),\\
S_{ij}(K) &= \frac{c}{2}\left(\langle\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^n},\frac{\partial}{\partial u^i}\rangle+\langle\frac{\partial}{\partial u^n},\nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i}-\langle\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^n},\frac{\partial}{\partial u^j}\rangle-\langle\frac{\partial}{\partial u^n},\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}\rangle\right)\\
&= \frac{c}{2}\left(\langle\nabla_{\frac{\partial}{\partial u^n}}\frac{\partial}{\partial u^j},\frac{\partial}{\partial u^i}\rangle-\langle\nabla_{\frac{\partial}{\partial u^n}}\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\rangle+\langle\frac{\partial}{\partial u^n}\left[\frac{\partial}{\partial u^j},\frac{\partial}{\partial u^i}\right]\rangle\right),\\
S_{ij}(K) &= \frac{c}{2}\left(\langle\Gamma^h_{nj}(K)\,\frac{\partial}{\partial u^h},\,\frac{\partial}{\partial u^i}\rangle-\langle\Gamma^h_{ni}\,\frac{\partial}{\partial u^h},\,\frac{\partial}{\partial u^j}\rangle\right)\\
&= \frac{c}{2}\left(\Gamma^i_{nj}-\Gamma^j_{ni}\right),
\end{aligned}$$

$$\begin{aligned}
S_{ij}(K) &= \frac{c}{4}\Big\{\left(-\langle[w_n,\,w_j]_{\mathfrak{m}},\,w_i\rangle+\langle[w_i,\,w_n]_{\mathfrak{m}},\,w_j\rangle+\langle[w_i,\,w_j]_{\mathfrak{m}},\,w_n\rangle\right)\\
&\quad-\left(-\langle[w_n,\,w_i]_{\mathfrak{m}},\,w_j\rangle+\langle[w_j,\,w_n]_{\mathfrak{m}},\,w_i\rangle+\langle[w_j,\,w_i]_{\mathfrak{m}},\,w_n\rangle\right)\Big\}, \qquad (4.13)\\
&= \frac{c}{2}\langle[w_i,\,w_j]_{\mathfrak{m}},\,w_n\rangle. \qquad (4.14)
\end{aligned}$$

Further,

$$s^i_j(K)=a^{ih}(K)s_{hj}(K)=\sum_{k=1}^{n}\delta_{ih}s_{hj}(K)=s_{ij}(K), \tag{4.15}$$

and

$$s_i(K)=b_h(K)s^h_i(K)=cs^n_i=cs_{ni}(K). \tag{4.16}$$

Therefore, for $v = v^i w_i \in \mathfrak{m}$, we have

$$\begin{aligned} s_o(v) &= s_i(K)v^i \\ &= cs_{ni}(K)v^i, \\ s_o(v) &= \frac{c^2}{2}v^i\langle[w_n,\, w_i]_{\mathfrak{m}},\, w_n\rangle \\ &= \frac{1}{2}\langle[cw_n,\, v^i w_i]_{\mathfrak{m}},\, cw_n\rangle \end{aligned} \tag{4.17}$$

$$s_o(v) = \frac{1}{2}\langle[w,\, v]_{\mathfrak{m}},\, w\rangle. \tag{4.18}$$

Further

$$\begin{aligned} r_{ij} &= \frac{1}{2}\bigg(b_{i|j} + b_{j|i}\bigg) \\ &= \frac{1}{2}\bigg(\frac{\partial b_i}{\partial u^j} - b_h\Gamma^h_{ij} + \frac{\partial b_i}{\partial u^j} - b_h\Gamma^h_{ji}\bigg), \end{aligned}$$

$$\begin{aligned} r_{ij} &= \frac{1}{2}\bigg(\frac{\partial b_i}{\partial u^j} + \frac{\partial b_j}{\partial u^i} - 2b_n\Gamma^n_{ij}\bigg) \\ &= \frac{1}{2}\bigg(\frac{\partial b_j}{\partial u^i} + \frac{\partial b_j}{\partial u^i}\bigg) - c\Gamma^n_{ij}, \end{aligned}$$

$$\begin{aligned} r_{ij} &= \frac{c}{2}\bigg(\langle\nabla_{\frac{\partial}{\partial u^j}}\ \frac{\partial}{\partial u^n}, \frac{\partial}{\partial u^i}\rangle + \langle\frac{\partial}{\partial u^n}, \nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i} + \langle\nabla_{\frac{\partial}{\partial u^i}}\ \frac{\partial}{\partial u^n}, \frac{\partial}{\partial u^j}\rangle + \langle\frac{\partial}{\partial u^n}, \nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}\rangle\bigg) - c\Gamma^n_{ij} \\ &= \frac{c}{2}\bigg(\Gamma^i_{nj} + \Gamma^j_{ni} + 2\Gamma^n_{ij}\bigg) - c\Gamma^n_{ij} \\ &= \frac{c}{2}\bigg(\Gamma^i_{nj} + \Gamma^j_{ni}\bigg), \end{aligned}$$

$$\begin{aligned} r_{ij} &= \frac{c}{4}\bigg\{\bigg(-\langle[w_n,\, w_j]_{\mathfrak{m}},\, w_i\rangle + \langle[w_i,\, w_n]_{\mathfrak{m}},\, w_j\rangle + \langle[w_i,\, w_j]_{\mathfrak{m}},\, w_n\rangle\bigg) \\ &\quad + \bigg(-\langle[w_n,\, w_i]_{\mathfrak{m}},\, w_j\rangle + \langle[w_j,\, w_n]_{\mathfrak{m}},\, w_i\rangle + \langle[w_j,\, w_i]_{\mathfrak{m}},\, w_n\rangle\bigg)\bigg\}, \end{aligned}$$

$$r_{ij} = -\frac{c}{2}\bigg(\langle[w_n,\, w_i]_{\mathfrak{m}},\, w_j\rangle + \langle[w_n,\, w_j]_{\mathfrak{m}},\, w_i\rangle\bigg). \tag{4.19}$$

therefore at the origin r_{oo} becomes

$$r_{oo} = r_{ij}v^i v^j \tag{4.20}$$

$$= -\langle[w,\, v]_{\mathfrak{m}}, v\rangle. \tag{4.21}$$

After, substituting all the above values into the equation (4.5), we obtain the formula for S-curvature, which is summarized in the following theorem:

Theorem 4.1. *Let $F = \alpha\phi(s)$ be a G-invariant (α, β)-metric on the reductive homogeneous Finsler space G/K with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then the S-curvature is given by*

$$S(K,\, v) = \frac{\Phi}{2\alpha\Delta^2}\Big(\langle [w,\, v]_{\mathfrak{m}},\, v\rangle + \alpha Q \langle [w,\, v]_{\mathfrak{m}},\, w\rangle\Big), \tag{4.22}$$

where $w \in \mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_K(G/K)$ of G/K at the origin K.

Next, we compute the S-curvature of homogeneous Finsler space with special (α, β)-metric $F = \frac{\beta^{m+1}}{\alpha^m} = \alpha(s^{m+1}) = \alpha\phi(s)$, where $\phi(s) = s^{m+1}$.
For (α, β)-metric, the identities given in the equation (4.4) reduce to the following:

$$Q = \frac{\phi'}{\phi - s\phi'} = -\frac{(m+1)}{ms}, \tag{4.23}$$

$$Q' = \frac{(m+1)}{ms^2}, \tag{4.24}$$

$$Q'' = \frac{-2(m+1)}{ms^3}, \tag{4.25}$$

$$\begin{aligned} \Delta &= 1 + sQ + (b^2 - s^2)Q' \\ &= \frac{1}{m}\left[(b^2 - s^2)\frac{(m+1)}{s^2} - 1\right], \\ \Phi &= (sQ' - Q)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'' \\ &= \frac{2(m+1)}{m^2 s^3}\Big\{b^2[n(m+1) - 1] - [n(m+2)]s^2\Big\}, \end{aligned}$$

Substituting the above values in the equation (4.22), we obtain the formula for S-curvature of the homogeneous Finsler space with special (α, β)-metric and obtain the following result.

Theorem 4.2. *Let $F = \frac{\beta^{m+1}}{\alpha^m}$ be a G-invariant (α, β)-metric on the reductive homogeneous Finsler space G/K with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then the S-curvature is given by*

$$\begin{aligned} S(K,\, v) = &\left[\frac{b^2(m+1)[n(m+1) - 1] - [n(m+1)(m+2)]s^2}{b^4(m+1)^2 + (m+2)^2 s^4 - [2b^2(m+1)(m+2)]s^2}\right] \\ &\left(\frac{s}{\alpha}\langle [w, v]_{\mathfrak{m}}, v\rangle - \frac{m+1}{m}\langle [w, v]_{\mathfrak{m}}, w\rangle\right), \end{aligned} \tag{4.26}$$

where $w \in \mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_K(G/K)$ of G/K at the origin K.

Definition 4.1. An n-dimentional Finsler space (M, F) is said to have almost isotropic S-curvature, if there exists a smooth function $c(u)$ on M and a closed 1-form u such that

$$S(u,\, v) = (n+1)\Big(c(u)F(v) + u(v)\Big), \quad u \in M,\ v \in T_u(M). \tag{4.27}$$

In addition, if u is zero, then (M, F) is said to have isotropic S-curvature. Also, if u is zero and $c(u)$ is constant, then (M, F) is said to have constant S-curvature.

Next, we give an application of theorem (4.2)

Theorem 4.3. *Let $F = \frac{\beta^{m+1}}{\alpha^m}$ be a G-invariant special (α, β)-metric on the reductive homogeneous Finsler space G/K with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then $(G/K, F)$ has isotropic S-curvature if and only if it has vanishing S-curvature.*

Proof. We only need to prove the necessary part. Suppose G/K has isotropic S-curvature, then we have

$$S(u, v) = (n+1)c(u)F(v), \quad u \in G/K, \ v \in T_u(G/K). \tag{4.28}$$

Letting $u = K$ and $v = w$ in the equation (4.26), we get $c(K) = 0$ and hence $S(K, v) = 0 \ \ \forall \ v \in T_u(G/K)$. Further, since F is a homogeneous metric, we have $S = 0$ everywhere. Therefore, G/K has vanishing S-curvature. □

5 Mean Berwald curvature of homogeneous Finsler space with (α, β)-metric $F = \frac{\beta^{m+1}}{\alpha^m}$

In this section, we find the mean Berwald curvature of the homogeneous Finsler space with special (α, β)-metric. We first discuss the notion of the mean Berwald curvature [7] of a Finsler space (M, F). For this, let

$$E_{ij}(u, v) = \frac{1}{2}\frac{\partial^2 S(u, v)}{\partial v^i \partial v^j}. \tag{5.1}$$

The E-tensor is a family of symmetric forms $E_v : T_uM \times T_uM \longrightarrow \mathbb{R}$ defined by $E_v(w, z) = E_{ij}(x, y)w^i z^j$, where $w = w^i \frac{\partial}{\partial u^i}, \ z = z^i \frac{\partial}{\partial u^i} \ \in T_uM, \ u \in M$. Then $E = \{E_v : v \in TM \setminus \{0\}\}$ is called the E-curvature or the mean Berwald curvature. To find it, we require the following computations.
At the origin $a_{ij} = \delta^i_j$ and therefore $v_i = v^i$

$$\begin{aligned} \alpha_{v^i} &= \frac{v_i}{\alpha}, \\ \beta_{v^i} &= b_i, \\ s_{v^i} &= \frac{\partial}{\partial v^i}\left(\frac{\beta}{\alpha}\right), \\ &= \left(\frac{b_i\alpha - s v_i}{\alpha^2}\right), \end{aligned} \tag{5.2}$$

$$\begin{aligned} s_{v^i v^j} &= \frac{\partial}{\partial v^j}\left(\frac{b_i\alpha - s v_i}{\alpha^2}\right), \\ &= \frac{\alpha^2\left[b_i\frac{v_j}{\alpha} - \left(\frac{b_j\alpha - \alpha v_j}{\alpha^2}\right)v_i - s\delta^i_l\right] - (b_i\alpha - s v_i)2\alpha\frac{v_j}{\alpha}}{\alpha^4}, \\ &= \frac{-(b_i v_j + b_j v_i)\alpha + 3 s v_i v_j - \alpha^2 s \delta^i_j}{\alpha^4}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} P &= \frac{b^2(m+1)[n(m+1)-1]-n(m+1)(m+2)s^2}{b^4(m+1)^2-2b^2(m+1)(m+2)s^2+(m+2)^2s^4}, \\ \frac{\partial P}{\partial v^j} &= \left\{\frac{[-2n(m+1)(m+2)]s}{A^2}+\frac{4b^2(m+1)(m+2)[n(m+1)-1]s}{A^3}\right. \\ &\left. -\frac{4n(m+1)(m+2)s^3}{A^3}\right\}s_{v^j}, \end{aligned}$$

(5.4)
$$\begin{aligned} \frac{\partial^2 P}{\partial v^i \partial v^j} &= \frac{1}{A^4}\Big\{-2n(m+1)(m+2)[A^2+4A(m+2)s^2] \\ &+4b^2(m+1)(m+2)[n(m+1)-1][B+6(m+2)s^2] \\ &-4n(m+1)(m+2)^2[3s^2A+6(m+2)s^4]\Big\}s_{v^i}s_{v^j} \\ &+\frac{1}{A^3}\Big\{-2n(m+1)(m+2)sA+4b^2(m+1)(m+2) \\ &[n(m+1)-1]s-4n(m+1)(m+2)^2s^3\Big\}s\,v^iv^j. \end{aligned}$$

From the equation (4.26), S-curvature at the origin is given by

(5.5)
$$S(K,v)=P\left(\frac{s}{\alpha}\right)\langle[w,v]_{\mathfrak{m}},v\rangle-P\left(\frac{m+1}{m}\right)\langle[w,v]_{\mathfrak{m}},w\rangle.$$

Further, we can write

(5.6)
$$S(K,v)=\chi_1-\left(\frac{m+1}{m}\right)\chi_2,$$

where

(5.7)
$$\chi_1=P\left(\frac{s}{\alpha}\right)\langle[w,v]_{\mathfrak{m}},\,v\rangle \quad \textit{and} \quad \chi_2=P\langle[w,v]_{\mathfrak{m}},w\rangle.$$

Therefore, the mean Berwald curvature becomes

(5.8)
$$E_{ij}=\frac{1}{2}\,\frac{\partial^2 S}{\partial v^i\partial v^j}=\frac{1}{2}\left[\frac{\partial^2\chi_1}{\partial v^i\partial v^j}-\left(\frac{m+1}{m}\right)\frac{\partial^2\chi_2}{\partial v^i\partial v^j}\right].$$

First consider

$$\begin{aligned}
\frac{\partial\chi_1}{\partial v^j} &= \frac{\partial}{\partial v^j}\Big\{P\Big(\frac{s}{\alpha}\Big)\langle[w,v]_{\mathfrak{m}},v\rangle\Big\},\\
&= \frac{\partial P}{\partial v^j}\Big(\frac{s}{\alpha}\langle[w,\,v]_{\mathfrak{m}},\,v\rangle\Big)+P\Big(\frac{S_{v^j}}{\alpha}-\frac{s\,v_j}{\alpha^3}\Big)\langle[w,\,v]_{\mathfrak{m}},\,v\rangle\\
&\quad +P\frac{s}{\alpha}\Big(\langle[w,\,w_j]_{\mathfrak{m}},\,v\rangle+\langle[w,\,v]_{\mathfrak{m}},\,w_j\rangle\Big),
\end{aligned}$$

$$\begin{aligned}
(5.9)\quad \frac{\partial^2\chi_1}{\partial v^i\partial v^j} &= \Big\{\frac{\partial^2 P}{\partial v^i\partial v^j}\,\frac{s}{\alpha}+\frac{\partial P}{\partial v^j}\Big(\frac{s_{v^i}}{\alpha}-\frac{s\,v_i}{\alpha}\Big)+\frac{\partial P}{\partial v^i}\Big(\frac{S_{v^j}}{\alpha}-\frac{s\,v_j}{\alpha^3}\Big)\\
&\quad +P\Big(\frac{s_{v^iv^j}}{\alpha}-\frac{v_i}{\alpha^3}\,s_{v^j}-\frac{v_i}{\alpha^3}\,s_{v^j}-s\frac{\delta^i_j}{\alpha^3}+3s\frac{v_iv_j}{\alpha^4}\Big)\Big\}\langle[w,\,v]_{\mathfrak{m}},v\rangle\\
&\quad +\Big(\frac{\partial P}{\partial v^j}\,\frac{s}{\alpha}+P\frac{s_{v^i}}{\alpha}-P\,s\frac{v_j}{\alpha^3}\Big)\Big(\langle[w,\,w_i]_{\mathfrak{m}},v\rangle+\langle[w,\,v]_{\mathfrak{m}},w_i\rangle\Big)\\
&\quad +\Big(\frac{\partial P}{\partial v^i}\,\frac{s}{\alpha}+P\frac{s_{v^i}}{\alpha}-P\,s\frac{v_i}{\alpha^3}\Big)\Big(\langle[w,\,w_j]_{\mathfrak{m}},v\rangle+\langle[w,\,v]_{\mathfrak{m}},w_j\rangle\Big)\\
&\quad +P\frac{s}{\alpha}\Big(\langle[w,\,w_j]_{\mathfrak{m}},w_i\rangle+\langle[w,\,w_i]_{\mathfrak{m}},w_j\rangle\Big).
\end{aligned}$$

Similarly, the derivative of χ_2 is obtained as follows

$$\begin{aligned}
\frac{\partial\chi_2}{\partial v^j} &= \frac{\partial P}{\partial v^j}\,\langle[w,\,v]_{\mathfrak{m}},w\rangle+P\,\langle[w,\,w_j]_{\mathfrak{m}},w\rangle,\\
\frac{\partial^2\chi_2}{\partial v^i\partial v^j} &= \frac{\partial^2 P}{\partial v^i\partial v^j}\,\langle[w,\,v]_{\mathfrak{m}},w\rangle+\frac{\partial P}{\partial v^j}\langle[w,\,w_i]_{\mathfrak{m}},w\rangle\\
&\quad +\frac{\partial P}{\partial v^i}\langle[w,\,w_j]_{\mathfrak{m}},w\rangle+0,\\
(5.10)\qquad \frac{\partial^2\chi_2}{\partial v^i\partial v^j} &= \frac{\partial^2 P}{\partial v^i\partial v^j}\,\langle[w,\,v]_{\mathfrak{m}},w\rangle+\frac{\partial P}{\partial v^j}\langle[w,\,w_i]_{\mathfrak{m}},w\rangle\\
&\quad +\frac{\partial P}{\partial v^i}\langle[w,\,w_j]_{\mathfrak{m}},\,w\rangle.
\end{aligned}$$

The above computation regarding mean Berwald curvature of the homogeneous Finsler space with special (α,β)-metric and the above result is summarized in the following theorem.

Theorem 5.1. *Let $F=\frac{\beta^{m+1}}{\alpha^m}$ be a G-invariant special (α,β)-metric on the reductive homogeneous Finsler space G/K with a decomposition of the Lie algebra $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. Then the mean Berwald curvature of the homogeneous Finsler space with special*

(α,β)-metric is given by

$$
\begin{aligned}
(5.11)\quad E_{ij} \;=\; & \frac{1}{2}\Bigg\{\Bigg[\frac{\partial^2 P}{\partial v^i \partial v^j}\,\frac{s}{\alpha} + \frac{\partial P}{\partial v^j}\Bigg(\frac{s_{v^i}}{\alpha} - \frac{s\,v_i}{\alpha}\Bigg) + \frac{\partial P}{\partial v^i}\Bigg(\frac{S_{v^j}}{\alpha} - \frac{s\,v_j}{\alpha^3}\Bigg) \\
& + P\Bigg(\frac{s_{v^i v^j}}{\alpha} - \frac{v_i}{\alpha^3}\,s_{v^j} - \frac{v_i}{\alpha^3}\,s_{v^i} - s\frac{\delta^i_j}{\alpha^3} + 3s\frac{v_i v_j}{\alpha^4}\Bigg)\Bigg]\langle [w,\, v]_{\mathfrak{m}}, v\rangle \\
& + \Bigg(\frac{\partial P}{\partial v^j}\,\frac{s}{\alpha} + P\frac{s_{v^i}}{\alpha} - P\,s\frac{v_j}{\alpha^3}\Bigg)\Bigg(\langle [w,\, w_i]_{\mathfrak{m}}, v\rangle + \langle [w,\, v]_{\mathfrak{m}}, w_i\rangle\Bigg) \\
& + \Bigg(\frac{\partial P}{\partial v^i}\,\frac{s}{\alpha} + P\frac{s_{v^i}}{\alpha} - P\,s\frac{v_i}{\alpha^3}\Bigg)\Bigg(\langle [w,\, w_j]_{\mathfrak{m}}, v\rangle + \langle [w,\, v]_{\mathfrak{m}}, w_j\rangle\Bigg) \\
& + P\frac{s}{\alpha}\Bigg(\langle [w,\, w_j]_{\mathfrak{m}}, w_i\rangle + \langle [w,\, w_i]_{\mathfrak{m}}, w_j\rangle\Bigg) \\
& - \Bigg(\frac{m+1}{m}\Bigg)\Bigg[\frac{\partial^2 P}{\partial v^i \partial v^j}\,\langle [w,\, v]_{\mathfrak{m}}, w\rangle + \frac{\partial P}{\partial v^j}\langle [w,\, w_i]_{\mathfrak{m}}, w\rangle \\
& + \frac{\partial P}{\partial v^i}\langle [w,\, w_j]_{\mathfrak{m}}, w\rangle\Bigg]\Bigg\}.
\end{aligned}
$$

where $w \in \mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is defined with tangent space $T_K(G/K)$ of (G/K) at the origin K.

References

[1] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kulwer Academic Publishers, Fundamental Theories of Physics 58, The Netherlands, 1993.

[2] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemannian-Finsler Geometry*, GTM-200, Springer-verlag 2000.

[3] K. Chandru, S. K. Narasimhamurthy, *On curvatures of homogeneous Finsler-Kropina space,*, Gulf Journal of Mathematics 5, 1 (2017), 73-83.

[4] K. Chandru, S. K. Narasimhamurthy, *On curvatures of homogeneous Finsler spaces with special (α,β)-metric*, IOSR J. Math. 13, 2 (2017), 47-53.

[5] X. Cheng, Z. Shen, *A class of Finsler metrics with isotropic S-curvature*, Israel J. Math. 169 (2009), 317-340.

[6] S. S. Chern, *Finsler geometry is just Riemannian geometry without the quadratic restriction*, Notices Amer. Math. Soc. 43, 9 (1996), 959-963.

[7] S. S. Chern, Z. Shen, *Riemann-Finsler Geometry*, Nankai Tracts in Mathematics, Vol. 6, World Scientific Publishers, 2005.

[8] S. Deng, *The S-curvature of homogeneous Randers spaces*, Differ. Geom. Appl. 27 (2009), 75-84.

[9] S. Deng, *Homogeneous Finsler Spaces*, Springer Monographs in Mathematics New York, 2012.

[10] S. Deng, Z. Hou, *The group of isometries of a Finsler space*, Pacific J. Math. 207 (2002), 149-155.

[11] S. Deng, Z. Hou, *Invariant Randers metrics on homogeneous Riemannian manifolds*, J. Phys. A: Math, Gen. 37 (2004), 4353-4360; Corrigendum, ibid 39 (2006), 5249-5250.

[12] S. Deng, X. Wang, *The S-curvature of homogeneous (α, β)-metrics*, Balkan J. Geom. Appl. 15, 2 (2010), 39-48.

[13] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, 2nd ed., Academic Press, New York, 1978.

[14] J. Milnor, *Curvature of left invariant Riemannian metrics on Lie groups*, Adv. Math. 21 (1976), 293-329.

[15] S. B. Myers, N. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. Math. 40 (1939), 400-416.

[16] G. Shankar, K. Kaur, *Homogeneous Finsler space with infinite series (α, β)-metric*, Appl. Sci. 21 (2019), 220-236.

[17] G. Shankar, S. Rani, *On S-curvature of a homogeneous Finsler space with square metric*, Int. J. Geom. Methods Mod. Phys. 17, 2 (2020), 2050019.

[18] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances in Mathematics 128 (1997), 306-328.

Author's address:

Kallajara Shankrappa Venkatesha and Senajji Kampalapa Narasimhamurthy
Department of Mathematics, Kuvempu University, Jnana Shyadri Shankaraghatta, Karnataka, India.
E-mail:ksvenkateshmaths@gmail.com , nmurthysk@gmail.com

Keriyappa Chandru
Department of Mathematics, DVS College of Arts and Science, Karnataka, India.
E-mail: kchandru2510@gmail.com