Ricci-Bourguignon solitons and almost solitons with concurrent vector field

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Abstract. The purpose of this article is to study Ricci-Bourguignon solitons and almost solitons with concurrent potential vector on Riemannian manifolds. First, we give the classification theorems for Ricci-Bourguignon solitons and almost solitons with concurrent potential vector field. Then, we obtain some geometric properties of these solitons on submanifolds of a Riemannian manifold equipped with a concurrent potential vector field. Finally, we classify Ricci-Bourguignon solitons and almost solitons on Euclidean hypersurface whose the potential vector field arisen from the position vector field of Euclidean hypersurface.

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1 Introduction

Ricci solitons are special solutions of the Ricci flow equation

$$\frac{\partial g}{\partial t} = -2Ric, \tag{1.1}$$

where Ric is the Ricci curvature with respect to the metric g. A Ricci soliton is a natural generalization of Einstein metric such that

$$Ric + \frac{1}{2}L_X g = \lambda g, \tag{1.2}$$

where $L_X g$ denotes the Lie derivative of the metric g along X and λ is a constant. A Ricci soliton on a Riemannian manifold (M^m, g) is said to be shrinking, steady or expanding according as λ is positive, zero or negative, respectively. A trivial Ricci soliton is one for which X is zero or Killing, i.e., $L_X g = 0$ in which case the metric becomes Einstein. If the potential vector field X is of gradient type, $X = \nabla f$, for some smooth function f on M, then a Ricci soliton is called a gradient Ricci soliton.

During the last two decades, the geometry of Ricci solitons and their generalizations have been the focus of attention of many researchers. For example, if λ is a

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smooth function on (M^m, g), then (1.2) defines an almost Ricci soliton [10]. The study of the concept Ricci-Bourguignon and Ricci-Bourguignon almost solitons are introduced in a recent paper due to Dwivedi [6], which extends the Ricci and almost Ricci solitons, respectively. In this study, the author derived integral formulas for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons. Using the integral formula he showed that a compact gradient Ricci-Bourguignon almost soliton is isometric to an Euclidean sphere. In [9], Gomes, Wang and Xia proved that a compact nontrivial h-almost Ricci soliton of dimension $n \geq 3$ with h having defined signal and constant scalar curvature is isometric to a standard sphere. Moreover, they gave characterizations for a special class of gradient h-Ricci solitons.

On the other hand, in [4], Chen and Deshmukh classified the Ricci solitons with concurrent potential vector field and provided a necessary and sufficient condition for an isometric immersion to be a Ricci soliton into a manifold equipped with a concurrent vector field. The some results of [4] was generalized for h-almost Ricci solitons and almost Ricci solitons in [7] and [1], respectively.

Motivated by the above studies, the first aim of this paper is to give a classification for Ricci-Bourguignon solitons and almost solitons with concurrent potential vector field. Secondly, we obtain some geometric properties of Ricci-Bourguignon soliton and almost solitons on submanifolds of a Riemannian manifold equipped with a concurrent potential vector field. Finally, we classify these solitons on Euclidean hypersurfaces.

2 Preliminaries

In this section, we will present some definitions and formulas which will be useful for the establishment of the desired results. We start with the Ricci-Bourguignon solitons and almost solitons (RB solitons and almost solitons for short) which are extensions of the Ricci solitons.

2.1 RB solitons and almost solitons

Let (M^m, g) be an m-dimensional Riemannian manifold whose metric $g = g(t)$ is evolving according to the flow equation

$$\frac{\partial g}{\partial t} = -2(Ric - \rho R g), \tag{2.1}$$

where Ric is a Ricci tensor of the manifold, R its scalar curvature and ρ is a real constant. The flow in Eq. (2.1) is called a *Ricci-Bourguignon flow* (RB flow for short), see [2]. This family of geometric flows contains, the famous Ricci flow ($\rho = 0$), the Einstein flow ($\rho = \frac{1}{2}$), the traceless Ricci flow ($\rho = \frac{1}{m}$) and the Schouten flow ($\rho = \frac{1}{2(m-1)}$). As in the Ricci flow case, Dwivedi [6] give the following definition.

Definition 2.1. (Dwivedi [6]) An RB soliton is a Riemannian manifold (M^m, g) endowed with a vector field X on M that satisfies

$$Ric + \frac{1}{2} L_X g = \lambda g + \rho R g, \tag{2.2}$$

where $L_X g$ denotes the Lie derivative of the metric g with respect to the vector field X and $\lambda \in \mathbb{R}$ is a constant.

If X becomes the gradient of a smooth function $f \in \mathbb{C}^\infty$ then the RB soliton is called *gradient RB soliton* and is shrinking, steady or expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively.

Definition 2.2. (Almost solitons) A Riemannian manifold (M^m, g) is an RB almost soliton and an RB h-almost soliton, respectively, if there is a vector field X, a soliton function $\lambda : M^m \to \mathbb{R}$ and a function $h : M^m \to \mathbb{R}$ such that

$$Ric + \frac{1}{2} L_X g = \lambda g + \rho R g, \text{ (RB almost soliton, [6])} \tag{2.3}$$

$$Ric + \frac{h}{2} L_X g = \lambda g + \rho R g, \text{ (RB h-almost soliton, [11])}, \tag{2.4}$$

where $L_X g$ denotes the Lie derivative of the metric g along X, R is scalar curvature and ρ is a constant.

Moreover, if X is a gradient vector field of a smooth function f, i.e. $X = \nabla f$, an RB almost soliton is called a *gradient RB almost soliton*

$$Ric + \nabla^2 f = \lambda g + \rho R g \tag{2.5}$$

and RB h-almost soliton is called a *gradient RB h-almost soliton*

$$Ric + h \nabla^2 f = \lambda g + \rho R g, \tag{2.6}$$

respectively. The almost solitons are said to be shrinking, steady and expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$. Also if λ has no definitive sign, the RB h-almost soliton will be called indefinite.

2.2 Some basic facts for submanifolds

Let (M^m, g) be an m-dimensional submanifold of a k-dimensional Riemannian manifold $(\widetilde{M}^k, \widetilde{g})$ and $\varphi : M^m \to \widetilde{M}^k$ be an isometric immersion. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + II(X, Y) \tag{2.7}$$

and

$$\widetilde{\nabla}_X N = -A_N X + D_X N \tag{2.8}$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\widetilde{\nabla}$, ∇ and D are the Levi-Civita connections on M^m and $\widetilde{M}^k$ and normal connection of M^m, respectively. Also, II is the second fundamental form related to the shape operator A by

$$\widetilde{g}(II(X, Y), N) = g(A_N X, Y). \tag{2.9}$$

For vectors X, Y, Z, W tangent to M^m, the equation of Gauss is given by

$$\begin{aligned} g(R(X,Y)Z,W) &= \widetilde{g}(\widetilde{R}(X,Y)Z,W) \\ &\quad +\widetilde{g}(II(X,W),II(Y,Z)) \\ &\quad -\widetilde{g}(II(X,Z),II(Y,W)). \end{aligned} \tag{2.10}$$

The normalized trace of the second fundamental form

$$H = \frac{1}{m} trace(II), \tag{2.11}$$

where $m = \dim M$, is called the mean curvature vector of M^m in $\widetilde{M}^k$. The submanifold M^m is totally geodesic in $\widetilde{M}^k$ if $II = 0$, and minimal if $H = 0$. If $II(X,Y) = g(X,Y)H$ for all $X, Y \in TM$, then M^m is totally umbilical.

2.3 Concurrent vector fields

The notion of concircular vector fields on a Riemannian manifold M^m was introduced by Fialkow [8] which satisfies

$$\nabla_Z X = \nu Z, \quad Z \in TM, \tag{2.12}$$

where ∇ denotes the Levi-Civita connection, TM the tangent bundle of M^m and ν a non-trivial function on M^m. Also, Yano [12] proved that if the holonomy group of a Riemannian m-manifold M^m leaves a point invariant, then there exists a vector field X on M^m which satisfies

$$\nabla_Z X = Z \tag{2.13}$$

for any vector Z tangent to M^m. Such a vector field X is called a *concurrent vector field*.

2.4 Warped product

Let (A^ℓ, g_A) and (B^k, g_B) be two Riemannian manifolds with the dimension ℓ and k and let π and σ denote the natural projections of $A \times B$ onto A and B, respectively. The *warped product* $A \times_\beta B$ with respect to warping function $\beta \in \mathcal{C}^\infty_{>0}(A)$ is defined as the product manifold $A \times B$ endowed with the metric $g = \pi^* g_A + \beta^2 \sigma^* g_B$, where * denotes the pull-back operator on tensors [5].

Example. Let I be an open interval of the real line $\mathbb{R}$ with s as its arclength and E be a Riemannian manifold. We can consider $I \times_s E$ as a warped product manifold. In this case, the metric tensor g of $I \times_s E$ is given by $g = ds^2 + s^2 g_E$, where g_E is the metric tensor of the second factor E. Put $X = s\frac{\partial}{\partial s}$. Following Proposition 4.1 of ([3], page 79), we can say the vector field X satisfies Eq. (2.13). So X is a concurrent vector field.

3 Main results

For RB h-almost solitons with concurrent potential vector field, we have the following theorem:

Theorem 3.1. *An RB h-almost soliton $(M^m, g, X, h, \lambda, \rho)$ on a Riemannian m-manifold (M^m, g), $m \geq 3$, has concurrent potential field X if and only if the following two conditions hold:*

i. The RB h-almost soliton is shrinking, steady or expanding with $\lambda = h - \rho R$.

ii. M^m is an open part of a warped product manifold $(I \times_s E, ds^2 + s^2 g_E)$, where I is an open interval with arclength s and E is an Einstein $(m-1)$-manifold whose Ricci tensor satisfies $Ric_E = (m-2)g_E$.

Proof. If $(M^m, g, X, h, \lambda, \rho)$ is an RB h-almost soliton on a Riemannian m-manifold with a concurrent potential field X, then

$$\nabla_Y X = Y \tag{3.1}$$

holds for every $Y \in TM$. We also have the following equality from the above condition and the definition of Lie derivative

$$(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 2g(Y, Z) \tag{3.2}$$

for any Y, Z tangent to M^m. Combining this with (2.4), we obtain

$$Ric(Y, Z) = (\lambda + \rho R - h)g(Y, Z), \tag{3.3}$$

which shows that M^m is an Einstein $(m-1)$-manifold. Since $m \geq 3$ and M^m is Einstein, M^m has constant scalar curvature.

Straightforward calculations yield

$$K(X, Y) = 0 \tag{3.4}$$

for each unit vector Y orthogonal to X. Hence

$$Ric(X, X) = 0. \tag{3.5}$$

Comparing (3.3) and (3.5) we see that, M^m is a Ricci-flat. Therefore, we get $\lambda = h - \rho R$. Then since ρ and h are an arbitrary constant and an arbitrary real function, respectively, and M^m has constant scalar curvature it follows that the RB h-almost soliton is shrinking, steady or expanding one.

Now, put $X = \mu\xi_1$, where ξ_1 is a unit vector field tangent to M^m. By continuity, we can extend ξ_1 to local orthonormal frame $\{\xi_i\}_{i=1}^m$ on M^m. Then we can find

$$\xi_1 \mu = 1 \text{ and } \nabla_{\xi_1}\xi_1 = 0. \tag{3.6}$$

By using the second equation in (3.6) we conclude that the integral curves of ξ_1 are geodesics in M^m. Now, we set $\mathcal{C}_1 = \mathrm{Span}\{\xi_1\}$ and $\mathcal{C}_2 = \mathrm{Span}\{\xi_2, ..., \xi_m\}$. Therefore $\mathcal{C}_1 = \mathrm{Span}\{\xi_1\}$ is a totally geodesic distribution so that the leaves of $\mathcal{C}_1$ are geodesics of M^m. We also know that $\mathcal{C}_2$ is an integrable distribution and moreover, the second fundamental form H of each leaf L of $\mathcal{C}_2$ in M^m satisfies

$$H(\xi_1, \xi_2) = -\frac{\delta_{ij}}{\mu}\xi_1, \;\; 2 \leq i, j \leq m. \tag{3.7}$$

So, the mean curvature of each leaf L equals $-\mu^{-1}$. The above equation also implies that each leaf of $\mathcal{C}_2$ is a totally umbilical hypersurface of M^m. By definition of X we have

$$\xi_2\mu = ... = \xi_m\mu = 0. \tag{3.8}$$

Therefore, we conclude that $\mathcal{C}_2$ is a spherical distribution. By Theorem 4.4 in [3], we say that M^m is locally a warped product manifold $I \times_{\beta(s)} E$ endowed with the warped metric $g = ds^2 + \beta^2(s)g_E$ such that $\xi_1 = \frac{\partial}{\partial s}$. Then it follows that

$$K(X, Y) = -\frac{\beta''(s)}{\beta(s)} \tag{3.9}$$

for any unit vector Y orthogonal to X, where K is the sectional curvature of M^m. Using this equality together with Eq. (3.4) we arrive $\beta''(s) = 0$. Then we have $\beta(s) = k_1 s + k_2$ for some constants k_1, k_2.

If $k_1 = 0$, then we see that $I \times_{\beta(s)} E$ is a Riemannian product and so every leaf of $\mathcal{C}_2$ is totally geodesic in M^m. Then μ must be zero and this case leads to a contradiction with Eq. (3.7). Hence we must have $k_1 \neq 0$. In here, we can get $\beta(s) = s$. This shows that M^m is locally a warped product manifold $I \times_s E$.

Finally, since M^m is a Ricci flat, by Corollary 4.1 of [3], E is an Einstein manifold and implies $Ric_E = (m-2)g_E$. The converse statement is straightforward. This completes the proof. □

As a direct corollary of the above theorem, we have:

Corollary 3.2. *There exist shrinking, steady or expanding RB h-almost solitons with concurrent potential field.*

Moreover, in the above theorem;

1) if $h = 1$ and $\rho = 0$, then the *RB h*-almost soliton reduces to the standard result for the Ricci soliton (or almost Ricci soliton), in this case $\lambda = 1$. Therefore there is no steady or expanding soliton with concurrent potential field.

2) if $h = 1$ and λ is a constant then we obtain:

Proposition 3.3. *An RB soliton $(M^m, g, X, \lambda, \rho)$ on a Riemannian m-manifold (M^m, g), $m \geq 3$, has concurrent potential field X if and only if the following two conditions hold:*

i. The RB soliton is shrinking, steady or expanding with $\lambda = 1 - \rho R$.

ii. M^m is an open part of a warped product manifold $(I \times_s E, ds^2 + s^2 g_E)$, where I is an open interval with arclength s and E is an Einstein $(m-1)$-manifold whose Ricci tensor satisfies $Ric_E = (m-2)g_E$.

Therefore, in constrast of Ricci soliton case, there is shrinking, steady or expanding RB soliton with concurrent vector field.

3) if $h = 1$ then the RB h-almost soliton reduces to the RB almost soliton, in this case $\lambda = 1 - \rho R$. Therefore there is shrinking, steady or expanding RB almost soliton with concurrent vector field.

The next results include some properties of RB solitons and almost solitons on submanifolds of Riemannian manifolds equipped with a concurrent potential field. Suppose that we have an isometric immersion $\varphi : M^m \to \widetilde{M}^k$, where (M^m, g) is a Riemannian submanifold of $(\widetilde{M}^k, \widetilde{g})$. Here, $X^\top$ and $X^\perp$ are respectively the tangential and normal components of X on M^m. As before, let D, II and A be the normal connection, the second fundamental form and the shape operator of M^m in $\widetilde{M}^k$, respectively.

Theorem 3.4. *Let $\widetilde{M}^k$ be a Riemannian manifold endowed with a concurrent vector field X. Then submanifold M^m in $\widetilde{M}^k$ has an RB h-almost soliton structure $(M^m, g, X^\top, h, \lambda, \rho)$ if and only if the Ricci tensor of M^m satisfies*

$$Ric(V, W) = (\lambda + \rho R - h)g(V, W) - h\widetilde{g}(II(V, W), X^\perp) \tag{3.10}$$

for any V, W tangent to M^m.

Proof. Let $\varphi : M^m \to \widetilde{M}^k$ be an isometric immersion. We write

$$X = X^\top + X^\perp. \tag{3.11}$$

Because of X is a concurrent vector field on $\widetilde{M}^k$, by Eq. (3.11), Eq. (2.13) and the formulas of Gauss and Weingarten, we obtain

$$V = \nabla_V X^\top - A_{X^\perp} V + II(V, X^\top) + D_V X^\perp \tag{3.12}$$

for any V tangent to M^m. By comparing the tangential and normal components from the last equation we arrive

$$\nabla_V X^\top = A_{X^\perp} V + V, \tag{3.13}$$

$$II(V, X^\top) = -D_V X^\perp. \tag{3.14}$$

Therefore, by using Eq. (3.13) and the definition of Lie derivative we obtain the following equality

$$\begin{aligned} (L_{X^\top} g)(V, W) &= g(\nabla_V X^\top, W) + g(\nabla_W X^\top, V) \\ &= 2g(A_{X^\perp} V, W) + 2g(V, W) \\ &= 2\widetilde{g}(II(V, W), X^\perp) + 2g(V, W) \end{aligned} \tag{3.15}$$

for V, W tangent to M^m.

Finally, by using Eq. (2.4) and the above equation we conclude that $(M^m, g, X^\top, h, \lambda, \rho)$ is an RB h-almost soliton if and only if we have

$$\begin{aligned} &Ric(V, W) + h\widetilde{g}(II(V, W), X^\perp) + hg(V, W) \\ &\qquad = (\lambda + \rho R)g(V, W). \end{aligned} \tag{3.16}$$

This completes the proof. □

Corollary 3.5. *Under the hypotheses of Theorem 3.4, if we take RB soliton (or RB almost soliton) instead of RB h-almost soliton then the Ricci tensor of M^m satisfies*

$$Ric(V,W) = (\lambda + \rho R - 1)g(V,W) - \widetilde{g}(II(V,W), X^{\perp}) \tag{3.17}$$

for any V, W tangent to M^m.

As an application of Theorem 3.4, we can give a criterion for the scalar curvature of M^m.

Proposition 3.6. *Let $(M^m, g, X^{\top}, h, \lambda, \rho)$ be an RB h-almost soliton on minimal submanifold M^m in $\widetilde{M}^k$ and $\rho \neq 2/m$. Then the scalar curvature of M^m is given by $m(\lambda - h)/(2 - m\rho)$.*

Proof. The assumption in this proposition with Theorem 3.4 together give the following equality

$$Ric(V,W) = (\lambda + \rho R - h)g(V,W) - h\widetilde{g}(II(V,W), X^{\perp}). \tag{3.18}$$

Because of M^m is minimal in $\widetilde{M}^k$, the mean curvature vector is zero, hence $\widetilde{g}(H, X^{\perp}) = 0$. Then from the above equation we have

$$\sum_{j=1}^{m} Ric(\xi_j, \xi_j) = m(\lambda + \rho R - h). \tag{3.19}$$

Thus M^m has constant scalar curvature $m(\lambda - h)/(2 - m\rho)$. This completes the proof. □

Corollary 3.7. *Under the hypotheses of Proposition 3.6, if we take RB soliton (or RB almost soliton) instead of RB h-almost soliton then the scalar curvature of M^m is given by $m(\lambda - 1)/(2 - m\rho)$.*

Proposition 3.8. *Let $(M^m, g, X^{\top}, h, \lambda, \rho)$ be an RB h-almost soliton on a hypersurface of M^m of $(\mathbb{E}^{m+1}, g_0)$. Then at most two distinct principal curvatures of M^m are given by*

$$\kappa_1, \kappa_2 = \frac{h\mu + m\varepsilon \pm \sqrt{(h\mu + m\varepsilon)^2 - 4(\lambda + \rho R - h)}}{2}, \tag{3.20}$$

where ε is the mean curvature, i.e., $H = \varepsilon N$ and μ is the support function, i.e., $\mu = < N, X >$ with N being a unit normal vector field.

Proof. By the assumption in theorem, there exists an *RB h*-almost soliton on a hypersurface of M^m of $\mathbb{E}^{m+1}$ such that $X^{\top}$ is the tangential component of the position vector field X. Also, let $\{\xi_i\}_{i=1,\dots,m}$ be an orthonormal frame on M^m such that $\xi_1, ..., \xi_m$ are eigenvectors of the shape operator A_N. Therefore we have

$$A_N \xi_i = \kappa_i \xi_i, \quad i = 1, ...m. \tag{3.21}$$

By using the Eq. (2.10) we get

$$\begin{aligned} Ric(V,W) &= m g_0(II(V,W), H) \\ &\quad - \sum_{i=1}^{m} g_0(II(V,\xi_i), II(W,\xi_i)) \end{aligned} \tag{3.22}$$

for all V, W tangent to M^m. Combining this with the Eq. (3.21) and Theorem 3.4, we conclude that $(M^m, g, X^\top, h, \lambda, \rho)$ is an RB h-almost soliton if and only if we have

$$m\varepsilon\kappa_i\delta_{ij} - \kappa_i\kappa_j\delta_{ij} = (\lambda + \rho R - h)\delta_{ij} - h\mu\kappa_i\delta_{ij}. \tag{3.23}$$

We can rewrite this equality as follows

$${\kappa_i}^2 - (h\mu + m\varepsilon)\kappa_i + \lambda + \rho R - h = 0, \quad i = 1, ...m. \tag{3.24}$$

Solving this equation, we get the desired result. □

Corollary 3.9. *Under the hypotheses of Proposition 3.8, if we take RB soliton (or RB almost soliton) instead of RB h-almost soliton then at most two distinct principal curvatures of M^m are given by*

$$\kappa_1, \kappa_2 = \frac{\mu + m\varepsilon \pm \sqrt{(\mu + m\varepsilon)^2 - 4(\lambda + \rho R - 1)}}{2}. \tag{3.25}$$

Finally, in the last result we discuss the classification of the RB h-almost solitons on Euclidean hypersurfaces.

Theorem 3.10. *Let $(M^m, g, X^\top, h, \lambda, \rho)$ be an RB h-almost soliton on a hypersurface of M^m of $\mathbb{E}^{m+1}$ with $\lambda = h - \rho R$. Then M^m is an open portion of one of the following hypersurfaces of $\mathbb{E}^{m+1}$:*

i) A totally umbilical hypersurface;

ii) A flat hypersurface generated by lines through the origin o of $\mathbb{E}^{m+1}$;

iii) A spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{E}^{m-k}$, $2 \leq k \leq m-1$.

Proof. Suppose that $(M^m, g, X^\top, h, \lambda, \rho)$ is an RB h-almost soliton on a hypersurface of M^m of $\mathbb{E}^{m+1}$ with $\lambda = h - \rho R$. So from Proposition 3.8, we have

$$\begin{aligned} \kappa_1 &= \frac{h\mu + m\varepsilon + \sqrt{(h\mu + m\varepsilon)^2 - 4(\lambda + \rho R - h)}}{2}, \\ \kappa_2 &= \frac{h\mu + m\varepsilon - \sqrt{(h\mu + m\varepsilon)^2 - 4(\lambda + \rho R - h)}}{2}. \end{aligned} \tag{3.26}$$

If M^m has only one principal curvature, we say that M^m is a totally umbilical. If we combine $\lambda = h - \rho R$ with Eq. (3.26), we find $\kappa_1 = h\mu + m\varepsilon$ and $\kappa_2 = 0$, respectively. Also, let n and $m - n$ be multiplicities of κ_1 and κ_2, respectively; for some n with $1 \leq n < m$. Then we have $m\varepsilon = n\kappa_1$. Using this equality we deduce that

$$m(1 - n)\varepsilon = nh\mu. \tag{3.27}$$

In the case $n = 1$, we obtain $\mu = 0$. So, the concurrent vector field X is tangent to M^m. It follows from Eq. (2.13) that $\widetilde{\nabla}_Z X = Z$. Therefore integral curves of X are part of lines through the origin in $\mathbb{E}^{m+1}$. The rest of the proof is similar as in Theorem 6.1 of [4]. So we omit it here. □

Note that, in the Theorem 3.10, the same results hold for RB soliton (or RB almost soliton) on a hypersurface of M^m of $\mathbb{E}^{m+1}$ with $\lambda = 1 - \rho R$.

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